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On retraction problem concerning inclusion of $F-$contractions in almost contractions

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Abstract

We have established, in the context of metric spaces, that an $F-$contraction restricted to appropriate neighborhood of its fixed point is an almost contraction and obtain retraction problem on complete metric spaces. We have illustrated such an inclusion using an example studied in (D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. 2012, 2012:94) and in (Minak et al, *Recent developments about multivalued weakly Picard operators*, Bull. Belg. Math. Soc. Simon Stevin 22 (2015)).

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1 Introduction

In this work we desire to show that the difference between $F-$contractions and almost contractions is outside certain retracts of their respective domains. In other words, $F-$contractions restricted to these nontrivial retracts are, precisely, almost contractions. Benefits of such studies include the fact that while $F-$contractions, being *Picard operators*, restrict investigations to classes of operators with unique fixed points (or limited to problems with continuous models), the almost contractions, as *weakly Picard operators* accommodate certain discontinuities in models even in the absence of uniqueness of fixed points. We recall, Andres [3, 6],

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in his studies of multivalued Poincare-Andronov (translation) operators, identified a drawback inherent in classical approaches as that of being based on uniqueness of fixed points which renders their generalizations impossible in the absence of uniqueness especially when discontinuous nonlinearities are involved. This informs the need to extend uniqueness-based formulations into classes of discontinuous formulations (or problems that need not require uniqueness of solutions) in other to obtain more realistic generalizations of most classical formulations.

A very important problem in this direction turns out to be the *retraction problem* which is invaluable in the studies of fixed point properties of spaces with respect to classes of mappings. It is well known that *retraction problems* are important cases of extension problems concerned with existence of a nontrivial extension $r : X \longrightarrow A$ of the identity mapping $I : A \longrightarrow A$ on a subspace A of a metric space (X, d) such that certain properties of the identity mapping are preserved; like continuity, compactness, connectedness etc. In general, a retraction is a mapping $r : X \longrightarrow A$ such that $r(a) = a$ for all $a \in A$ denoted by $r : X \supset A$. In this case A is called a *retract* of X. Clearly, every set is its own retract and every singleton is a retract. Both are called *trivial retracts.* Important nontrivial retracts include subsets with nonempty interiors which satisfy the extension property. It is, also, well known that every retract A of a Hausdorff space X has the same fixed point property as the space X - so that it becomes very economical to concentrate on retracts for the purposes of existence and computations of fixed points. Its effectiveness in recent investigations and analysis of structures of fixed point sets can be found in [6, 7] and their bibliographies while theory of retracts can be found in [11] and bibliography.

Definition 1 *[4] Let (X, d) be a metric space, $\delta \in (0, 1)$ and $k \geq 0$, then a mapping $T : X \longrightarrow X$ is called $(\delta, k)-$weak contraction (or a weak contraction) if and only if*

$$d(Tx, Ty) \leq \delta d(x, y) + kd(y, Tx), \text{ for all } x, y \in X. \tag{1}$$

Remark 2 *A current terminology (see [4]) for the class of $(\delta, k)-$weak contractions is **almost contractions.***

In 2012, Wardowski [12] formulated a new kind of contractive self-mapping T, called F-contraction, on metric spaces (X, d). Let $\tau > 0$ and $F : (0, \infty) \longrightarrow \mathbb{R}$ a strictly increasing function such that (i) $\lim_{n\to\infty} \alpha_n = 0$ iff $\lim_{n\to\infty} F(\alpha_n) = -\infty$ for all sequences $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers and (ii) there exists $k \in (0, 1)$ such that $\lim_{\alpha\downarrow 0} \alpha^k F(\alpha) = 0$, then a mapping T is called $F-$contraction if $\tau + F(d(Tx, Ty)) < F(d(x, y))$. $F-$contractions are known to be Picard operators while the almost contractions are weakly Picard operators. We shall establish that given any $F-$contraction there exists a neighborhood of its fixed point on which the $F-$contraction is an almost contraction. We recall, an operator $T : X \longrightarrow X$, on a metric space (X, d), is called a weakly Picard operator (see [4, 10] for example) if and only if for all $x \in X$, the sequence of Picard iterations $\{T^n x\}_{n=0}^{\infty}$ converges to a limit $x^* \in Fix(T)$ where $Fix(T)$ is the fixed-point-set of T. T is called Picard operator if $Fix(T)$ is a singleton (i.e if the sequence of Picard iterations $\{T^n x\}_{n=0}^{\infty}$ converges to a unique fixed point $x^* \in X$). The introduction of weakly Picard operators by Rus (see [4, 10] for example) motivates the urgent need to investigate fixed point properties of spaces with respect to such operators taking into considerations their simplicity both theoretically and in applications. These facts motivated studies on inclusion of $F-$contractions in almost contractions [13, 14]. In a very recent submission the author [15] proposed the use of almost contraction principle to establish whether the class of weakly Picard operators is a dense subset of the class of operators of which their sequences of Krasnoselskii iterations $\lambda x + (1 - \lambda)Tx, \lambda \in [0, 1]$, converge to their respective fixed points in Banach spaces. In line with this, a method of almost contraction was applied in [14] to obtain fixed point results for a very wide class of Lipschitzian mappings. Very recently, an interesting analysis on the method used in [13, 14] is initiated in [2] and a complementary investigation of the method has been treated and reported in [16] while studies of inclusions for multivalued weakly Picard operators are mentioned in [8].

