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Application of a new fixed point condition for unification of studies of Fredholm-type alternatives

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Abstract

A unification of a new necessary condition and the Landesman-Lazer condition is established by using a new fixed point condition, leading to further extension of Fredholm alternatives.

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1 Introduction

In [?] Sanni obtained a nonlinear analogue of the Fredholm alternatives as a necessary condition below:

$$\int_\Omega g(u(x))\phi_k(x)dx = \int_\Omega h(x)\phi_k(x)dx \tag{1}$$

for solvability of the boundary value problem

$$\Delta u + \lambda_k u + g(u(x)) = h(x),\ x \in \Omega;\ u\ = 0 \text{ on } \partial\Omega, \tag{2}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $h \in L^2(\Omega)$ and ϕ_k is the eigenfunction corresponding to the eigenvalue λ_k of the eigenvalue problem:

$$\Delta u + \lambda_k u = 0,\ x \in \Omega;\ u\ = 0 \text{ on } \partial\Omega, \tag{3}$$

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It is important to observe that the condition (**??**) obtained in [**?**] neither requires boundedness of the nonlinearity g nor the existence of the limits $g(\pm\infty)$. This makes it a very significant contribution in the direction of improvement upon the celebrated Landesman-Lazer necessary and sufficient condition below:

$$g(-\infty)\int_{\Omega^+}\phi_k dx+g(\infty)\int_{\Omega^-}\phi_k dx<\int_{\Omega}h(x)\phi_k dx<g(\infty)\int_{\Omega^+}\phi_k dx+g(-\infty)\int_{\Omega^-}\phi_k dx \quad (4)$$

for existence of a weak solution of the boundary value problem (**??**), where $\Omega^+=\{x\in\Omega:\phi_k(x)>0\}$ and $\Omega^-=\{x\in\Omega:\phi_k(x)<0\}$. Here $g:\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function such that $g(-\infty)<g(\xi)<g(\infty)$ where $g(-\infty)=\lim_{s\to-\infty}g(s)$ and $g(\infty)=\lim_{s\to\infty}g(s)$ are finite and unique. The condition of Landesman and Lazer (**??**) which appeared in 1970 inspires quality contributions ever since (see, for example, [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**] and their references), firstly, for being a major breakthrough in the direction of Hilbert 20th problem concerned with difficulties in formulation of necessary and sufficient condition for solvability of (**??**) and, secondly, for being in urgent need of improvement based on the following two main reasons: (a) it restricts the theory of (**??**) to boundary value problem with bounded nonlinearities $g(u)$ and (b) the inherent requirement that the limits $\lim_{s\to\pm\infty}g(s)$ exist exclude important nonlinearities like cases when $g(-\infty)=g(\infty)$. We have made references to [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**] and their references concerning important improvements on the Landesman-Lazer condition (**??**).

It is of interest to view the necessary condition (**??**) as a significant contribution in the direction of nonlinear analogue of the linear Fredholm alternatives since it is equivalent to the following nonlinear alternatives:

Either

*(S1) the nonlinear problem (**??**) has a weak solution for each right-hand-side* $h\in L^2(\Omega)$*, or*

*(S2) its associated nonlinear homogeneous problem (**??**) below:*

$$\Delta u+\lambda_k u+g(u)=0,\ x\in\Omega;\ u\ =0 \ \textit{on}\ \partial\Omega, \quad (5)$$

has a weak solution.

We observe that this nonlinear alternatives assert that if $\int_\Omega h(x)\phi_k(x)dx=0$ then (**??**) is a necessary condition for either the nonlinear problem (**??**) to have a weak solution, or for its associated nonlinear homogeneous eigenvalue-problem (**??**) to have a weak solution. This shows that the condition (**??**) is not a sufficient condition for solvability of (**??**) since it excludes certain critical cases where $h(x)$ and ϕ_k are orthogonal. These situations are dipicted in an example due to Ahmad [?], which was considered by Drabec [?] and Nieto [?] below:

$$u'' + u + 3u = sin2x, \quad 0 < x < \pi, \\ u(0) = 0; \; u(\pi) = 0. \tag{6}$$

Here, $\lambda_k = \lambda_1 = 1, g(u) = 3u$ and $h(x) = sin2x$ and $\overline{\Omega} = [0, \pi]$. In this case $\int_\Omega h(x)\phi_1(x)dx = \int_0^\pi \sin 2x \sin x dx = 0$ and the nonlinear problem (??) has no solution because the associated homogeneous problem (??), given as:

