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SOME INEQUALITIES OF HERMITE-HADAMARD TYPE FOR CONVEX FUNCTIONS AND RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. In this paper we establish several upper and lower bounds for the functions

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{a+}^{\alpha}f(x)}{(x-a)^{\alpha}}+\frac{J_{b-}^{\alpha}f(x)}{(b-x)^{\alpha}}\right] \text{ and } \frac{1}{2}\Gamma(\alpha)\left[\frac{J_{x-}^{\alpha}f(a)}{(x-a)^{\alpha}}+\frac{J_{x+}^{\alpha}f(b)}{(b-x)^{\alpha}}\right]$$

in the case of Riemann-Liouville fractional integrals $J_{\cdot\pm}^{\alpha}$, for convex functions $f:[a,b]\to\mathbb{R}$, for $\alpha>0$ and $x\in(a,b)$. Some particular cases of interest are examined. Various Hermite-Hadamard type inequalities are also provided.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality*.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [8], the recent survey paper [7] and the references therein.

Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt$$

for $a<x\leq b$ and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt$$

for $a\leq x<b$, where Γ is the *Gamma function*. For $\alpha=0$, they are defined as

$$J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x) \text{ for } x\in(a,b).$$

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In [18] Sarikaya et al. established the following Hermite-Hadamard type inequality for $\alpha > 0$

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

provided $f : [a,b] \to \mathbb{R}$ is a convex function.

A different version was also obtained by Sarikaya and Yildirim in [19] as follows

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}+}^{\alpha} f(b) + J_{\frac{a+b}{2}-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

provided $f : [a,b] \to \mathbb{R}$ is a convex function.

For other Hermite-Hadamard type inequalities for the Riemann-Liouville fractional integrals, see [1]-[3], [10]-[23] and the references therein.

Motivated by the above results, we establish in this paper several upper and lower bounds for the functions

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{a+}^{\alpha} f(x)}{(x-a)^{\alpha}} + \frac{J_{b-}^{\alpha} f(x)}{(b-x)^{\alpha}}\right] \text{ and } \frac{1}{2}\Gamma(\alpha)\left[\frac{J_{x-}^{\alpha} f(a)}{(x-a)^{\alpha}} + \frac{J_{x+}^{\alpha} f(b)}{(b-x)^{\alpha}}\right]$$

in the case of convex functions $f : [a,b] \to \mathbb{R}$ for $\alpha > 0$ and $x \in (a,b)$. Some particular cases of interest are examined. Other Hermite-Hadamard type inequalities are also provided.

2. Some Preliminary Facts

In 1906, Fejér [9], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 1 (Fejér's Inequality)**.** *Consider the integral* $\int_a^b h(x) g(x) dx$, *where* h *is a convex function in the interval* (a,b) *and* g *is a positive function in the same interval such that*

$$g(a+t) = g(b-t), \quad 0 \leq t \leq \frac{1}{2}(b-a),$$

i.e., $g(\cdot)$ *is symmetric. Under those conditions the following inequalities are valid:*

$$h\left(\frac{a+b}{2}\right)\int_a^b g(t) dt \leq \int_a^b h(t) g(t) dx \leq \frac{h(a)+h(b)}{2}\int_a^b g(t) dt. \tag{2.1}$$

If h *is concave on* (a,b), *then the inequalities reverse in (2.1).*

Clearly, for $g(x) \equiv 1$ on $[a,b]$ we get (1.1).

Since we have the representation

$$J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) = \frac{1}{\Gamma(\alpha)}\int_a^b \left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right] f(t) dt, \tag{2.2}$$

the function $g : [a,b] \to \mathbb{R}$

$$g(t) = \frac{1}{\Gamma(\alpha)}\left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right]$$

is positive and symmetric on $[a,b]$ and

$$\int_a^b g(t) dt = \frac{1}{\Gamma(\alpha)}\int_a^b \left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right] dt = \frac{2}{\Gamma(\alpha+1)}(b-a)^{\alpha},$$

then by Fejér's inequality (2.1) we have (1.2). This is a simpler proof than the one in [18].