In a recent contribution by Minak et al [8], an example of $F-$contractions (see Example 6 below) due to Wardowski [12] is used to illustrate independence of the class of $F-$contractions from the class of almost contractions introduced by V. Berinde [4]. Minak et al [8] remarked that Udo-utun's result in [13] establishes inclusion of $F-$contractions in almost contractions in the context of Banach spaces (see also [12]) implying that the inclusion may not hold in metric spaces. We emphasize that the inclusion studied in [13] applies when the $F-$contractions are restricted to certain neighborhoods of their unique fixed points, so the results in [13] are not expected to hold and may not be generalized on the whole metric space (X, d) but on certain subspaces (X_1, d). The present comments address the inclusion, in the context of complete metric spaces, of $F-$contractions in the class of almost contractions in the sense used in [13]. It should be recalled that in [13] Udo-utun initiated studies of inclusion of $F-$contractions in the class of almost contractions on Banach spaces by employing arguments concerning existence of a bounded deleted neighbohood $(X_1, d) \subset (X, d)$ of the fixed point of an $F-$contraction $T : (X, d) \longrightarrow (X, d)$ on which $T : (X_1, d) \longrightarrow (X_1, d)$ is an almost contraction. In this article, we have obtained a generalization of this inclusion in the context of metric spaces, obtained a related retraction problem in metric spaces and used the same example studied in [8, 12] to explain the inclusion in the sense of Udo-utun [13] on (X_1, d) and their independence on (X, d) due to Minak et al [8]. So, this article constitute an extension of this aspect of results in [13] from Banach spaces to complete metric spaces among other objectives mentioned above.

2 Main Results

Proposition 3 *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ an $F-$contraction with F a differentiable function. Then there exist $\alpha \in (0, 1), M > 0,$ and $(X_1, d) \subset (X, d)$ such that $T : (X_1, d) \longrightarrow (X_1, d)$ and*

$$\alpha d(Tx, Ty) - \frac{\tau}{F'(c_{xy})} \leq Md(y, Tx) \tag{2}$$

for all distinct $x \neq Tx$ and $y \neq Ty$ in (X_1, d) where c_{xy} are some constants satisfying $d(Tx, Ty) < c_{xy} < d(x, y)$ and the constant τ satisfies $\tau + F(d(Tx, Ty)) < F(d(x, y))$.

PROOF
Since T is a Picard operator, there exists a deleted neighborhood X_1 of the unique fixed point of T such that $T : (X_1, d) \longrightarrow (X_1, d)$ ie, T is a self mapping of X_1. Given that $T : X \longrightarrow X$ is an $F-$contraction on a complete metric space (X, d) with $F : [0, \infty) \longrightarrow \mathbb{R}$ a differentiable function. Since $\tau < F(d(x, y)) - F(d(Tx, Ty))$, the mean value theorem yields $\frac{\tau}{F'(c_{xy})} < d(x, y) - d(Tx, Ty)$ where $d(Tx, Ty) < c_{xy} < d(x, y)$. We observe that $F'(c_{xy}) > 0$ since F is strictly increasing. This, in turn, yields $d(Tx, Ty) - d(x, y) < -\frac{\tau}{F'(c_{xy})}$. Clearly, we obtain:

$$d(Tx, Ty) - d(x, y) < \alpha d(Tx, Ty) - \frac{\tau}{F'(c_{xy})}, \text{ for some } \alpha \in (0, 1), \tag{3}$$

with the possibility of (3) taking on negative values on its right hand side. Suppose, on the contrary, that for each $M > 0$ and for all $\alpha \in (0, 1)$ there exists $x_0, y_0 \in (X_1, d)$ such that $\frac{\alpha d(Tx_0, Ty_0) - \frac{\tau}{F'(c_{x_0 y_0})}}{M} > d(y_0, Tx_0)$, then it follows that, since $d(y_0, Tx_0) \geq 0$, we have $0 < \alpha d(Tx_0, Ty_0) - \frac{\tau}{F'(c_{xy})}$ for all $\alpha \in (0, 1)$. This violates possible negative values on the right hand side of the last inequality leading to contradiction since α can be made as small as we please. $\square$

An immediate consequence of (2) is the following corollary:

Corollary 4 *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ an $F-$contraction with F a differentiable function. Then there exists $\alpha \in (0, 1)$ and $(X_1, d) \subset (X, d)$ such that*

$T : (X_1, d) \longrightarrow (X_1, d)$ and

$$d(y, Tx) = 0 \Longrightarrow \alpha d(Tx, Ty) \le \frac{\tau}{F'(c_{xy})} \tag{4}$$

for all distinct $x \neq Tx$ and $y \neq Ty$ in (X_1, d) where c_{xy} are some constants satisfying $d(Tx, Ty) < c_{xy} < d(x, y)$ and the constant τ satisfies $\tau + F(d(Tx, Ty)) < F(d(x, y))$.

Theorem 5 *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ an $F-$contraction. Then there exists $(X_1, d) \subset (X, d)$ such that $T : (X_1, d) \longrightarrow (X_1, d)$ is an almost contraction. In other words, the class of $F-$contractions is a proper subclass of almost contractions on some subspace $(X_1, d) \subset (X, d)$.*

PROOF

Since T has a unique fixed point there exists a closed bounded neighbohood $X_1 \subset (X, d)$ of the fixed point of T such that $T : (X_1, d) \longrightarrow (X_1, d)$. We shall show that T restricted to X_1 is an $F-$contraction. Let $0 < \theta \le \alpha < 1$, where α is as gauranteed in Proposition 3, taking into consideraion the defition of $F-$contraction, $\tau + F(d(Tx, Ty)) < F(d(x, y))$, and making use of its equivalent form, $\frac{\tau}{F'(c_{xy})} < d(x, y) - d(Tx, Ty)$, we proceed as follows:

$$\begin{aligned}
d(Tx, Ty) &\le \theta d(Tx, Ty) + (1-\theta) d(Tx, Ty), \\
&< \theta d(Tx, Ty) + (1-\theta)\left[d(x, y) - \frac{\tau}{F'(c_{xy})}\right] \\
&= \theta d(Tx, Ty) + d(x, y) - \theta d(x, y) - \frac{\tau}{F'(c_{xy})} + \theta \frac{\tau}{F'(c_{xy})} \\
&< \theta d(Tx, Ty) + d(x, y) - \theta d(x, y) - \frac{\tau}{F'(c_{xy})} \\
&\quad + \theta\left[d(x, y) - d(Tx, Ty)\right] \\
&\le (1+\theta) d(x, y) + \theta d(Tx, Ty) - \frac{\tau}{F'(c_{xy})} - 2\theta d(Tx, Ty) \\
&\le (1+\theta) d(x, y) + M d(y, Tx) - 2\theta d(Tx, Ty) \text{ (by Proposition 3)} \\
\Longrightarrow d(Tx, Ty) &< \frac{1+\theta}{1+2\theta} d(x, y) + \frac{M}{1+2\theta} d(y, Tx) \qquad (5)
\end{aligned}$$

To conclude, we recall a striking theorem of Lebesgue which asserts that: "If the function $F : [0, \infty) \longrightarrow \mathbb{R}$ is monotone on the open interval $(a, b) \subset [0, \infty)$, then it is differentiable almost everywhere on (a, b)". This implies that any monotone function can be approximated by a differentiable function as close as we please. So, we can always assume the function F is differentiable for any $F-$contraction T which means that Proposition 3 applies to all $F-$contractions T. End of proof. $\square$

Independence/Inclusion via Example 1 in [8]

We can now apply Proposition 3 and Theorem 5 to resolve and illustrate independence/inclusion issues between the classes of $F-$contractions and almost contractions using the same illustrative example in [8].