$$u'' + u + 3u = 0 \qquad i.e; \; u'' + 4u = 0, \; 0 < x < \pi, \\ u(0) = 0; \; u(\pi) = 0, \tag{7}$$

has a weak solution. To ensure existence of weak solution of the problem

$$u'' + u + g(u) = h(x), \quad 0 < x < \pi, \\ u(0) = 0; \; u(\pi) = 0, \tag{8}$$

i.e a particular case of (??) (in which $g(u)$ in (??) is arbitrary), Ahmad and Nieto assumed a growth condition on the nonlinearity g as: $|g(u)| \leq \gamma|u| + c$ and reported best estimate for γ as $\gamma \in [0, 3)$. A similar estimate was reported by Iannacci and Nkashama [?] while the estimate, $0 < \gamma < (k+1)^2 - k^2 < 3$, which generalizes it, due to Arias [?] for solvability of the following problem

$$u'' + k^2u + g(u) = h(x) \quad i.e; \; u'' + \lambda_k u + g(u) = h(x); \; 0 < x < \pi, \\ u(0) = 0; \; u(\pi) = 0, \tag{9}$$

can be shown to be in general compliance with our hypothesis. For similar and related contributions see [?, ?, ?, ?, ?, ?] and the references there-in.

So unlike the Landesman-Lazer condition, (??) is not a sufficient condition for existence of weak solution of (??). The Lipschitz condition on g which serves as sufficient condition in [?] is relaxed by the sufficient condition reported here-in. This new sufficient condition which aids to unify and extend earlier necessary and sufficient conditions is merely a characterization of certain class of L^2-Caratheodory composition functions $F(x, u(x))$ for solvability of (??).

2 Preliminary

The method employed in [?] made use of Schaefer fixed point theorem which is based on compactness of associated operator which hinders extensions of the problem to arbitrary contexts. The method employed in this report neither requires compactness of ambient sets nor compactness of associated operators and promises extensions to abitrary situations in which compactness might be lacking. The following are results in [?] studied in this article:

Theorem 1 *The necessary condition that $u \in H_0^1(\Omega)$ be a weak solution of (??) is that condition (??) holds.*

Theorem 2 *Let condition (??) holds. Then there exists a weak solution to the problem*

$$-\nabla.(\phi_k^2 \nabla v) = \phi_k g(\phi_k v) - \phi_k h(x), \;\; \phi_k v = 0 \;\; on \;\partial\Omega. \tag{10}$$

Theorem 3 *Let $v \in H_0^1(\Omega)$ be the solution of*

$$Lv = -\nabla.(\phi_k^2 \nabla v) + \mu \phi_k^2 v = \mu \phi_k^2 s + \phi_k g(\phi_k s) - \phi_k h; \;\; \phi_k v = 0 \;\; on \;\partial\Omega \tag{11}$$

Then the solution $u = \phi_k v$ of (??) belongs to $H_0^1(\Omega)$ and we have the following estimate

$$\|u\|_{H_0^1(\Omega)} \leq C \|v\|_{H_0^1(\Omega)}. \tag{12}$$

We shall apply the following recent fixed point theorem due to Udo-utun et al in the sequel.

Theorem 4 *Let K be a closed convex subset of a Banach space E and $T : K \longrightarrow K$ an $L-$Lipschitzian operator. Suppose there exists an open subset $K_1 \subset K$ such that T satisfies the condition below:*

$$\|y - Tx\| \quad \leq \quad M\|x - y\| \;\; whenever \;\|x - y\| \leq \|y - Tx\| \tag{13}$$

for some $M \geq 1$ for all $x, y \in K_1; x \neq y, x, y \notin Fix(T)$. Then T has a fixed point in K and the Krasnoselskii iteration scheme $x_{n+1} = \lambda x_n + (1 - \lambda)T^n x_0, n \geq 0; x_0 \in K_1$ converges to a fixed point of T in K.

Further, condition (??) generalizes contraction condition in Banach spaces.