We have

$$\begin{aligned} & J^{\alpha}_{\frac{a+b}{2}+} f(b) + J^{\alpha}_{\frac{a+b}{2}-} f(a) \\ & = \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^{b} (b-t)^{\alpha-1} f(t)\, dt + \frac{1}{\Gamma(\alpha)} \int_{a}^{\frac{a+b}{2}} (t-a)^{\alpha-1} f(t)\, dt. \end{aligned}$$

Using the change of variable $t = a + b - u$, we have $dt = -du$ and

$$\int_{\frac{a+b}{2}}^{b} (b-t)^{\alpha-1} f(t)\, dt = \int_{a}^{\frac{a+b}{2}} (u-a)^{\alpha-1} f(a+b-u)\, du.$$

Therefore we have the representation

$$\begin{aligned} & \frac{1}{2}\left[J^{\alpha}_{\frac{a+b}{2}+} f(b) + J^{\alpha}_{\frac{a+b}{2}-} f(a) \right] \\ & = \frac{1}{\Gamma(\alpha)} \int_{a}^{\frac{a+b}{2}} (t-a)^{\alpha-1} \frac{f(t) + f(a+b-t)}{2} dt \end{aligned} \tag{2.3}$$

for $\alpha > 0$.

Since f is convex on $[a, b]$ then, see for instance [6],

$$f\left(\frac{a+b}{2}\right) \le \frac{f(t) + f(a+b-t)}{2} \le \frac{f(a) + f(b)}{2} \tag{2.4}$$

for any $t \in [a, b]$.

If we multiply this inequality by $\frac{1}{\Gamma(\alpha)} (t-a)^{\alpha-1}$ and integrate on $\left[a, \frac{a+b}{2}\right]$, then we get

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \frac{1}{\Gamma(\alpha)} \int_{a}^{\frac{a+b}{2}} (t-a)^{\alpha-1}\, dt \\ & \le \frac{1}{2}\left[J^{\alpha}_{\frac{a+b}{2}+} f(b) + J^{\alpha}_{\frac{a+b}{2}-} f(a) \right] \\ & \le \frac{f(a) + f(b)}{2} \frac{1}{\Gamma(\alpha)} \int_{a}^{\frac{a+b}{2}} (t-a)^{\alpha-1}\, dt \end{aligned} \tag{2.5}$$

and since

$$\frac{1}{\Gamma(\alpha)} \int_{a}^{\frac{a+b}{2}} (t-a)^{\alpha-1}\, dt = \frac{1}{\alpha\Gamma(\alpha)} \left(\frac{b-a}{2}\right)^{\alpha} = \frac{1}{\Gamma(\alpha+1)\, 2^{\alpha}} (b-a)^{\alpha},$$

then by (2.5) we recapture (1.3). This is a different and perhaps a simpler proof than the one from [19].

3. Main Results

The following result holds:

Theorem 2. *Let f be a convex function on the interval $[a,b]$. If $\alpha>0$, then we have*

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{a+}^{\alpha}f(x)}{(x-a)^{\alpha}}+\frac{J_{b-}^{\alpha}f(x)}{(b-x)^{\alpha}}\right]\geq\int_{0}^{1}(1-s)^{\alpha-1}f\left(sx+(1-s)\frac{a+b}{2}\right)ds \tag{3.1}$$
$$\geq\frac{1}{\alpha}f\left(\frac{\alpha}{\alpha+1}\left(\frac{x}{\alpha}+\frac{a+b}{2}\right)\right)$$

and

$$\frac{1}{\alpha+1}\left[\frac{1}{\alpha}f(x)+\frac{f(a)+f(b)}{2}\right]\geq\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{a+}^{\alpha}f(x)}{(x-a)^{\alpha}}+\frac{J_{b-}^{\alpha}f(x)}{(b-x)^{\alpha}}\right] \tag{3.2}$$

for any $x\in(a,b)$.

As a particular case of interest we have:

Corollary 1. *With the assumptions in Theorem 2, we have*

$$\frac{f(a)+f(b)}{2}\geq\frac{\alpha}{\alpha+1}\left[\frac{1}{\alpha}f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \tag{3.3}$$
$$\geq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]$$
$$\geq f\left(\frac{a+b}{2}\right)$$

for any $\alpha>0$.