Example 6 *[8, 12] Let (X,d) be defined by $X = \left\{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\right\}$, $d(x,y) = |x-y|$ and the mapping $T : X \longrightarrow X$ given by:*

$$Tx = \begin{cases} x_1, & x = x_1 \\ x_{n-1}, & x = x_n. \end{cases}$$

Using the facts that $d(y,Tx) = d(x_{n-1}, Tx_n) = |x_{n-1} - Tx_n| = 0$ and

$$\lim_{n\to\infty} \frac{d(Tx_n, Tx_{n-1})}{d(x_n, x_{n-1})} = \lim_{n\to\infty} \frac{|x_{n-1} - x_{n-2}|}{|x_n - x_{n-1}|} = \lim_{n\to\infty} \frac{2n-2}{2n} = 1$$

we shall show that for any $n_0 \in \mathbb{N}$ the $F-$contraction $T : (X,d) \longrightarrow (X,d)$ restricted to $\{x_n\}_{n\le n_0}$ is an almost contraction. But this does not counter the claim in [8] that the $F-$contraction $T : (X,d) \longrightarrow (X,d)$ is not an almost contraction since one cannot find fixed constants $\delta \in (0,1)$ and $k \ge 0$ such that $d(Tx,Ty) \le \delta d(x,y) + kd(y,Tx)$ when $n \longrightarrow \infty$. In this case Corollary 4 is not applicable based on the fact that (4) is violated with respect to the full space X since $d(y,Tx) = 0$ and there is no fixed $\alpha > 0$ such that $\alpha d(Tx,Ty) \le \frac{\tau}{F'(c_{xy})}$ when n_0 varies as shown shortly:

We observe that, in X, $d(y,Tx) = 0 \ge \alpha d(Tx,Ty) - \frac{\tau}{F'(c_{xy})}$ can not hold for all $x,y \in X$ since given any $\alpha > 0$ there exists $n_0 \in \mathbb{N}$ such that $\alpha|x_{n-1} - x_{n-2}| - \frac{1}{1+F'(|x_{n-1}-x_{n-2}|+\eta)} > 0$ for all $n > n_0$. Here, by mean value theorem, η satisfies $|x_{n-1} - x_{n-2}| < \eta < |x_n - x_{n-1}|$. Hence such an α is given by $\alpha = \alpha(n)$ but not fixed. This follows directly using $|x_{n-1} - x_{n-2}| = \frac{n-1}{2}$ and $\frac{1}{1+F'(|x_{n-1}-x_{n-2}|+\eta)} = \frac{1}{1+\frac{2}{n-1+\eta}}$ to obtain $d(y,Tx) = 0 \le \alpha\frac{n-1}{2} - \frac{n+\eta-1}{n+\eta+1}$ which holds for all $n > n_0 \ge \frac{2}{\alpha}$ and fixed $\alpha \in (0,1)$.

We desire to show that such independence between $F-$contractions and almost contractions breaks down in relevant bounded neighbohoods $(X_1,d) \subset (X,d)$ of the fixed point of an $F-$contractions. This is accomplished by showing that given a fixed $N > 1$ the $F-$contraction restricted to (X_1,d) is an almost contraction where

$$X_1 = \{x_n \in X : n < N \in \mathbb{N}\} \tag{6}$$

by using $d(y,Tx) = 0$ and $\alpha = \frac{1}{N^2}$ in the computation of (2) (in Lemma 3) as shown below. Here, $x = x_n$, $y = x_m$, $n = N-p, m = N-q$ where p and $q \in \mathbb{N}$ are such that

$p < q$, i.e $n > m$. In this case $\alpha d(Tx, Ty) \leq \frac{\tau}{F'(d(Tx,Ty)+\eta)}$ becomes:

$$\begin{aligned}\alpha d(Tx_n, Tx_m) &\leq \frac{1}{F'(d(Tx_n, Tx_m) + \eta)} \\ i.e\ \frac{1}{2N^2}|(n-1)n - (m-1)m| &\leq \frac{1}{1 + \frac{2}{|(n-1)n-(m-1)m|+\eta}}. \end{aligned} \tag{7}$$

where ,

$$0 < \eta < |x_n - x_m| - |Tx_n - Tx_m| \implies 0 < \eta < 2(q-p). \tag{8}$$

From (7) we obtain

$$\frac{(q-p)(2N-p-q-1)}{2N^2} \leq \frac{(q-p)(2N-p-q-1)+\eta}{(q-p)(2N-p-q-1)+\eta+2}. \tag{9}$$

