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A SURVEY FOR GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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ABSTRACT. The generalized trigonometric functions which have a short history, were introduced by Lindqvist two decades ago. Since 2010, many mathematician began to study their classical inequalities, general convexity and concavity, multiple-angle formulas and parameter convexity and concavity. A number of results have been obtained. This is a survey. Some new refinements, generalizations, applications, and related problems are summarized.

1. INTRODUCTION

It is well known from calculus that

$$\arcsin x = \int_0^x \frac{1}{(1-t^2)^{1/2}} dt$$

for $0 \le x \le 1$ and

$$\frac{\pi}{2} = \arcsin 1 = \int_0^1 \frac{1}{(1-t^2)^{1/2}} dt.$$

For $1 < p < \infty$ and $0 \le x \le 1$, the arcsine may be generalized as

$$\arcsin_p x = \int_0^x \frac{1}{(1-t^p)^{1/p}} dt \tag{1.1}$$

and

$$\frac{\pi_p}{2} = \arcsin_p 1 = \int_0^1 \frac{1}{(1-t^p)^{1/p}} dt. \tag{1.2}$$

The inverse of $\arcsin_p$ on $[0, \frac{\pi_p}{2}]$ is called the generalized sine function, denoted by $\sin_p$ and may be extended to $(-\infty, \infty)$. See [29] and closely related references therein.

For $x \in [0, \frac{\pi_p}{2}]$, the generalized cosine function $\cos_p x$ is defined by

$$\cos_p x = \frac{\mathrm{d} \sin_p x}{\mathrm{d}x}. \tag{1.3}$$

It is easy to see that

$$\cos_p x = (1 - \sin_p^p x)^{1/p} \tag{1.4}$$

and

$$\frac{\mathrm{d} \cos_p x}{\mathrm{d}x} = -\cos_p^{2-p} x \sin_p^{p-1} x. \tag{1.5}$$

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The generalized tangent function $\tan_p x$ is defined as

$$\tan_p x = \frac{\sin_p x}{\cos_p x}, \quad x \in \mathbb{R}\backslash\left\{k\pi_p + \frac{\pi_p}{2} : k \in \mathbb{Z}\right\}. \tag{1.6}$$

From (1.6), it follows that

$$\frac{\mathrm{d}\tan_p x}{\mathrm{d}x} = 1 + |\tan_p x|^p, \quad x \in \left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right). \tag{1.7}$$

The generalized secant function $\sec_p x$ is defined as

$$\sec_p x = \frac{1}{\cos_p x}, \quad x \in \left[0, \frac{\pi_p}{2}\right). \tag{1.8}$$

It follows from (1.6) and (1.7) that

$$\sec_p^p x = 1 + \tan_p^p x, \quad x \in \left(0, \frac{\pi_p}{2}\right) \tag{1.9}$$

and

$$\frac{\mathrm{d}\sec_p x}{\mathrm{d}x} = \sec_p x \tan_p^{p-1} x, \quad x \in \left[0, \frac{\pi_p}{2}\right). \tag{1.10}$$

The generalized cosecant function $\csc_p x$ may be defined as

$$\csc_p x = \frac{1}{\sin_p x}, \quad x \in \left(0, \frac{\pi_p}{2}\right]. \tag{1.11}$$

It is clear that

$$\csc_p^p x = 1 + \frac{1}{\tan_p^p x}, \quad x \in \left(0, \frac{\pi_p}{2}\right) \tag{1.12}$$

and

$$\frac{\mathrm{d}\csc_p x}{\mathrm{d}x} = -\frac{\csc_p x}{\tan_p x}, \quad x \in \left(0, \frac{\pi_p}{2}\right). \tag{1.13}$$

The generalized inverse hyperbolic sine function $\mathrm{arcsinh}_p x$ is defined by

$$\mathrm{arcsinh}_p(x) = \begin{cases} \displaystyle\int_0^x \frac{1}{(1+t^p)^{1/p}} dt, & x \in [0, \infty), \\ -\mathrm{arcsinh}_p(-x), & x \in (-\infty, 0). \end{cases} \tag{1.14}$$

The inverse of $\mathrm{arcsinh}_p$ is called the generalized hyperbolic sine function and denoted by $\sinh_p$.

The generalized hyperbolic cosine function $\cosh_p x$ is defined as

$$\cosh_p x = \frac{\mathrm{d}\sinh_p x}{\mathrm{d}x}. \tag{1.15}$$

It is easy to show that

$$(\cosh_p^p x) - |\sinh_p x|^p = 1, \quad x \in \mathbb{R} \tag{1.16}$$

and

$$\frac{\mathrm{d}\cosh_p x}{\mathrm{d}x} = \cosh_p^{2-p} x \sinh_p^{p-1} x, \quad x \geq 0. \tag{1.17}$$

The generalized hyperbolic tangent function and the generalized hyperbolic secant function are defined as

$$\tanh_p x = \frac{\sinh_p x}{\cosh_p x} \tag{1.18}$$

and

$$\mathrm{sech}_p x = \frac{1}{\cosh_p x}. \tag{1.19}$$

Their derivatives are

$$\frac{\mathrm{d} \tanh_p x}{\mathrm{d}x} = 1 - \tanh_p^p x = \mathrm{sech}_p^p x, \ \ x \geq 0 \tag{1.20}$$

and

$$\frac{\mathrm{d}\mathrm{sech}_p x}{\mathrm{d}x} = -\mathrm{sech}_p x \tanh_p^{p-1} x. \tag{1.21}$$

Recently, Takeuchi [46] studied the (p, q)-trigonometric functions depending on two parameters. For $p = q$, these functions reduce to the so-called p-trigonometric functions introduced by Lindqvist in his highly cited paper [34]. In present, there has been a vivid interest on the generalized trigonometric and hyperbolic functions, numerous papers have been published on the studies of generalized trigonometric functions and their inequalities. The following (p, q)-eigenvalue problem with Dirichlét boundary condition was considered by Drábek and Manásevich [23]. Let $\phi_p(x) = |x|^{p-2}x$. For $T, \lambda > 0$ and $p, q > 1$

$$\begin{cases} (\phi_p(u'))' + \lambda\, \phi_q(u) = 0, \quad t \in (0, T), \\ u(0) = u(T) = 0. \end{cases}$$

They found the complete solution to this problem. The solution of this problem also appears in [46, Thm 2.1]. In particular, for $T = \pi_{p,q}$ the function $u(t) = \sin_{p,q}(t)$ is a solution to this problem with $\lambda = p/q(p-1)$, where

$$\pi_{p,q} = \int_0^1 (1 - t^q)^{-1/p}\, dt = \frac{2}{q} B\left(1 - \frac{1}{p}, \frac{1}{q}\right). \tag{1.22}$$

For $p = q$, $\pi_{p,q}$ reduces to π_p, see [6]. In order to give the definition of the function $\sin_{p,q}$, first we define its inverse function $\mathrm{arcsin}_{p,q}$, then the function itself. For $x \in [0, 1]$, set

$$F_{p,q}(x) = \mathrm{arcsin}_{p,q} = \int_0^x (1 - t^q)^{-1/p}\, dt\,. \tag{1.23}$$

The function $F_{p,q} : [0, 1] \to [0, \pi_{p,q}/2]$ is an increasing homeomorphism, and

$$\sin_{p,q} = F_{p,q}^{-1}$$

is defined on the the interval $[0, \pi_{p,q}/2]$. The function $\sin_{p,q}$ can be extended to $[0, \pi_{p,q}]$ by

$$\sin_{p,q}(x) = \sin_{p,q}(\pi_{p,q} - x), \quad x \in [\pi_{p,q}/2, \pi_{p,q}].$$

By oddness, the further extension can be made to $[-\pi_{p,q}, \pi_{p,q}]$. Finally, the functions $\sin_{p,q}$ is extended to whole $\mathbb{R}$ by $2\pi_{p,q}$-periodicity, see [25].