We also have the dual results:

Theorem 3. *Let f be a convex function on the interval $[a,b]$. If $\alpha>0$, then we have*

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{x-}^{\alpha}f(a)}{(x-a)^{\alpha}}+\frac{J_{x+}^{\alpha}f(b)}{(b-x)^{\alpha}}\right]\geq\int_{0}^{1}s^{\alpha-1}f\left(sx+(1-s)\frac{a+b}{2}\right)ds \tag{3.4}$$
$$\geq\frac{1}{\alpha}f\left(\frac{\alpha}{\alpha+1}\left(x+\frac{1}{\alpha}\frac{a+b}{2}\right)\right)$$

and

$$\frac{1}{\alpha+1}\left[f(x)+\frac{1}{\alpha}\frac{f(a)+f(b)}{2}\right]\geq\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{x-}^{\alpha}f(a)}{(x-a)^{\alpha}}+\frac{J_{x+}^{\alpha}f(b)}{(b-x)^{\alpha}}\right] \tag{3.5}$$

for any $x\in(a,b)$.

As a particular case of interest we also have:

Corollary 2. *With the assumptions in Theorem 2, we have*

$$\frac{f(a)+f(b)}{2}\geq\frac{\alpha}{\alpha+1}\left[f\left(\frac{a+b}{2}\right)+\frac{1}{\alpha}\frac{f(a)+f(b)}{2}\right] \tag{3.6}$$
$$\geq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}-}^{\alpha}f(a)+J_{\frac{a+b}{2}+}^{\alpha}f(b)\right]\geq f\left(\frac{a+b}{2}\right),$$

for any $\alpha>0$.

The first inequality in (3.6) is improving the second inequality in (1.3).

From a different perspective we also have:

Theorem 4. *Let f be a convex function on the interval $[a,b]$. If $\alpha > 0$, then we have*

$$\frac{1}{2}\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(a+b-x)\right]\geq f\left(\frac{a+b}{2}\right) \tag{3.7}$$

and

$$\begin{aligned}\frac{f(a)+f(b)}{2} &\geq \frac{\alpha}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{f(x)+f(a+b-x)}{2\alpha}\right] \\ &\geq \frac{1}{2}\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(a+b-x)\right]\end{aligned} \tag{3.8}$$

for $a < x \leq b$.

We observe that if we take $x = b$ in Theorem 4, then we get the inequality (1.2). If we take $x = \frac{a+b}{2}$ in Theorem 4, then we also get the inequality (3.3). We also observe that, by swapping x with $a+b-x$ in (3.7) and (3.8), we get

$$\frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}\left[J_{a+}^{\alpha}f(a+b-x)+J_{b-}^{\alpha}f(x)\right]\geq f\left(\frac{a+b}{2}\right) \tag{3.9}$$

and

$$\begin{aligned}\frac{f(a)+f(b)}{2} &\geq \frac{\alpha}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{f(x)+f(a+b-x)}{2\alpha}\right] \\ &\geq \frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}\left[J_{a+}^{\alpha}f(a+b-x)+J_{b-}^{\alpha}f(x)\right]\end{aligned} \tag{3.10}$$

for $a \leq x < b$ and $\alpha > 0$.

4. Proofs

If we use the change of variable $t = (1-s)a + sx$ for $s \in [0,1]$ and $a < x \leq b$ then we have $dt = (x-a)\,ds$, $x - t = (1-s)(x-a)$ and

$$\begin{aligned}J_{a+}^{\alpha}f(x) &= \frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)\,dt \\ &= \frac{1}{\Gamma(\alpha)}(x-a)^{\alpha}\int_{0}^{1}(1-s)^{\alpha-1}f((1-s)a+sx)\,ds\end{aligned} \tag{4.1}$$

for $a < x \leq b$.

If we use the change of variable $t = (1-u)x + ub$ for $u \in [0,1]$ and $a \leq x < b$, then we have $dt = (b-x)\,ds$, $t - x = u(b-x)$ and

$$\begin{aligned}J_{b-}^{\alpha}f(x) &= \frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)\,dt \\ &= \frac{1}{\Gamma(\alpha)}(b-x)^{\alpha}\int_{0}^{1}u^{\alpha-1}f((1-u)x+ub)\,du\end{aligned} \tag{4.2}$$

for $a \leq x < b$.