Remark 5 *It is very important to comment that apart from generalizing the contraction condition [?], the condition (??) of Theorem ?? has been shown to include so many weak contractive conditions like the quasicontraction and almostcontraction conditions as special cases [?]. In [?, ?] it was proved that a nonexpasive mapping has a fixed point if and only if (??) is satisfied. Some important consequences of this include the following comments and deductions:*

(I) *All contraction mappings satisfy (**??**) since they constitute a proper subclass of non-expansive mappings with unique fixed points. Specifically, if T is a contraction mapping with contraction constant α, it can be shown that if $\|x-y\| \leq \|y-Tx\|$ and $\|x-y\| \leq \|x-Tx\|$ then for $\mu \in [1-\alpha, 1)$ we have:*

$$(1-\mu)\|x-Tx\| \leq \|Tx-Ty\|. \tag{14}$$

*In view of symmetry considerations in (**??**), we may assume that $\|x-Ty\| \leq \|y-Tx\|$ for two points x and y in a small neighborhood U of the fixed point p of T. Firstly, if $\|x-Tx\| < \|x-y\|$ then we have*

$$\|y-Tx\| \leq \|x-y\| + \|x-Tx\| \leq 2\|x-y\|. \tag{15}$$

*On the other hand, if $\|x-y\| \leq \|x-Tx\|$ then we apply (**??**) as follows: Let T be a contraction, since Tx and Ty can be made very close to each other in appropriate neighjborhood U of p then for any $\mu \in [1-\alpha, 1)$ we have:*

$$\begin{aligned}
(1-\mu)\|y-Tx\| &\leq (1-\mu)\|x-y\| + (1-\mu)\|x-Tx\| \\
&\leq (1-\mu)\|x-y\| + \|Tx-Ty\| \text{ (application of (\textbf{??}))} \\
\|y-Tx\| &\leq \left[1+\frac{\alpha}{1-\mu}\right]\|x-y\| \qquad (16)
\end{aligned}$$

*Combination of (**??**) and (**??**) verifies that for a contraction T the condition (**??**) is satisfied in the open set $K_1 = U$ above with $M = \max\left\{2, 1+\frac{\alpha}{1-\mu}\right\}$.*

(II) *If an operator T satisfies (**??**) then T is Lipschitz in some open set since*

$$\|Tx-Ty\| \leq \|y-Tx\| + \|y-Ty\| \leq (2M+1)\|x-y\| \tag{17}$$

*for all x and y in the subset K_1 of the convex set K in Theorem **??**.*

3 Main Results

Important applications of the comments in Remark **??** are the following lemmas needed in the sequel:

Lemma 6 *Let h be the image of a constant map $\chi : K \longrightarrow K$ where K is appropriate subset of the Banach space $L^2(\Omega)$; i.e $\chi z = h$ for all $z \in K$. Then there exist an open set $K_h \subset K$ and a constant $M_h \geq 1$ such that*

$$\|z_2 - h\|_{L^2(\Omega)} = \|z_2 - \chi z_1\|_{L^2(\Omega)} \leq M_h \|z_1 - z_2\|_{L^2(\Omega)} \tag{18}$$

for all distinct $z_1, z_2 \in K_h$ with $z_1 \neq h$ and $z_2 \neq h$.

The operator $\chi : L^2(\Omega) \longrightarrow L^2(\Omega)$ being a contraction has a fixed point $h = \chi h$ since $\chi z = h$ for all $z \in K \subseteq L^2(\Omega)$ where $h \in L^2(\Omega)$ is a fixed function. We shall verify that χ satisfies (**??**) by proving (**??**) in some deleted open neighborhood $K_h \subset L^2(\Omega)$ of h. In this case using $\|z_2 - \chi z_1\|_{L^2(\Omega)} = \|z_2 - h\|_{L^2(\Omega)}$ we observe that for $\mu \in (0,1)$ the mapping $(1-\mu)h + \mu z$ is also a contraction, so we have:

$$\begin{aligned}\|z_2 - h\|_{L^2(\Omega)} &= \|z_2 - \chi z_1\|_{L^2(\Omega)} \\ &\leq \|z_2 - (1-\mu)h - \mu z_1\|_{L^2(\Omega)} + \|z_2 - (1-\mu)z_1 + \mu h\|_{L^2(\Omega)} + \|z_1 - z_2\|_{L^2(\Omega)} \\ &\leq M_h \|z_1 - z_2\|_{L^2(\Omega)}\end{aligned}$$