In this survey, we give an account of the work in the generalized trigonometric and hyperbolic functions. In many of these results, the l'Hôspital Monotone Rule is a very useful tool. Because of practical constraints, we have to exclude many fine papers and have limited our bibliography to those papers most closely connected to our work.

This survey is organized as follows: In Section 1, we give the introduction. Section 2 gives multiple-angle formulas of generalized trigonometric functions. Section 3 presents classical inequalities for generalized trigonometric and hyperbolic functions. In Section 4, we focus on general convexity and concavity for generalized trigonometric and hyperbolic functions. In section 5, Some Turán type inequalities have been obtained. Section 6 shows some new results about generalized elliptic integrals. Finally, we gives some open problems in Section 7.

2. Multiple-angle formulas of generalized trigonometric functions

Motivated by addition formula for sine function, Edmunds, Gurka and Lang obtained a very beautiful result named by Edmunds-Gurka-Lang identity:

$$\sin_{4/3,4}(2x) = \frac{2\sin_{4/3,4} x(\cos_{4/3,4} x)^{1/3}}{(1+4(\sin_{4/3,4} x)^4(\cos_{4/3,4} x)^{4/3})^{1/2}} \tag{2.1}$$

for $x \in [0, \pi_{4/3,4}/4]$ in [25]. The proof of formula (2.1) applied the addition formula of the Jacobian elliptic function.

Later, in 2012, Bhayo and Vuorinen gave two sub-additive inequalities. For $p, q > 1$, then

$$\sin_{p,q}(r+s) \le \sin_{p,q}(r) + \sin_{p,q}(s), \quad r, s \in (0, \pi_{p,q}/4); \tag{2.2}$$

and

$$\sinh_{p,q}(r+s) \ge \sinh_{p,q}(r) + \sinh_{p,q}(s), \quad r, s \in (0, \infty). \tag{2.3}$$

See Lemma 2.14 of reference [13] in detail.

Recently, Takeuchi [50] gave an alternative proof of formula (2.1) based on multiple-angle formula of lemniscate function slx in 2016. In the paper, he also presented multiple-angle formulas between two kind of the generalized trigonometric functions with parameters $(2, p)$ and $(p*, p)$ where $p* = \frac{p}{p-1}$.

Theorem 2.1 (Theorem 1.1 [50])**.** *For $p \in (1, \infty)$ and $x \in [0, 2^{-2/p}\pi_{2,p}] = [0, \pi_{p*,p}/2]$, we have*

$$\sin_{2,p}(2^{2/p}x) = 2^{2/p} \sin_{p*,p} x \cos_{p*,p}^{p*-1} x \tag{2.4}$$

and

$$\cos_{2,p}(2^{2/p}x) = \cos_{p*,p}^{p*} x - \sin_{p*,p}^{p} x = 1 - 2\sin_{p*,p}^{p} x = 2\cos_{p*,p}^{p*} x - 1. \tag{2.5}$$

Moreover, for $x \in \mathbb{R}$, we have

$$\sin_{2,p}(2^{2/p}x) = 2^{2/p} \sin_{p*,p} x |\cos_{p*,p} x|^{p*-2} \cos_{p*,p} x \tag{2.6}$$

and

$$\cos_{2,p}(2^{2/p}x) = |\cos_{p*,p} x|^{p*} - |\sin_{p*,p} x|^{p} = 1 - 2|\sin_{p*,p} x|^{p} = 2|\cos_{p*,p} x|^{p*} - 1. \tag{2.7}$$

The general multiple-angle formulas of generalized trigonometric functions with single and two parameters are till open.

3. Classical inequalities for generalized trigonometric and hyperbolic functions

3.1. Mitrinović-Adamović-type inequalities and Lazarević-type inequalities.

In 2010, Klén, Vuorinen and Zhang [32] obtained Mitrinović-Adamović inequality and Lazarević inequality for generalized trigonometric and hyperbolic functions, showing that, for all $p \in (1, \infty)$ and $x \in (0, \frac{\pi_p}{2})$

$$(\cos_p(x))^{\alpha} < \frac{\sin_p(x)}{x} < 1 \tag{3.1}$$

with the best constant $\alpha = \frac{1}{p+1}$, and that, for all $p \in (1, \infty)$ and $x \in (0, \infty)$,

$$(\cosh_p(x))^\alpha < \frac{\sinh_p(x)}{x} < (\cosh_p(x))^\beta, \tag{3.2}$$

with the best constants $\alpha = \frac{1}{p+1}$ and $\beta = 1$.

In 2013, Bhayo and Yin solved conjecture 3.12 posed by Klén, Vuorinen and Zhang [32]. In [19], they gave the following inequalities:

For $p \in [2, \infty)$ and $x \in (0, \frac{\pi_p}{2})$, then

$$\left(\frac{x}{\sinh_p(x)}\right)^p < \frac{\sin_p(x)}{x} < \frac{x}{\sinh_p(x)}, \tag{3.3}$$

and

$$\frac{1}{(\cosh_p(x))^\beta} < \frac{\sin_p(x)}{x} < \frac{1}{(\cosh_p(x))^\alpha}, \tag{3.4}$$

with the best constants $\alpha = \frac{1}{p+1}$ and $\beta = \frac{\log(\frac{\pi_p}{2})}{\log(\cosh_p(\frac{\pi_p}{2}))}$.

The inequality (3.4) had also been obtained by Yang. See Theorem 1.6 of reference [52].

3.2. **Huygens-type inequalities.** In 2010, Klén, Vuorinen and Zhang [32] obtained the following inequalities of Huygens type for the generalized trigonometric and hyperbolic functions

$$\frac{p\sin_p(x)}{x} + \frac{\tan_p(x)}{x} > 1 + p, \tag{3.5}$$

for $p > 1$ and $x \in (0, \frac{\pi_p}{2})$;

$$\frac{p\sinh_p(x)}{x} + \frac{\tanh_p(x)}{x} > 1 + p, \tag{3.6}$$

for $p > 1$ and $x > 0$.

In the same paper, they also showed that

$$(p+1)\frac{\sin_p(x)}{x} + \frac{1}{\cos_p(x)} > p + 2, \ \ for\ p > 1, x \in (0, \frac{\pi_p}{2}), \tag{3.7}$$

and

$$(p+1)\frac{\sinh_p(x)}{x} + \frac{1}{\cosh_p(x)} > p + 2, \ \ for\ p > 1, x > 0. \tag{3.8}$$

In 2014, Yin, Huang and Qi [58] obtained the second Huygens-type inequalities.

$$\frac{px}{\sin_p(x)} + \frac{x}{\tan_p(x)} > 1 + p, \ \ for\ p \in (1, 2], x \in (0, \frac{\pi_p}{2}), \tag{3.9}$$

and

$$\frac{px}{\sinh_p(x)} + \frac{x}{\tanh_p(x)} > 1 + p, \ \ for\ p \in (1, 2], x \in (0, \infty). \tag{3.10}$$

The formulas (3.5) and (3.9) had also been obtained by Neumann in 2014. See formulas (41) and (43) of references [39]. A particular case $p = 2$ of formulas (3.5)(3.9) and (3.10) also appeared [37] in 2014.