If we make the change of variable $s = 1-u$ we also have

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}(b-x)^{\alpha}\int_{0}^{1}(1-s)^{\alpha-1}f(sx+(1-s)b)\,ds \tag{4.3}$$

for $a \leq x < b$.

From (4.1) and (4.3) we get

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{a+}^{\alpha}f(x)}{(x-a)^{\alpha}}+\frac{J_{b-}^{\alpha}f(x)}{(b-x)^{\alpha}}\right] = \int_0^1 (1-s)^{\alpha-1}\left[\frac{f((1-s)a+sx)+f(sx+(1-s)b)}{2}\right]ds \tag{4.4}$$

for any $x\in(a,b)$.

By using the convexity of f we have

$$\frac{f((1-s)a+sx)+f(sx+(1-s)b)}{2}\geq f\left(sx+(1-s)\frac{a+b}{2}\right) \tag{4.5}$$

and

$$sf(x)+(1-s)\frac{f(a)+f(b)}{2}\geq\frac{f((1-s)a+sx)+f(sx+(1-s)b)}{2} \tag{4.6}$$

for any $s\in[0,1]$ and $x\in(a,b)$.

If we multiply (4.5) by $(1-s)^{\alpha-1}$ and integrate over $s\in[0,1]$ we get by (4.4) that

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{a+}^{\alpha}f(x)}{(x-a)^{\alpha}}+\frac{J_{b-}^{\alpha}f(x)}{(b-x)^{\alpha}}\right]\geq\int_0^1(1-s)^{\alpha-1}f\left(sx+(1-s)\frac{a+b}{2}\right)ds \tag{4.7}$$

for any $x\in(a,b)$. This proves the first inequality in (3.1).

Using Jensen's weighted integral inequality for the convex function f and the nonnegative weight $w(s)=(1-s)^{\alpha-1}$, $s\in[0,1]$ we have

$$\frac{\int_0^1(1-s)^{\alpha-1}f\left(sx+(1-s)\frac{a+b}{2}\right)ds}{\int_0^1(1-s)^{\alpha-1}ds} \geq f\left(\frac{\int_0^1(1-s)^{\alpha-1}\left(sx+(1-s)\frac{a+b}{2}\right)ds}{\int_0^1(1-s)^{\alpha-1}ds}\right) \tag{4.8}$$

for any $x\in(a,b)$.

Since $\int_0^1(1-s)^{\alpha-1}ds=\frac{1}{\alpha}$,

$$\int_0^1(1-s)^{\alpha-1}sds=\int_0^1 s^{\alpha-1}(1-s)ds=\frac{1}{\alpha(\alpha+1)}$$

and

$$\int_0^1(1-s)^{\alpha-1}\left(sx+(1-s)\frac{a+b}{2}\right)ds=\frac{1}{\alpha+1}\left(\frac{x}{\alpha}+\frac{a+b}{2}\right).$$

Then by (4.8) we get

$$\int_0^1(1-s)^{\alpha-1}f\left(sx+(1-s)\frac{a+b}{2}\right)ds\geq\frac{1}{\alpha}f\left(\frac{\alpha}{\alpha+1}\left(\frac{x}{\alpha}+\frac{a+b}{2}\right)\right) \tag{4.9}$$

or any $x\in(a,b)$. This proves the second inequality in (3.1).

If we multiply (4.6) by $(1-s)^{\alpha-1}$ and integrate over $s\in[0,1]$ we get by (4.4) that

$$\int_0^1(1-s)^{\alpha-1}\left[sf(x)+(1-s)\frac{f(a)+f(b)}{2}\right]ds \geq\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{a+}^{\alpha}f(x)}{(x-a)^{\alpha}}+\frac{J_{b-}^{\alpha}f(x)}{(b-x)^{\alpha}}\right],$$

which is equivalent to the desired result (3.2).