for all distinct $z_1, z_2 \in K_h$ with $z_1 \neq h$ and $z_2 \neq h$. □

Lemma 7 *Let $F : \Omega \times \mathbb{R} \longrightarrow L^2(\Omega)$ given by $F(x,u) = -g(u) + h(x)$ where $-g$ satisfies (**??**) on some open set $K_g \subset L^2(\Omega)$ and $h \in L^2(\Omega)$ a fixed function. Then F satisfies (**??**), that is; if there exists $M_g \geq 1$ such that $\|z_2 - [-g(z_1)]\|_{L^2(\Omega)} \leq M_g \|z_1 - z_2\|_{L^2(\Omega)}$ on K_g then there exists $M_F \geq 1$ such that*

$$\|z_2 + h - F(x, z_1 + h)\|_{L^2(\Omega)} \leq M_F \|z_1 - z_2\|_{L^2(\Omega)},\ z_1, z_2 \in K_g; z_1, z_2 \notin Fix(F). \quad (19)$$

PROOF

We begin by showing that if $\|z_2 - [-g(z_1)]\|_{L^2(\Omega)} \leq M_g \|z_1 - z_2\|_{L^2(\Omega)}$ on K_g then

$$\left\| z_2 \pm \frac{h}{2} \right\|_{L^2(\Omega)} \leq M_{h12} \|z_1 - z_2\|_{L^2(\Omega)} \text{ on } K_g \quad (20)$$

for some $M_{h12} \in \{M_{h1}, M_{h2}\}$ where $M_{h1} \geq 1$ and $M_{h2} \geq 1$ are constants. Now,

$$(1-\lambda) \left\| z_2 - \frac{h}{2} \right\| = \left\| (1-\lambda) z_2 + \lambda \frac{h}{2} - \frac{h}{2} \right\|. \quad (21)$$

We can make $z = (1-\lambda)z_2 + \lambda \frac{h}{2}$ belong to any small deleted neighborhood U_h of $\frac{h}{2}$ by appropriately choosing $\lambda \in (0,1)$. Since the constant operator $\chi_2 x = \frac{h}{2}$ for all x is a contraction, application of (**??**) in (**??**) and putting $\frac{h}{2} = \chi_2[(1-\lambda)z_1 + \lambda \frac{h}{2}]$ yields:

$$\begin{aligned}(1-\lambda) \left\| z_2 - \frac{h}{2} \right\| &= \left\| (1-\lambda) z_2 + \lambda \frac{h}{2} - \chi_2 \left[(1-\lambda) z_1 + \lambda \frac{h}{2} \right] \right\| \\ &\leq M_1 \left\| (1-\lambda) z_2 + \lambda \frac{h}{2} - \left[(1-\lambda) z_1 + \lambda \frac{h}{2} \right] \right\| \leq M_{h1} (1-\lambda) \|z_1 - z_2\|.\end{aligned}$$

This gives $\left\| z_2 - \frac{h}{2} \right\| \leq M_{h1} \|z_1 - z_2\|$ as desired. Similar argument yields $\left\| z_2 - \left[\frac{-h}{2} \right] \right\| \leq M_{h2} \|z_1 - z_2\|$.

To prove that F satisfies (**??**) we choose the open set K_g as the ball $B(p;R)$ for some $R>0$ where p is a fixed point of $-g$. We obsserve that for $z_1, z_2 \in B(p;R)$, the difference $\|u - F(x,v)\|_{L^2(\Omega)}$ in the ball $B(h+p;R)$ takes the form $\|z_2 + h - F(x, z_1+h)\|_{L^2(\Omega)}$ $= \|z_2 - [-g(z_1+h)]\|_{L^2(\Omega)}$. Application of (**??**) and (**??**) to $\|z_2 - [-g(z_1+h)]\|_{L^2(\Omega)} \le M_g[\|z_1 - [-\frac{h}{2}]\|_{L^2(\Omega)} + \|z_2 - \frac{h}{2}\|_{L^2(\Omega)}$, gives $\|z_2 + h - F(x, z_1+h)\|_{L^2(\Omega)} \le M_F\|z_1 - z_2\|_{L^2(\Omega)}$ for all $z_1, z_2 \in K_g = B(p;R)$.