3.3. **Wilker-type inequalities.** In 2010, Klén, Vuorinen and Zhang [32] obtained Wilker-type inequalities for generalized hyperbolic functions

$$\left(\frac{\sinh_p(x)}{x}\right)^p + \frac{\tanh_p(x)}{x} > 2, \tag{3.11}$$

for $p > 1$ and $x > 0$.

In 2014, Yin, Huang and Qi proved Wilker-type inequalities involving the generalized sine and tangent functions: For $p > 1$ and $x \in (0, \frac{\pi_p}{2})$, then

$$\left(\frac{\sin_p(x)}{x}\right)^p + \frac{\tan_p(x)}{x} > 2. \tag{3.12}$$

In the same paper, they also proved the second Wilker-type inequalities, showing that, for $x \in (0, \frac{\pi_p}{2}), p \in (1, 2]$,

$$\left(\frac{x}{\sin_p x}\right)^p + \frac{x}{\tan_p x} > 2 \tag{3.13}$$

and that, for $x > 0, p \in (1, 2]$,

$$\left(\frac{x}{\sinh_p(x)}\right)^p + \frac{x}{\tanh_p(x)} > 2. \tag{3.14}$$

Later, Yin and Huang [56] generalized above the first and second Wilker-type inequalities, showing that, for $x \in (0, \frac{\pi_p}{2}), p > 1, \alpha - p\beta \le 0, \beta > 0$,

$$\left(\frac{\sin_p x}{x}\right)^\alpha + \left(\frac{\tan_p x}{x}\right)^\beta > 2 \tag{3.15}$$

and that, for $p > 1, x > 0, \alpha - p\beta \le 0, \beta > 0$,

$$\left(\frac{\sinh_p x}{x}\right)^\alpha + \left(\frac{\tanh_p x}{x}\right)^\beta > 2. \tag{3.16}$$

Using different method, Neumann [37] and Yin el. [58] proved the following inequality

$$\left(\frac{t}{\sin_p(t)}\right)^p + \frac{t}{\tan_p t} < \left(\frac{\sin_p(t)}{t}\right)^p + \frac{\tan_p t}{t} \tag{3.17}$$

for $p > 1$ and $t \in (0, \frac{\pi_p}{2})$. Applying AGM inequality, Yin, Huang and Qi had proved that, for $p \ge 2, t > 0$ and $x \in (0, \frac{\pi_p}{2})$,

$$\left(\frac{x}{\sin_p(x)}\right)^{pt} + \left(\frac{x}{\sinh_p(x)}\right)^t > 2 \tag{3.18}$$

and

$$p\left(\frac{x}{\sin_p(x)}\right)^t + \left((\frac{x}{\sinh_p(x)}\right)^t > p + 1. \tag{3.19}$$

3.4. **Cusa-Huygens-type inequalities.** In 2010, Klén, Vuorinen and Zhang proved the following Cusa-Huygens type inequalities for generalized trigonometric and hyperbolic functions, showing that, for $p \in (1,2]$ and $x \in (0, \frac{\pi_p}{2}]$,

$$\frac{\sin_p(x)}{x} < \frac{\cos_p(x)+p}{1+p} \leq \frac{\cos_p(x)+2}{3} \tag{3.20}$$

and that, for $p \in (1,2]$ and $x > 0$,

$$\frac{\sinh_p(x)}{x} < \frac{\cosh_p(x)+p}{1+p}. \tag{3.21}$$

Later, Yin and Huang [56] obtained the following version of (3.20): For $p \in (1,2]$ and $x \in (0, \frac{\pi_p}{2}]$,

$$\left(\frac{p+\cos_p x}{p+1}\right)^{\alpha} < \frac{\sin_p(x)}{x} < \left(\frac{p+\cos_p x}{p+1}\right)^{\beta}. \tag{3.22}$$

The constrants $\alpha = \frac{\ln(\frac{2}{\pi_p})}{\ln(\frac{p}{p+1})}$ and $\beta = 1$ are best possible.

In 2013, Yin and Huang [55] also obtained the following inequality

$$\left(\frac{2+\cos x}{3}\right)^{\alpha} < \frac{\sin_p x}{x} < \left(\frac{2+\cos_p x}{3}\right)^{\beta} \tag{3.23}$$

for $p \in (1,2]$ and $x \in (0, \frac{\pi_p}{2}]$. The constants $\alpha = \frac{\ln(\frac{2}{\pi_p})}{\ln(\frac{2}{3})}$ and $\beta = \frac{3}{p+1}$ are best possible.

3.5. **Neumann inequality.** In 2014, by using Schwab-Borchadt mean, Neumann proved that

$$(\cos_p t)^{\frac{1}{p+1}} < \left[\frac{\sin_p t}{\tanh^{-1}(\sin_p t)}\right]^{\frac{1}{p}} < \frac{\sin_p t}{t}, \ for\ p > 1, x \in (0, \frac{\pi_p}{2})$$

and

$$(\cosh_p t)^{\frac{1}{p+1}} < \left[\frac{\sinh_p t}{\tanh^{-1}(\sinh_p t)}\right]^{\frac{1}{p}} < \frac{\sinh_p t}{t}, \ for\ p > 1, x > 0.$$

3.6. **Bounds of generalized trigonometric and hyperbolic functions.** In 2013, Bhayo and Vuorinen [14] gave some bounds of generalized trigonometric and hyperbolic functions by using properties of hypergeometric function. Their results read as follows

Theorem 3.1 (Theorem 1.1 [14]). *For $p > 1$ and $x \in (0,1)$, we have*

$$\left(1+\frac{x^p}{p(1+p)}\right)x < \operatorname{arcsin}_p x < \frac{\pi_p}{2}x,$$

$$\left(1+\frac{1-x^p}{p(1+p)}\right)(1-x^p)^{1/p} < \operatorname{arccos}_p x < \frac{\pi_p}{2}(1-x^p)^{1/p},$$

$$\frac{(p(1+p)(1+x^p)+x^p)x}{p(1+p)(1+x^p)^{1+1/p}} < \operatorname{arctan}_p x < 2^{1/p} b_p \left(\frac{x^p}{1+x^p}\right)^{1/p}.$$

Theorem 3.2 (Theorem 1.2 [14])**.** *For $p>1$ and $x\in(0,1)$, we have*

$$z\left(1+\frac{\log(1+x^p)}{1+p}\right)<\operatorname{arcsinh}_p x<z\left(1+\frac{1}{p}\log(1+x^p)\right),z=\left(\frac{x^p}{1+x^p}\right)^{1/p},$$

$$x\left(1-\frac{1}{1+p}\log(1-x^p)\right)<\operatorname{arctanh}_p x<x\left(1-\frac{1}{p}\log(1-x^p)\right).$$

Later, in [13], they also gave bounds of generalized trigonometric and hyperbolic functions with two parameters, showing that for $p,q>1$ and $x\in(0,1)$,

(1) $x\left(1+\frac{x^q}{p(1+q)}\right)<\arcsin_{p,q}x<\min\left\{\frac{\pi_{p,q}}{2}x,(1-x^q)^{-1/(p(1+q))}x\right\}$,

(2) $\left(\frac{x^p}{1+x^q}\right)^{1/p}L(p,q,x)<\operatorname{arcsinh}_{p,q}x<\left(\frac{x^p}{1+x^q}\right)^{1/p}U(p,q,x)$,

where $L(p,q,x)=\max\left\{\left(1-\frac{qx^q}{p(1+q)(1+x^q)}\right)^{-1},(x^q+1)^{1/p}\left(\frac{pq+p+qx^q}{p(q+1)}\right)^{-1/q}\right\}$, and $U(p,q,x)=\left(1-\frac{x^q}{1+x^q}\right)^{-q/(p(q+1))}$.