We claim that (9) holds since the contrary leads to contradction as follows:

Suppose $\frac{(q-p)(2N-p-q-1)}{2N^2} > \frac{(q-p)(2N-p-q-1)+\eta}{(q-p)(2N-p-q-1)+\eta+2}$ holds then we have

$$1 + \frac{2}{(q-p)(2N-p-q-1)+\eta} > \frac{2N^2}{(q-p)(2N-p-q-1)}.$$

This yields

$$\begin{aligned}\frac{2(q-p)(2N-p-q-1)}{(q-p)(2N-p-q-1)+\eta} &> 2N^2 - (q-p)(2N-p-q-1) \\ 2 &> 2N^2 - (q-p)(2N-p-q-1).\end{aligned}$$

On application of (9) we obtain

$$\begin{aligned} 2 &> 2N^2 - (q-p)(2N-p-q-1) \\ 2Nq + 2 &> 2N^2 + p(2N-p-q-1) + pq + q^2 + q \\ 2Nq + 2 &> 2N^2 + 2Np + q^2 - p^2 + p + q \\ 2Nq &> 2N^2 + 2Np + q^2 - p^2 + p + q - 2 \\ q &> N + p + \frac{q^2 - p^2 + p + q - 2}{2N}. \end{aligned} \tag{10}$$

This gives $q > N$ contradicting $n, m < N$ in (6) since the third term on the right hand side of (10) is a positive quantity. Therefore the $F-$contractions $T : (X_1, d) \longrightarrow (X_1, d)$ is an almost contraction since there exist α ($\alpha = \frac{1}{N^2}$) and $(X_1, d) \subset (X, d)$ such that $T : (X_1, d) \longrightarrow (X_1, d)$ satisfies Proposition 3 for all $M > 0$. $\square$

Applications in Retraction Problems

In this section we shall prove and illustrate applications of our results in construction of absolute neighborhood retracts. Our emphasis is on determination of a retraction problem on a complete metric space using the notion of inclusion of of $F-$contractions in almost contractions presented so far. On the other hand, we shall identify the open subset $K_1 = (X_1, d)$, considered above, as the absolute neighborhood retract derived from the the retraction problem. The concept of a retract (or retraction) is central in the formulation of the concepts of absolute retracts, absolute neigborhood retracts, approximative absolute neighborhood retracts etc. They are very useful and important in the study of coincidence points of multivalued mappings which in turn are invaluable in investigation of structures of solution sets of non-uniqueness problems like Poincare transition operators and for analysis of integro-differential inclusions in automatic control theory. We refer to the monographs [1, 3, 6, 7] and their rich bibliographies for more readings. It is worth mentioning that the retraction problem studied here generalizes and complements the notion considered by Rus [9] and his followers [5]. We shall make use of the results below:

Definition 7 *A subspace A of a metric space (X, d) is called an absolute retract (written $A \in AR$) if and only if A has an extension property in the sense that; for any space Y and any closed set $B \subset Y$ and any continuous map $f : B \longrightarrow A$, there is a continuous map $g : Y \longrightarrow X$ satisfying $g(x) = f(x)$ for all $x \in B$.*

Definition 8 *A subspace A of a metric space (X, d) is called an absolute neighbourhood retract (written $X \in ANR$) provided A has neighborhood extension property in the sense that; for every space Y and any closed set $B \subset Y$ and any continuous map $f : B \longrightarrow A$, there is an open neighbourhood U of B in Y and a continuous map $g : U \longrightarrow A$ satisfying $g(x) = f(x)$ for all $x \in B$.*

It is obvious that every absolute retract is retract and every absolute retract is an absolute neighborhood retract. Also, it is of significance to observe that an absolute neighborhood retract is necessarily the intersection of all retracts of X and is closed since all retracts are closed sets. In the case when $A \subset X = Y$ is a closed subset of a metric space X, it follows that if A is absolute retract, then for any closed subset $B \subset X$ and a continuous map $f : B \longrightarrow A$ there exists a continuous extension $g : X \longrightarrow A$ such that $g(x) = f(x)$ for all $x \in B$. So, if $X_1 \subset X$ is a retract of X then an absolute retract $A \subset X$ is also a subset of X_1. Hence, $A \in ANR$ since X_1 has the extension property $g : X_1 \longrightarrow A$ such that $g(x) = f(x)$ for all $x \in B$.