Theorem 8 *Define $F : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ by $F(x, u(x)) = -g(u(x)) + h(x),\ x \in \Omega$. A sufficient condition for existence of weak solution of (**??**) is that there exist a constant $M_F \in [1,3)$ and an open set $K_1 \subset H_0^1$ such that for any distinct elements $z_1, z_2 \ \in K_1$ condition (**??**) below is satisfied:*

$$\| - z_2 - g(z_1)\|_{L^2(\Omega)} \le M_F\|z_1 - z_2\|_{L^2(\Omega)} \tag{22}$$

provided z_1 and z_2 are not fixed points of g.

PROOF

Following the method employed in [**?**] we let $A : H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$ be a nonlinear operator defined by

$$\begin{aligned} &A\hat{z} = z; \qquad z, \hat{z} \ \in H_0^1(\Omega) \\ &\text{whenever} \quad \Delta\hat{z} + \lambda\hat{z} = -g(z) + h(x). \end{aligned} \tag{23}$$

That the operator A is well defined on $H_0^1(\Omega)$ follows from the existence and uniqueness of the solution $u = L_k^{-1}\xi,\ \xi = F(x, z_1)$, to the linear operator equation

$$L_k u = \xi;\ \xi \in \mathrm{L}^2(\Omega) \tag{24}$$

where the linear operator $L_k : H_0^1(\Omega) \longrightarrow L^2(\Omega)$ is an injection given by $L_k u = \Delta u + \lambda_k u = F(x, z_1);\ x \in \Omega;\ u = 0,$ on $\partial\Omega$. Further, it follows from uniqueness of solutions of (**??**), for each $\xi \in L^2(\Omega)$, that the inverse A^{-1} of operator A defined by (**??**) is so well defined that both A and A^{-1} exist and have common fixed point-set $Fix(A)$. It turns out that it is easier to prove existence of fixed points of A^{-1} than proving for A in the sequel.

To apply Theorem **??** on A^{-1} we take $H_0^1(\Omega)$ as a convex subset K of the Banach space $L^2(\Omega)$ and infer that the operator A has a fixed point by showing that its inverse A^{-1} has a fixed point. To prove sufficiency we only need to show that A^{-1} is a Lipschitzian operator and satisfies $\|z_2 - A^{-1}z_1\| \le M_A\|z_1 - z_2\|$ for some $M_A \ge 1$. Given that h is a Caratheodory function and g satisfies Lipschitz condition in the open set K_1 with Lipschitz constant γ, let F be defined by $F(x,z) = -g(z) + h(x)$ and $z_1, \hat{z}_1 \in H_0^1(\Omega)$ be such that $\Delta\hat{z}_1 + \lambda_k\hat{z}_1$ $= F(x, z_1)$. In view of (**??**) this yields:

$$A\hat{z}_1 = z_1. \tag{25}$$

To prove sufficiency, we assume the hypothesis (??) holds and using

$$A^{-1}z_1 = \hat{z}_1 = L^{-1}(F(x, z_1)), \tag{26}$$

we shall show that A^{-1} is a Lipschitzian operator, thereafter we estimate the constant M_A as follows:

$$\begin{aligned} \|A^{-1}z_1 - A^{-1}z_2\| &= \|\hat{z}_1 - \hat{z}_2\| = \|L^{-1}(F(x, z_1)) - L^{-1}(F(x, z_2))\| \\ &\leq \gamma\|L^{-1}\|\|z_1 - z_2\|. \end{aligned} \tag{27}$$

In (??) we have used the fact that L_λ^{-1} is a bounded linear operator since the operator L_λ is an unbounded linear operator. Next, choosing the open set $K_A = B(p + h; R)$ where p is the fixed point of $-g$ we estimate M_A as follows:

$$\begin{aligned} \|z_2 - A^{-1}z_1\| &= \|\hat{z}_1 + h - (z_2 + h)\| = \|z_2 + h - L^{-1}(F(x, z_1 + h))\| \\ &\leq M_L\|z_2 + h - F(x, z_1 + h)\| \\ &\leq M_A\|z_1 - z_2\| \text{ (by application of (??))} \end{aligned} \tag{28}$$

where $M_A = M_L M_F$ and (??) follows as an application of the last statement of Theorem ??. This shows that A^{-1} (and A) has a fixed point u. We claim that a fixed point u of A is the desired solution of (??) since if $u = Au$, then $L_\lambda u = F(x, u)$. End of proof. □

Theorem 9 *A necessary condition for solvability of the problem (??), for any RHS* $h \in L^2[0, 1]$*, is that the constant* M_F *in the condition (??) satisfies* $1 \leq M_F < 3$.