In 2014, Baricz, Bhayo and Pogány presented some new lower and upper bounds for the functions $\arctan_p(x)$ and $\operatorname{arctanh}_p(x)$ in [5].

Theorem 3.3 (Theorem 6 [5])**.** *For $p>1,x\in(0,1)$, there holds*

$$\operatorname{arctanh}_p(x)<\frac{x}{2}\left(1-\frac{2}{p}\log(1-x^{\frac{p}{2}})+\frac{2^{\frac{2}{p}}b_{\frac{p}{2}}}{(1+x^{\frac{p}{2}})^{\frac{2}{p}}}\right),$$

$$\arctan_p(x)<x\left(1-\frac{1}{p(1+p)}\log(1-x^p)-\frac{1}{p}\log(1+x^p)\right)=:R_p(x),$$

where

$$b_s:=\frac{1}{2s}\left\{\psi\left(\frac{1+s}{2s}\right)-\psi\left(\frac{1}{2s}\right)\right\}.$$

Moreover, we have

$$\operatorname{arctanh}_p(x)>\frac{x}{2}\left(1-\frac{2}{2+p}\log(1-x^{\frac{p}{2}})+\frac{p(2+p)(1+x^{\frac{p}{2}})+4x^{\frac{p}{2}}}{p(2+p)(1+x^{\frac{p}{2}})^{1+\frac{2}{p}}}\right),$$

and

$$\arctan_p(x)>x\left(1+\frac{1}{p(1+p)}\log(1-x^p)-\frac{2}{1+2p}\log(1+x^p)\right)=:L_p(x).$$

In addition, they also proved that

$$xF\left(\frac{1}{p},1+\frac{1}{p};2+\frac{1}{p};-x^p\right)<\operatorname{arcsinh}_p x<xF\left(-1+\frac{1}{p},\frac{1}{p};\frac{1}{p};-x^p\right),p,x\in(0,1) \tag{3.24}$$

and

$$\arctan_p(x)>xF(2,\frac{1}{p};2+\frac{1}{p};-x^p) \tag{3.25}$$

by proving that the function

$$x\mapsto\frac{\operatorname{arcsinh}_p(x)}{xF(-1+\frac{1}{p},\frac{1}{p};\frac{1}{p};-x^p)}$$

is decreasing on $(0,1)$ for all $p \in (0,1)$, while the functions

$$x \mapsto \frac{xF(\frac{1}{p}, 1+\frac{1}{p}; 2+\frac{1}{p}; -x^p)}{\operatorname{arcsinh}_p(x)}$$

and

$$x \mapsto \frac{xF(2, \frac{1}{p}; 2+\frac{1}{p}; -x^p)}{\arctan_p(x)}$$

are increasing on $(0,1)$ for all $p > 0$.

3.7. **Grünbaum-type inequalities.** In 2014, Baricz, Bhayo and Pogány gave Grünbaum-type inequalities for generalized inverse trigonometric functions.

Theorem 3.4 (Theorem 5 [5])**.** *Let $x, y, z \in (0,1)$ be such that $z^2 = x^2 + y^2$. If $p \geq 1$, then the following Grünbaum type inequalities are true*

$$1 + \frac{\arcsin_p(z^2)}{z^2} \geq \frac{\arcsin_p(x^2)}{x^2} + \frac{\arcsin_p(y^2)}{y^2},$$

$$1 + \frac{\operatorname{arctanh}_p(z^2)}{z^2} \geq \frac{\operatorname{arctanh}_p(x^2)}{x^2} + \frac{\operatorname{arctanh}_p(y^2)}{y^2}.$$

Moreover, if $p \geq 2$, then we have

$$1 + \frac{\arctan_p(z^2)}{z^2} \leq \frac{\arctan_p(x^2)}{x^2} + \frac{\arctan_p(y^2)}{y^2},$$

$$1 + \frac{\operatorname{arcsinh}_p(z^2)}{z^2} \leq \frac{\operatorname{arcsinh}_p(x^2)}{x^2} + \frac{\operatorname{arcsinh}_p(y^2)}{y^2},$$

and the last inequality is reversed when $p \in (0,1]$.

Recently, Yin and Huang generalized these inequalities to generalized inverse trigonometric function with two parameters in 2015. See [57].

4. General convexity and concavity for generalized trigonometric and hyperbolic functions

For two distinct positive real numbers x and y, the Arithmetic mean, Geometric mean, Logarithmic mean, Harmonic mean and the Power mean of order $p \in \mathbb{R}$ are respectively defined by

$$A(x,y) = \frac{x+y}{2},\ G(x,y) = \sqrt(xy),$$

$$L(x,y) = \frac{x-y}{\log(x) - \log(y)},\ x \neq y,$$

$$H(x,y) = \frac{1}{A(1/x, 1/y)},$$

and

$$M_t = \begin{cases} \left(\frac{x^t+y^t}{2}\right)^{1/t}, & t \neq 0, \\ \sqrt{xy}, & t = 0. \end{cases}$$

Let $f : I \to (0,\infty)$ be continuous, where I is a sub-interval of $(0,\infty)$. Let M and N be the means defined above, then we call that the function f is MN-convex (concave) if

$$f(M(x,y)) \leq (\geq) N(f(x), f(y)) \quad for \quad all \quad x, y \in I.$$

Recently, generalized convexity/concavity with respect to general mean values has been studied by Anderson et al. in [4]. We recall one of their results as follows.

Lemma 4.1 ([4], Theorem 2.4). *Let I be an open sub-interval of $(0,\infty)$ and let $f : I \to (0,\infty)$ be differentiable. Then f is HH-convex (concave) on I if and only if $x^2 f'(x)/f(x)^2$ is increasing (decreasing).*

In [4], Baricz studied that if the functions f is differentiable, then it is $(a,b)-$convex (concave) on I if and only if $x^{1-a}f'(x)/f(x)^{1-b}$ is increasing (decreasing).

It is important to mention that (1,1)-convexity means the AA-convexity, $(1,0)-$convexity means the $AG-$convexity, and (0,0)-convexity means the AG-convexity, and (0,0)-convexity means $GG-$convexity.

Recently, Bhayo and Yin considered extensively LL-convex, II-convex by using Chebshev inequality in [17, 18]. They presented the following results.