Theorem 9 *If a Hausdorff metric space X has the fixed point property and A is a retract of X, then A has the fixed point property.*

It follows from the above that $r(Tx) = x \in A$ if and only if x is a fixed point of T and if $z \in A$ is not a fixed point of $T : X \longrightarrow X$ then $r(Tz) \neq z$.

Theorem 10 *Let $T : X \longrightarrow X$ be an $F-$contraction defined on a complete (Hausdorff) metric space (X, d) and $A \subset X$ a closed subset of X. Then T is also an almost contraction on A if and only if $A \in ANR$.*

PROOF

The proof is based on showing that the subspace (X_1, d), in the proof of Theorem 5, is a retract of X. From Theorem 5, there exists a neighborhood $(X_1, d) \subset X$ of the fixed point of T on which T is an almost contraction. Since T is a Picard operator, there exists $N \in \mathbb{N}$ such that $T^n x \in X_1, n \geq N$ for all $x \in X$. Let $n_r \in \mathbb{N}$ be the smallest natural number such that $T^{n_r} x \in X_1$ for all $x \in X$, then we define a retraction $r : (X, d) \longrightarrow (X_1, d)$ by $r(x) = T^{j_x}(x)$ where $j_x \in \{0, 1, ..., n_r\}$ is the smallest natural number for a particular member $x \in X$ such that $T^{j_x}(x) \in X_1$. This means that r satisfies the following:

$$r(x) = \begin{cases} x, & x \in A \\ T^{j_x}(x) \in A, & x \notin A, j_x \in \{1, ..., n_r\} \end{cases} \tag{11}$$

Continuity of r follows from continuity of the $F-$contraction T showing that X_1 is a retract of X. Next, given that the $F-$contraction T is an almost contraction on a closed subset A, it implies that $A \subseteq X_1$. Let $B \subset X$ be a closed subset and $f : B \longrightarrow A$ be a continuous map. Without loss of generality, we may assume that $f : B \longrightarrow X_1$ (i.e $A = X_1$). Since X is a Hausdorf space, we can always construct a continuous extension $g : U \longrightarrow A$ on a neighborhood $U \supset B$ of X_1 by $g(x) = f(\lim_{n\to\infty} x_n)$ where $\{x_n\}$ is a sequence with initial point $x_1 = x \in U$ such that $\lim_{n\to\infty} x_n \in B$. Hence a closed subset $A \in ANR$ if A contains the fixed point of an $F-$contraction which is also an almost contraction on A.

On the other hand, suppose a closed subset $A \subset X$ is such that $A \in ANR$ and T is an $F-$contraction on X. Since X_1 described above is a retract of X, we have that T is an almost contraction on A since $A \subset X_1$ and by Theorem 5 T is also an almost contraction on X_1. $\square$.

3 Conclusion

A consequence of Theorem 10 is that if we can find an absolute neighborhood retract on which an almost contraction is an $F-$contraction then we are assured of uniqueness of fixed point in some neighborhood of the absolute neighborhood retract. It is of interest to observe that when $n_r \longrightarrow \infty$ the retraction $r(x)$ defined above in (11) approximates the retraction studied in [9] and [5]. This suggests the possibility modifying the *retraction-displacement condition*, studied there-in, in order to formulate further generalizations and extensions of related results.

Let (X, d) be a metric space, then a mapping $T : X \longrightarrow X$ is said to be *weakly pseudo-Picard (WPP) operator* if and only if there exist $x_0 \in X$ such that the sequence of Picard iterations $\{T^n x_0\}_{n=1}^{\infty}$ converges to a fixed point of T. A weakly pseudo-Picard operator T is called a *pseudo-Picard operator* if its fixed point is unique. It should be mentioned that the class of operators satisfying the hypothesis of Theorem 2.1 in [13] constitute a very large example of weakly pseudo-Picard operator of which multivalued analogues were considered in [8]. Moreover, it would be fruitful to investigate whether weakly pseudo-Picard operators restricted to appropriate retracts of their domains of definition coincide with weakly Picard operators.

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