PROOF

It suffices to justify the situation $\int_\Omega h(x)\phi_k(x)dx = 0$ for certain $h \in L^2[0, 1]$ when $g(u) \neq 0$. We shall prove that if there is no nonnontrivial solution $x_0 \in L^2[0, 1]$ such that $g(x_0(t)) = [\lambda_{k+j} - \lambda_k]x_0(t)$; $j, k \in \mathbb{N}$, then g satisfies (??), with $M_F \in [1, 3)$. In other words if (??) has a weak solution then (??) holds with $M_F \in [1, 3)$. Observe that if $g(x_0(t)) = [\lambda_{k+j} - \lambda_k]x_0(t)$ then by Fredholm alternatives the problem (??) has no solution since the corresponding eigenvalue problem below:

$$\Delta u + \lambda_{k+j}u = 0,\ x \in \Omega;\ u\ = 0 \text{ on } \partial\Omega,$$

is solvable with solution $u(t) = x_0(t) = \phi_{k+j}(t)$. On the contrary, if g is such that there exists a point $x_0 \in L^2[0, 1]$ at which $g(x_0(t)) = [\lambda_{k+j} - \lambda_k]x_0(t)$. Let $U \subset L^2[0, 1]$ be an open set on which g satisfies (??). Since, by (??), g is Lipschitz on U, without loss of generality we assume that g is nonexpansive on U so that g satisfies (??) on U. Then

Theorem **??** guarantees that g has a fixed point in a closed convex set K containing the closure $\overline{U}$ of U and the points $[\lambda_{k+j} - \lambda_k]x_0$ and x_0. But then the point x_0 is a fixed point of the mapping $\frac{1}{\lambda_{k+j}-\lambda_k}g$ which is a contraction and so there are $u, v \in K$ close to x_0 such that $\|u - \frac{1}{\lambda_{k+j}-\lambda_k}g(v)\| \leq M_\lambda\|u - v\|$; $M_\lambda \geq 1$. This yields $\|[\lambda_{k+j} - \lambda_k]u - g(v)\| \leq [\lambda_{k+j} - \lambda_k]\|u - v\|$, for approprite choice of $j \in \mathbb{N}$ with $\lambda_{k+j} - \lambda_k \geq M_\lambda$. Since we can make u and v as close to x_0 as we please, it follows that x_0 is also a fixed point of g. Therefore x_0 must be the trivial solution since this yields $x_0 = g(x_0) = [\lambda_{k+j} - \lambda_k]x_0$. Therefore if (**??**) is solvable then $\|u - g(v)\| \leq M_F\|u - v\|$ for some $M_F \in [1, 3)$ and for some distinct u and v in some open subset.

Corollary 10 *A necessary and sufficient condition for solvability of (**??**) is that if $u = \phi_{k+\nu}$; $\nu \in \mathbb{N}$, is a weak solution of the homogeneous problem (**??**) viz;*

$$\Delta u + \lambda_k u + g(u) = 0,\ x \in \Omega;\ u\ = 0 \ on\ \partial\Omega,$$

*then the image of $u = \phi_{k+\nu}$ under g be not given by $g(\phi_{k+\nu}) = [\lambda_{k+\nu} - \lambda_k]u$, where ϕ_k is a solution of (**??**).*

Remark 11 *It is of a significant advantage that condidition (**??**) is independent of Hilbert spaces' properties so it applies to Banach spaces other than Hilbert spaces. Further, an appreciation of the method of proof reveals that the solution so obtained are strong solutions which is an edge over usual methods which require regularity considerations for strong solutions.*

To demonstrate how general our method is we now show that for nonlinearities satisfying our hypothesis (**??**) the proof of Theorem **??** follows as a special case of our Theorem **??** which we apply to prove the following result:

Theorem 12 *Let there exist a constant $M \geq 1$ such that the nonlinearity g satisfy the following condition:*

$$\|\phi_k v_2 - g(\phi_k v_1)\| \leq M\|v_1 - v_2\| \ whenever\ \|v_1 - v_2\| < \|\phi_k v_2 - g(\phi_k v_1)\|,\ \phi_k v_1, \phi_k v_2 \notin Fix(g) \tag{29}$$

for distinct points v_1 and v_2 in an open subset $K_1 \subset H_0^1$. Then there exists a weak solution to the problem

$$-\nabla.(\phi_k^2 \nabla v) = -\phi_k g(\phi_k v) + \phi_k h(x),\ \phi_k v = 0 \ on\ \partial\Omega.$$