The first inequality in (3.3) is equivalent to

$$\frac{f(a)+f(b)}{2} \geq f\left(\frac{a+b}{2}\right).$$

Now, if we take $x=\frac{a+b}{2}$ in (3.1) and (3.2) we get

$$\begin{aligned}
&2^{\alpha-1}\Gamma(\alpha)\left[\frac{J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right)}{(b-a)^{\alpha}}+\frac{J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)}{(b-a)^{\alpha}}\right] \\
&\geq \int_0^1 (1-s)^{\alpha-1} f\left(s\frac{a+b}{2}+(1-s)\frac{a+b}{2}\right) ds=\frac{1}{\alpha} f\left(\frac{a+b}{2}\right)
\end{aligned}$$

and

$$\frac{1}{\alpha+1}\left[\frac{1}{\alpha} f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \geq 2^{\alpha-1}\Gamma(\alpha)\left[\frac{J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right)}{(b-a)^{\alpha}}+\frac{J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)}{(b-a)^{\alpha}}\right]$$

for any $\alpha>0$. If we multiply these inequalities by α and take into account that $\alpha\Gamma(\alpha)=\Gamma(\alpha+1)$, we get the desired result (3.3).

Using the definition of fractional integrals we have

$$J_{x-}^{\alpha} f(a)=\frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1} f(t)\, dt$$

for $a<x\leq b$ and

$$J_{x+}^{\alpha} f(b)=\frac{1}{\Gamma(\alpha)}\int_x^b (b-t)^{\alpha-1} f(t)\, dt$$

for $a\leq x<b$.

Performing in the first integral the change of variable $t=(1-s)a+sx$, $s\in[0,1]$ we have $dt=(x-a)\,ds$, $t-a=s(x-a)$ and

$$J_{x-}^{\alpha} f(a)=\frac{1}{\Gamma(\alpha)}(x-a)^{\alpha}\int_0^1 s^{\alpha-1} f((1-s)a+sx)\, ds \tag{4.10}$$

for $a<x\leq b$.

By the change of variable $t=(1-u)x+ub$, $u\in[0,1]$ we have $dt=(b-x)\,du$, $b-t=b-(1-u)x-ub=(1-u)(b-x)$ and

$$J_{x+}^{\alpha} f(b)=\frac{1}{\Gamma(\alpha)}(b-x)^{\alpha}\int_0^1 (1-u)^{\alpha-1} f((1-u)x+ub)\, dt.$$

Moreover, by changing the variable $u=1-s$, we also have

$$J_{x+}^{\alpha} f(b)=\frac{1}{\Gamma(\alpha)}(b-x)^{\alpha}\int_0^1 s^{\alpha-1} f(sx+(1-s)b)\, dt, \tag{4.11}$$

for $a\leq x<b$.

If we multiply the inequality (4.5) by $s^{\alpha-1}$ and integrate over s on $[0,1]$ we get

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{x-}^{\alpha} f(a)}{(x-a)^{\alpha}}+\frac{J_{x+}^{\alpha} f(b)}{(b-x)^{\alpha}}\right] \geq \int_0^1 s^{\alpha-1} f\left(sx+(1-s)\frac{a+b}{2}\right) ds \tag{4.12}$$

for $a<x<b$.

Using Jensen's weighted integral inequality for the convex function f and the nonnegative weight $w(s)=s^{\alpha-1}$, $s\in[0,1]$ we have

$$\begin{aligned}\frac{\int_0^1 s^{\alpha-1}f\left(sx+(1-s)\frac{a+b}{2}\right)ds}{\int_0^1 s^{\alpha-1}ds} &\geq f\left(\frac{\int_0^1 s^{\alpha-1}\left(sx+(1-s)\frac{a+b}{2}\right)ds}{\int_0^1 s^{\alpha-1}ds}\right)\\ &= f\left(\frac{\frac{1}{\alpha+1}x+\frac{1}{\alpha(\alpha+1)}\frac{a+b}{2}}{\frac{1}{\alpha}}\right)\\ &= f\left(\frac{\alpha}{\alpha+1}\left(x+\frac{1}{\alpha}\frac{a+b}{2}\right)\right),\end{aligned}$$

namely

$$\int_0^1 s^{\alpha-1}f\left(sx+(1-s)\frac{a+b}{2}\right)ds\geq\frac{1}{\alpha}f\left(\frac{\alpha}{\alpha+1}\left(x+\frac{1}{\alpha}\frac{a+b}{2}\right)\right)$$

for $a<x<b$. These prove (3.4).