Lemma 4.2 ([17], Theorem 1). *Let $f : I \to (0,\infty)$ be a continuous and $I \subseteq (0,\infty)$, then*

1. $L(f(x), f(y)) \geq (\leq) f(L(x,y))$,
2. $L(f(x), f(y)) \geq (\leq) f(A(x,y))$,

if f is increasing and log-convex(concave).

Lemma 4.3 ([18], Theorem 1). *Let $f : I \to (0,\infty)$ and $I \subseteq (0,\infty)$. Then the following inequalities holds true:*

$$I(f(x), f(y)) \geq f(I(x,y)) \ (I(f(x), f(y)) \leq f(A(x,y))).$$

If the function $f(x)$ is a continuous differentiable, increasing and log-convex(concave).

Other results of MN-convexity may see references [59, 18]. When these results applied to generalized trigonometric and hyperbolic functions, we can obtain a number of inequalities.

In 2015, [15], Bhayo and Vuorinen proved some power mean inequalities for generalized trigonometric functions with single parameter.

Theorem 4.1 ([15] Theorem 1.1). *For $p > 1, t \geq 0$ and $r, s \in (0,1)$, we have*

(1) $\operatorname{arcsin}_p(M_t(r,s)) \leq M_t(\operatorname{arcsin}_p(r), \operatorname{arcsin}_p(s))$,
(2) $\operatorname{arctanh}_p(M_t(r,s)) \leq M_t(\operatorname{arctanh}_p(r), \operatorname{arctanh}_p(s))$,
(3) $\operatorname{arctan}_p(M_t(r,s)) \geq M_t(\operatorname{arctan}_p(r), \operatorname{arctan}_p(s))$,
(4) $\operatorname{arcsinh}_p(M_t(r,s)) \geq M_t(\operatorname{arcsinh}_p(r), \operatorname{arcsinh}_p(s))$.

Theorem 4.2 ([15] Theorem 1.2). *For $p > 1, t \geq 0$ and $r, s \in (0,1)$, the following relations hold*

(1) $\sin_p(M_t(r,s)) \geq M_t(\sin_p(r), \sin_p(s))$,
(2) $\cos_p(M_t(r,s)) \leq M_t(\cos_p(r), \cos_p(s))$,
(3) $\tan_p(M_t(r,s)) \leq M_t(\tan_p(r), \tan_p(s))$,
(4) $\tanh_p(M_t(r,s)) \geq M_t(\tanh_p(r), \tanh_p(s))$,
(5) $\sinh_p(M_t(r,s)) \leq M_t(\sinh_p(r), \sinh_p(s))$.

Using the same method, Baricz, Bhayo and Klén obtained some power mean inequalities for generalized trigonometric functions with two parameters.

Theorem 4.3 ([7] Theorem 1). *If $p, q > 1$ and $a \geq 1$, then $\operatorname{arcsin}_{p,q}$ is $(a,a)-$convex on (0,1), $\operatorname{arctan}_{p,q}$ is $(a,a)-$convex on (0,1), while $\operatorname{arcsinh}_{p,q}$ is $(a,a)-$convex on $(0,\infty)$. In other words, if $p, q > 1$ and $a \geq 1$, then we have*

$$\operatorname{arcsin}_{p,q}(M_a(r,s)) \leq M_a(\operatorname{arcsin}_{p,q}(r), \operatorname{arcsin}_{p,q}(s)), \quad r, s \in (0,1),$$

$$\operatorname{arctan}_{p,q}(M_a(r,s)) \geq M_a(\operatorname{arctan}_{p,q}(r), \operatorname{arctan}_{p,q}(s)), \quad r,s \in (0,1),$$
$$\operatorname{arcsinh}_{p,q}(M_a(r,s)) \geq M_a(\operatorname{arcsinh}_{p,q}(r), \operatorname{arcsinh}_{p,q}(s)), \quad r,s > 0.$$

Theorem 4.4 ([7] Theorem 2)**.** *If $p,q>1$ and $a \geq 1$, then $\sin_{p,q}$ is $(a,a)-$concave, and $\cos_{p,q}, \tan_{p,q}, \sinh_{p,q}$ are $(a,a)-$convex on (0,1). In other words, if $p,q>1, a \geq 1$ and $r,s \in (0,1)$, then the next inequalities are valid*

$$\sin_{p,q}(M_a(r,s)) \geq M_a(\sin_{p,q}(r), \sin_{p,q}(s)),$$
$$\cos_{p,q}(M_a(r,s)) \leq M_a(\cos_{p,q}(r), \cos_{p,q}(s)),$$
$$\tan_{p,q}(M_a(r,s)) \leq M_t(\tan_{p,q}(r), \tan_{p,q}(s)),$$
$$\sinh_{p,q}(M_a(r,s)) \leq M_t(\sinh_{p,q}(r), \sinh_{p,q}(s)).$$

The next theorems improve some of the above results.

Theorem 4.5 ([7] Theorem 3)**.** *If $p,q>1, a \leq 0$ and $b \in \mathbb{R}$ or $0<a \leq b$ and $b \leq 1$, then $\operatorname{arcsin}_{p,q}$ is $(a,b)-$convex on (0,1), and in particular if $p=q$, then the function $\operatorname{arcsin}_p = \operatorname{arcsin}_{p,p}$ is $(a,b)-$convex on (0,1). In other words, if $p,q>1, a \leq 0$, and $b \in \mathbb{R}$ or $0<a \leq b$ and $b \leq 1$, then for all $r,s \in (0,1)$ we have*

$$\operatorname{arcsin}_{p,q}(M_a(r,s)) \leq M_b(\operatorname{arcsin}_{p,q}(r), \operatorname{arcsin}_{p,q}(s)).$$

Theorem 4.6 ([7] Theorem 4)**.** *If $p,q>1, a \leq 0 \geq b$ or $0<a \leq b$ and $a \leq 1$, then $\operatorname{arcsinh}_{p,q}$ is $(a,b)-$convex on $(0,\infty)$, and in particular if $p=q$, then the function $\operatorname{arcsinh}_p = \operatorname{arcsinh}_{p,p}$ is $(a,b)-$concave on $(0,\infty)$. In other words, if $p,q>1, a \leq 0 \geq b$ or $0<b \leq a$ and $a \leq 1$, then for all $r,s \in (0,\infty)$ we have*

$$\operatorname{arcsinh}_{p,q}(M_a(r,s)) \geq M_b(\operatorname{arcsinh}_{p,q}(r), \operatorname{arcsinh}_{p,q}(s)).$$

Due to geometric convexity (concavity), Bhayo and Vuorinen [13] posed a conjecture in 2012:

Conjecture 4.1. *For $p,q \in (1,\infty)$ and $r,s \in (0,1)$, we have*
(1) $\sin_{p,q}(\sqrt{rs}) \leq \sqrt{\sin_{p,q}(r)\sin_{p,q}(s)}$,
(2) $\sinh_{p,q}(\sqrt{rs}) \geq \sqrt{\sinh_{p,q}(r)\sinh_{p,q}(s)}$.