PROOF

The proof is based on showing that the problem (**??**) is equivalent to the problem (**??**) with

$u = \phi_k v$ using following:

$$\nabla.(\phi_k^2 \nabla v) = \phi_k^2 \Delta v + 2\nabla\phi_k.\nabla v \tag{30}$$

$$\phi_k \Delta \phi_k v = \phi_k^2 \Delta v + 2\nabla\phi_k.\nabla v + v.\Delta\phi_k \tag{31}$$

Combining (**??**) and (**??**) the problem (**??**) becomes

$$\begin{aligned} -\phi_k \Delta v + v\Delta\phi_k &= -\phi_k g(\phi_k v) + \phi_k h(x) \\ \Longrightarrow\ -\phi_k \Delta v - \mu\phi_k v + v\Delta\phi_k + \phi_k v &= -\phi_k g(\phi_k v) + \phi_k h(x) \end{aligned}$$

This yields

$$-\Delta u - \mu u = -\phi_k g(u) + \phi_k h(x)$$

so that Theorem **??** applies where $u = \phi_k v$ and $\Delta\phi + \mu\phi_k = 0$ since ϕ_k is an eigenfunction corresponding to μ. End of proof. □

4 Illustrative Examples

For illustration we investigate cases when $g(u) = sinu$, when $g(u) = -u^2$ and the situation when $g(u) = 2u$ considered in [**?**].

Example 13 *For solvability of $-u'' - \lambda_k u = sinu + h(x)$, $x \in [0,1]$ with $u(0) = u(1) = 0$, we claim that there exist a deleted neighborhood $B(0, \frac{\epsilon}{2})$ such that whenever $x, y \in B(0, \frac{\epsilon}{2})$ then $\frac{\|sinx - y\|}{\|x-y\|} \leq M$ for some $M \geq 1$. This is verified by constructing two sequences $\{x_n\}$ and $\{y_n\}$ in $B(0, \frac{\epsilon}{2})$ with the following properties: $x_n \longrightarrow 0$, $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and $\frac{\|sinx_n - y_n\|}{\|x_n - y_n\|} > K_n$ for some K_n. We desire to show existence of $M \geq 1$. The hypothesis holds trivially if either x_n or y_n is zero, so we assume that x_n and y_n are nonzero and obtain the following:*

$$\lim_{n\to\infty} \frac{\|sinx_n - y_n\|}{\|x_n - y_n\|} = \frac{\left\|\lim_{n\to\infty} \frac{sinx_n}{x_n} - \lim_{n\to\infty} \frac{y_n}{x_n}\right\|}{\left\|1 - \lim_{n\to\infty} \frac{y_n}{x_n}\right\|} = 1.$$

This shows there exists $M \geq 1$ such that $\frac{\|sinx - y\|}{\|x-y\|} \leq M$ for all $x, y \in B(0, \epsilon)$; $0 < \epsilon < 1$.

In some cases it may be difficult to verify bounedness of the sequences $\left\{\frac{\|g(x_n) - y_n\|}{\|x_n - y_n\|}\right\}_{n=1}^{\infty}$ in some deleted $\epsilon-$neighborhood of fixed points x_0 of g. For instance, a simple and shorter approach to to verify that $g(x) = sinx$ satisfies (**??**) is to apply the comments (I) in Remark **??** by showing that there exists a deleted neighborhood, $B(0,\epsilon) \subset L^2(\Omega)$, of the zero function on which $sinx$ is a contraction mapping so that (**??**) follows. This method is more general and it is easy to verify this in the following examples.

Example 14 *[**?**] Given the problem,* $-u''-\lambda_k u = 2u+h(x)$, $x \in (0,1)$ *with* $u(0) = u(1) = 0$.

Example 15 $-u'' - \lambda_k u = u^2 + h(x)$, $x \in (0,1)$ *with* $u(0) = u(1) = 0$

In addition to $B(0, \epsilon)$ it is also straightforward to construct a deleted neighborhood $B(I, \epsilon)$ of the identity function I on which the mapping $g(u) = u^2$, in Example **??**, is a contraction.

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