Very quickly, the conjecture has been proved to be correct by Jiang et. in [29].
In 2014, Bhayo and Yin gave some logarithmic mean inequalities for generalized trigonometric functions by using Lemma 4.2. Their results read as follows:

Theorem 4.7 ([17] Theorem 2)**.** *For $x,y \in (0,\pi_p/2)$, the following inequalities*
1. $L(\sin_p(x), \sin_p(y)) \leq \sin_p(L(x,y)), \quad p>1$,
2. $L(\cos_p(x), \cos_p(y)) \leq \cos_p(L(x,y)), \quad p \geq 2$.

Theorem 4.8 ([17] Theorem 3)**.** *For $p>1$, we have*
1. $L(\frac{1}{\sin_p(x)}, \frac{1}{\sin_p(y)}) \geq \frac{1}{\sin_p(A(x,y))}, \quad x,y \in (0,\pi_p/2)$,
2. $L(\frac{1}{\cos_p(x)}, \frac{1}{\cos_p(y)}) \geq \frac{1}{\cos_p(L(x,y))}, \quad x,y \in (0,\pi_p/2)$,
3. $L(\tanh_p(x), \tanh_p(y)) \leq \tanh_p(A(x,y)), \quad x,y \in (0,\infty)$,
4. $L(\operatorname{arcsinh}_p(x), \operatorname{arcsinh}_p(y)) \leq \operatorname{arcsinh}_p(A(x,y)), \quad x,y \in (0,1)$,
5. $L(\operatorname{arctan}_p(x), \operatorname{arctan}_p(y)) \leq \operatorname{arctan}_p(A(x,y)), \quad x,y \in (0,1)$.

Later, in 2014, Cui and Yin [22] obtained logarithmic mean inequalities for generalized trigonometric functions with two parameters.

5. Parameter convexity and concavity for generalized trigonometric and hyperbolic functions

In 2015, Baricz, Bhayo and Vuorinen began to discuss parameter convexity and concavity of generalized trigonometric functions in [6]. Their main results read as follows.

Theorem 5.1 ([6] Theorem 1)**.** *For all $x \in (0,1)$ fixed, the following hold:*

(1) The functions $p \mapsto \operatorname{arcsin}_p(x)$ and $p \mapsto \operatorname{arctanh}_p(x)$ are strongly decreasing and log-convex on $(1,\infty)$. Moreover, $p \mapsto \operatorname{arcsin}_p(x)$ is strictly geometrically convex on $(1,\infty)$.

(2) The function $p \mapsto \operatorname{arctan}_p(x)$ is strictly increasing and concave on $(1,\infty)$. In particular, the following Turán type inequalities are valid for all $p > 2$ and $x \in (0,1)$

$$\operatorname{arcsin}_p^2(x) < \operatorname{arcsin}_{p-1}(x) \operatorname{arcsin}_{p+1}(x),$$

$$\operatorname{arctanh}_p^2(x) < \operatorname{arctanh}_{p-1}(x) \operatorname{arctanh}_{p+1}(x),$$

$$\operatorname{arctan}_p^2(x) > \operatorname{arctan}_{p-1}(x) \operatorname{arctan}_{p+1}(x).$$

Theorem 5.2 ([6] Theorem 2)**.** *For all $x \in (0,1)$ fixed, the following hold:*

(1) $p \mapsto \operatorname{arcsin}_{p,q}(x)$ is completely monotonic and log-convex on $(1,\infty)$ for $q > 1$.

(2) $p \mapsto \operatorname{arcsin}_{p,q}(x)$ is strictly geometrically convex on $(1,\infty)$ for $q > 1$.

(3) $q \mapsto \operatorname{arcsin}_{p,q}(x)$ is completely monotonic and log-convex on $(1,\infty)$ for $p > 1$.

(4) $p \mapsto \operatorname{arcsinh}_{p,q}(x)$ is strictly increasing and concave on $(1,\infty)$ for $q > 1$.

(5) $q \mapsto \operatorname{arcsinh}_{p,q}(x)$ is strictly increasing and concave on $(1,\infty)$ for $p > 1$.

In particular, the following Turán type inequalities are valid for all $p > 2, q > 1$ and $x \in (0,1)$

$$\operatorname{arcsin}_{p,q}^2(x) < \operatorname{arcsin}_{p-1,q}(x) \operatorname{arcsin}_{p+1,q}(x),$$

$$\operatorname{arcsinh}_{p,q}^2(x) > \operatorname{arcsinh}_{p-1,q}(x) \operatorname{arcsinh}_{p+1,q}(x).$$

Moreover, for $p > 1, q > 2$ and $x \in (0,1)$, we have the next Turán type inequalities

$$\operatorname{arcsin}_{p,q}^2(x) < \operatorname{arcsin}_{p,q-1}(x) \operatorname{arcsin}_{p,q+1}(x),$$

$$\operatorname{arcsinh}_{p,q}^2(x) > \operatorname{arcsinh}_{p,q-1}(x) \operatorname{arcsinh}_{p,q+1}(x).$$

In the same paper, they also posed two conjectures.

Conjecture 5.1. *For $x \in (0,1)$ fixed, the function $p \mapsto \operatorname{arcsinh}_p(x)$ is strictly concave on $(1,\infty)$. In particular, the following Turán type inequality is valid for all $p > 2$ and $x \in (0,1)$*

$$\operatorname{arcsinh}_p^2(x) > \operatorname{arcsinh}_{p-1}(x) \operatorname{arcsinh}_{p+1}(x).$$

Conjecture 5.2. *The following Turán type inequalities hold for all $p > 2$ and $x \in (0,1)$*

$$\sin_p^2(x) > \sin_{p-1}(x) \sin_{p+1}(x),$$

$$\cos_p^2(x) > \cos_{p-1}(x) \cos_{p+1}(x),$$

$$\tan_p^2(x) < \tan_{p-1}(x) \tan_{p+1}(x),$$

$$\sinh_p^2(x) < \sinh_{p-1}(x) \sinh_{p+1}(x),$$

$$\tanh_p^2(x) > \tanh_{p-1}(x) \tanh_{p+1}(x).$$

Later, Karp and Prilepkina [31] studied extensively the conjectures in 2015. Using an auxiliary Lemma, they obtained the following results, showing that, for each fixed $y \in (0,1)$, the function $p \mapsto \sin_p(y)$ is strictly log-concave on $(0,\infty)$, and that, for each fixed $y \in (0,\log 2)$, the function $p \mapsto \tan_p(y)$ is strictly convex on $(1,\infty)$, and the function $p \mapsto \cos_p(y)$ is strictly concave on $(1,\infty)$ respectively, and that, for each fixed $y \in (0,\infty)$, the functions $p \mapsto \sinh_p(y)$ and $p \mapsto \cosh_p(y)$ are strictly log-concave on $(0,\infty)$, the function $p \mapsto \tanh_p(y)$ is strictly concave on $(0,\infty)$.

6. GENERALIZED COMPLETE ELLIPTIC INTEGRALS

We may define all kinds of general complete elliptic integrals via generalized trigonometric functions.

6.1. **Complete p-elliptic integrals.** In 2016, Takeuchi [49] defined a new form of the generalized complete elliptic integrals via generalized trigonometric functions with single parameter. We repeat the definition of complete $p-$elliptic integrals of the first kind $K_p(k)$ and of the second kind $E_p(k)$: for $k \in (0,1)$

$$K_p(k) := \int_0^{\frac{\pi_p}{2}} \frac{d\theta}{(1-k^p \sin_p^p \theta)^{1-\frac{1}{p}}} = \int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}(1-k^p t^p)^{1-\frac{1}{p}}}, \tag{6.1}$$

$$E_p(k) := \int_0^{\frac{\pi_p}{2}} (1-k^p \sin_p^p \theta)^{\frac{1}{p}}\, d\theta = \int_0^1 \left(\frac{1-k^p t^p}{1-t^p}\right)^{\frac{1}{p}} dt. \tag{6.2}$$

In the paper, he showed Legendre's relation for $K_p(k)$ and $E_p(k)$

$$K_p'(k)E_p(k) + K_p(k)E_p'(k) - K_p(k)K_p'(k) = \frac{\pi_p}{2}, \quad for\ k \in (0,1), \tag{6.3}$$

where $k' := (1-k^p)^{\frac{1}{p}}$, $K_p'(k) = K_p(k')$ and $E_p'(k) := E_p(k')$, and observed relationship between the complete p-elliptic integrals and the Gaussian hyperbolic functions. As applications of complete p-elliptic, Takeuchi also gave a computation formula of π_p with $p = 3$ and an elementary proof of Ramanujan's cubic transformation.

Later, Yin and Mi [59] presented some Landen type inequalities related to $K_p(k)$ as follows.

Theorem 6.1 ([59] Theorem 2.1)**.** *Let $a, b, c \in \mathbb{R}, p > 1$ such that c is not a negative integer or zero and consider the function $H : (0,1) \mapsto (0,\infty)$, defined by $H(x) = \frac{F(a,b;c;x)}{F(\frac{1}{p},1-\frac{1}{p};1;x)}$. Then the following results are true.*

(1) If $a+b-c \geq 0$ and $p^2 ab \geq \max\{(p-1)c,(p-1)\}$, then $H(x)$ is increasing, and

$$\frac{F(a,b;c;r^p)}{F\left(a,b;c;\frac{p^p r}{(1+r)^p}\right)} \leq \frac{K_p(r)}{K_p\left(\frac{p\sqrt[p]{r}}{1+r}\right)}, \tag{6.4}$$

$$\frac{F\left(a,b;c;\left(\frac{1-r}{1+r}\right)^p\right)}{F(a,b;c;1-r^p)} \leq \frac{K_p\left(\frac{1-r}{1+r}\right)}{K_p\left((1-r^p)^{1/p}\right)} \tag{6.5}$$

hold true for each other $r \in (0,1)$.

(2)If $a+b-c \leq 0$ and $p^2 ab \leq \max\{(p-1)c,(p-1)\}$, then $H(x)$ is increasing, and

$$\frac{F(a,b;c;r^p)}{F\left(a,b;c;\frac{p^p r}{(1+r)^p}\right)} \geq \frac{K_p(r)}{K_p\left(\frac{p\sqrt[p]{r}}{1+r}\right)}, \tag{6.6}$$

$$\frac{F\left(a,b;c;\left(\frac{1-r}{1+r}\right)^p\right)}{F(a,b;c;1-r^p)}\geq\frac{K_p\left(\frac{1-r}{1+r}\right)}{K_p\left((1-r^p)^{1/p}\right)}\tag{6.7}$$

hold true for each other $r\in(0,1)$.

6.2. **Complete (p,q)-elliptic integrals.** In 2015, for all $p,\in(1,\infty)$ and $r\in(0,1)$, the complete (p,q)-elliptic integrals of the first and second kinds [20, 47] are defined by

$$K_{p,q}(r):=\int_0^{\frac{\pi_{p,q}}{2}}(1-r^q\sin_{p,q}^q t)^{(1/p-1)}dt, K'_{p,q}=K'_{p,q}(r)=K_{p,q}(r')$$

and

$$E_{p,q}(r):=\int_0^{\frac{\pi_{p,q}}{2}}(1-r^q\sin_{p,q}^q t)^{1/p}dt, E'_p=E'_{p,q}(r)=E_{p,q}(r'),$$

respectively. Here, $p,q>1,r\in(0,1)$ and $r'=(1-r^p)^{1/p}$.

In [20], Bhayo and Yin studied Turán type inequalities and series representation of complete $(p,q)-$elliptic integrals in detail. Their main results read as follows.

Theorem 6.2 ([18] Theorem 2.6)**.** *For $p,q>1$ and $r\in(0,1)$, we have*

(1) The function $r\mapsto K_{p,q}(r)$ is strictly increasing and log-convex. Moreover, $r\mapsto K_{p,q}(r)$ is strictly geometrically convex on $(0,1)$.

(2) The function $r\mapsto E_{p,q}(r)$ is strictly decreasing and geometrically concave on $(0,1)$.

Theorem 6.3 ([18] Theorem 2.7)**.** *For fixed $r\in(0,1)$ and $q>0$,*

(1) The functions $p\mapsto K_{p,q}(r)$ is strictly increasing and log-concave on $(0,\infty)$,

(2) The function $p\mapsto E_{p,q}(r)$ is strictly increasing and log-concave on $(0,\infty)$.

For fixed $r\in(0,1)$ and $p>0$,

(3) The functions $q\mapsto K_{p,q}(r)$ is strictly decreasing and log-convex on $(0,\infty)$,

(4) The function $q\mapsto E_{p,q}(r)$ is strictly decreasing and log-convex on $(0,\infty)$.

In particular, for $r\in(0,1)$, the following Turán type inequalities hold true

$$K_{p,q}(r)^2\geq K_{p-1,q}(r)K_{p+1,q}(r),\quad p>1,q>0,$$

$$E_{p,q}(r)^2\geq E_{p-1,q}(r)E_{p+1,q}(r),\quad p>1,q>0,$$

$$K_{p,q}(r)^2\leq K_{p,q-1}(r)K_{p,q+1}(r),\quad p>0,q>1,$$

$$E_{p,q}(r)^2\leq E_{p,q-1}(r)K_{p,q+1}(r),\quad p>0,q>1.$$

Theorem 6.4 ([18] Theorem 2.9)**.** *For $p,q>1$ and $r\in(0,1),\lambda<\frac{1}{2}$, we have*

$$K_{p,q}(r)=\frac{\pi_{p,q}}{2}\sum_{n=0}^{\infty}\begin{pmatrix}\frac{1}{p}-1\\ n\end{pmatrix}\frac{1}{(1-\lambda)^{n+1-\frac{1}{p}}}\sum_{j=0}^{n}(-1)^j\begin{pmatrix}n\\ j\end{pmatrix}\begin{pmatrix}-\frac{1}{q}\\ j\end{pmatrix}\begin{pmatrix}\frac{1}{p}-1-\frac{1}{q}\\ j\end{pmatrix}\lambda^{n-j}r^{qj},\tag{6.8}$$

and

$$E_{p,q}(r)=\frac{\pi_{p,q}}{2}\sum_{n=0}^{\infty}\begin{pmatrix}\frac{1}{p}\\ n\end{pmatrix}\frac{1}{(1-\lambda)^{n-\frac{1}{p}}}\sum_{j=0}^{n}(-1)^j\begin{pmatrix}n\\ j\end{pmatrix}\begin{pmatrix}-\frac{1}{q}\\ j\end{pmatrix}\begin{pmatrix}\frac{1}{p}-1-\frac{1}{q}\\ j\end{pmatrix}\lambda^{n-j}r^{qn}.\tag{6.9}$$

Later, Bhayo and Yin [20] gave two interesting inequalities. First of all, they denoted the function

$$\Delta_{p,q}(r) = \frac{E_{p,q} - (r')^p K_{p,q}}{r^p} - \frac{E'_{p,q} - r^p K'_{p,q}}{(r')^p}$$

and obtained following theorems.

Theorem 6.5 ([20] Theorem 1.3). *The function $\Delta_{p,q}$ is strictly increasing and strictly convex from $(0,1)$ onto $\left(\frac{(1-\frac{1}{p})\pi_{p,q}}{2(1+\frac{1}{q}-\frac{1}{p})} - 1, 1 - \frac{(1-\frac{1}{p})\pi_{p,q}}{2(1+\frac{1}{q}-\frac{1}{p})}\right)$ for p, q satisfy the following conditions:*
(i) $2 + \frac{1}{p} + \frac{1}{p^2} \leq \frac{5}{p} + \frac{1}{q} < 3 + \frac{1}{p^2}$;
(ii) $\varepsilon(p,q) > 0$;
where

$$\varepsilon(p,q) = 20 - \frac{42}{p} + \frac{6}{q} + \frac{21}{p^2} - \frac{2}{q^2} - \frac{20}{pq} + \frac{9}{p^2 q} - \frac{3}{p^3} - \frac{1}{p^3 q}.$$

Moreover, for all $r \in (0,1)$, we have

$$\frac{(1-\frac{1}{p})\pi_{p,q}}{2(1+\frac{1}{q}-\frac{1}{p})} - 1 + \alpha(r) < \Delta_{p,q}(r) < \frac{(1-\frac{1}{p})\pi_{p,q}}{2(1+\frac{1}{q}-\frac{1}{p})} - 1 + \beta r \tag{6.10}$$

with best possible constants $\alpha = 0$ and $\beta = 2 - \frac{(1-\frac{1}{p})\pi_{p,q}}{(1+\frac{1}{q}-\frac{1}{p})}$.

Theorem 6.6 ([20] Theorem 1.4). *For all $r, s \in (0,1)$ and p, q satisfying conditions (i) and (ii), we have*

$$\frac{(1-\frac{1}{p})\pi_{p,q}}{2(1+\frac{1}{q}-\frac{1}{p})} - 1 < \Delta_{p,q}(rs) - \Delta_{p,q}(r) - \Delta_{p,q}(s) < 1 - \frac{(1-\frac{1}{p})\pi_{p,q}}{2(1+\frac{1}{q}-\frac{1}{p})}. \tag{6.11}$$

Theorem 6.5 and 6.6 generalized results of Alzer and Richards in [2]. It is worth to note that Yin and Huang also denoted another $(p,q)-$elliptic integrals in 2015. The reader may see the reference [55] for more. Very recently, Takeuchi [51] gave a new complete $(p,q,r)-$ elliptic integrals with three parameters. These integrals are defined by

$$K_{p,q,r}(k) := \int_0^1 \frac{dt}{(1-t^q)^{\frac{1}{p}}(1-k^q t^q)^{\frac{1}{r}}} \tag{6.12}$$

and

$$E_{p,q,r}(k) := \int_0^1 \frac{1-k^q t^q{}^{1/r^*}}{1-t^q{}^{\frac{1}{p}}} dt, \tag{6.13}$$

where$p \in \mathbb{P}^* := (-\infty, 0) \cup (1, \infty], q, r \in (1, \infty)$ and $1/r + 1/r^* = 1$.

For $p \in \mathbb{P}^*$ and $q, r \in (1, \infty)$, using $\sin_{p,q}\theta$ and $\pi_{p,q}$, we can express $K_{p,q,r}(k)$ and $E_{p,q,r}(k)$ as follows.

$$K_{p,q,r}(k) = \int_0^{\pi_{p,q}/2} \frac{d\theta}{(1-k^q \sin_{p,q}^q \theta)^{1/r}}, \tag{6.14}$$

$$E_{p,q,r}(k) = \int_0^{\pi_{p,q}/2} (1-k^q \sin_{p,q}^q \theta)^{1/r*} d\theta. \tag{6.15}$$

In the paper, he proved Legendre type relation:
Let $p \in \mathbb{P}^*, q, r \in (1, \infty)$ and $k \in (0,1)$. Then

$$E_{p,q,r^*}(k)K_{p,r,q^*}(k') + K_{p,q,r^*}(k)E_{p,r,q^*}(k') - K_{p,q,r^*}(k)K_{p,r,q^*}(k') = \frac{\pi_{p,q}\pi_{s,r}}{4}, \tag{6.16}$$

where $k' := (1-k^q)^{1/r}$ and $1/s = 1/p - 1/q$.

The research has just begun, and there are a lot of work remains to be further research.

7. Open problems

Here, we enumerate several open problems or unsolve problems.

Open Problem 7.1. *(conjecture 3.29 [32]) For $p \in (2, \infty)$ and $x \in (0, \pi_p/2)$,*

$$\frac{\sinh_p(x)}{x} < \frac{p+1}{p+\cos_p(x)}. \tag{7.1}$$

Open Problem 7.2. *(conjecture [31]) There exists $p_0 \in (0,1)$ such that the function $p \mapsto \sin_p(y)$ is strictly concave on (p_0, ∞) for all $y \in (0,1)$. If $p \in (0, p_0)$, concavity is violated for some $y \in (0,1)$.*

Open Problem 7.3. *(open problem 3.1 [53]) For all $p \in (1,2]$ and $x \in (0, \pi_p)$, then*

$$\frac{\ln(1-\sin_p(x))}{\ln\cos_p(x)} < \frac{x+p}{x}. \tag{7.2}$$

Open Problem 7.4. *(conjecture 3.8 [14]) For a fixed $x \in (0,1)$, the functions $\sin_p\left(\frac{\pi_p x}{2}\right), \tan_p\left(\frac{\pi_p x}{2}\right), \sinh_p(c_p x)$ are monotone in $p \in (1,\infty)$. For fixed $x > 0$, $\tanh_p(x)$ is increasing in $p \in (1,\infty)$.*

Open Problem 7.5. *(open problem 4.1 [58]) For $p \in (1, +\infty)$,*

$$\frac{p\sin_p x}{x} + \frac{\tan_p x}{x} > \frac{px}{\sin_p x} + \frac{x}{\tan_p x} \tag{7.3}$$

is valid on $(0, \frac{\pi_p}{2})$.

Open Problem 7.6. *For $p \in [2, +\infty)$, the function $\frac{\frac{K_p(\sqrt{r})}{K_p(r)} - 1}{r}$ is strictly decreasing for $x \in (0,1)$.*

Open Problem 7.7. *For $\lambda \leq 0$ and $p \geq 2$, then the inequality*

$$M_\lambda(m(x), m(y)) \leq m(M_\lambda(x,y)) \tag{7.4}$$

holds true for all $x, y \in (0,1)$, where $M_\lambda(x,y)$ is the power mean and $m(r)$ is defined by $m(r) = \frac{p}{\pi_p}(1-r^p)K_p(r)K'_P(r)$.

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