If we multiply the inequality (4.6) by $s^{\alpha-1}$ and integrate over $s\in[0,1]$, then we get

$$\int_0^1 s^{\alpha-1}\left[sf(x)+(1-s)\frac{f(a)+f(b)}{2}\right]ds\geq\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{x-}^{\alpha}f(a)}{(x-a)^{\alpha}}+\frac{J_{x+}^{\alpha}f(b)}{(b-x)^{\alpha}}\right],$$

which is equivalent to (3.5).

The proof of inequalities in (3.6) follow by Theorem 3 for $x=\frac{a+b}{2}$.

Now, observe that by the representations (4.1) and (4.2) we have for $a<x\leq b$ that

$$\begin{aligned}&\frac{1}{2}\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(a+b-x)\right]\\ &=\frac{1}{2\Gamma(\alpha)}(x-a)^{\alpha}\int_0^1(1-s)^{\alpha-1}f((1-s)a+sx)\,ds\\ &+\frac{1}{2\Gamma(\alpha)}(x-a)^{\alpha}\int_0^1(1-s)^{\alpha-1}f(s(a+b-x)+(1-s)b)\,ds\\ &=\frac{1}{\Gamma(\alpha)}(x-a)^{\alpha}\\ &\times\int_0^1(1-s)^{\alpha-1}\left[\frac{f((1-s)a+sx)+f(s(a+b-x)+(1-s)b)}{2}\right]ds,\end{aligned}\tag{4.13}$$

were $\alpha>0$.

By the convexity of f we have

$$\begin{aligned}&\frac{f((1-s)a+sx)+f(s(a+b-x)+(1-s)b)}{2}\\ &\geq f\left[\frac{(1-s)a+sx+s(a+b-x)+(1-s)b}{2}\right]=f\left(\frac{a+b}{2}\right)\end{aligned}$$

for any $a<x\leq b$ and $s\in[0,1]$.

Then by the representation (4.13) we get

$$\begin{aligned}&\frac{1}{2}\left[J_{a+}^{\alpha}f\left(x\right)+J_{b-}^{\alpha}f\left(a+b-x\right)\right]\\&\geq\frac{1}{\Gamma\left(\alpha\right)}\left(x-a\right)^{\alpha}f\left(\frac{a+b}{2}\right)\int_{0}^{1}\left(1-s\right)^{\alpha-1}ds\\&=\frac{1}{\alpha\Gamma\left(\alpha\right)}\left(x-a\right)^{\alpha}f\left(\frac{a+b}{2}\right)=\frac{1}{\Gamma\left(\alpha+1\right)}\left(x-a\right)^{\alpha}f\left(\frac{a+b}{2}\right),\end{aligned}$$

for any $a<x\leq b$, which proves (3.7).

By the convexity of f we also have

$$\begin{aligned}&\frac{f\left(\left(1-s\right)a+sx\right)+f\left(s\left(a+b-x\right)+\left(1-s\right)b\right)}{2}\\&\leq\frac{1}{2}\left[\left(1-s\right)f\left(a\right)+sf\left(x\right)+sf\left(a+b-x\right)+\left(1-s\right)f\left(b\right)\right]\\&=\left(1-s\right)\frac{f\left(a\right)+f\left(b\right)}{2}+s\left[\frac{f\left(x\right)+f\left(a+b-x\right)}{2}\right]\end{aligned}$$

for any $a<x\leq b$ and $s\in\left[0,1\right]$.

Then by the representation (4.13) we get

$$\begin{aligned}&\frac{1}{2}\left[J_{a+}^{\alpha}f\left(x\right)+J_{b-}^{\alpha}f\left(a+b-x\right)\right]\\&\leq\frac{1}{\Gamma\left(\alpha\right)}\left(x-a\right)^{\alpha}\\&\times\int_{0}^{1}\left(1-s\right)^{\alpha-1}\left[\left(1-s\right)\frac{f\left(a\right)+f\left(b\right)}{2}+s\left[\frac{f\left(x\right)+f\left(a+b-x\right)}{2}\right]\right]\\&=\frac{1}{\Gamma\left(\alpha\right)}\left(x-a\right)^{\alpha}\\&\times\left[\frac{f\left(a\right)+f\left(b\right)}{2}\int_{0}^{1}\left(1-s\right)^{\alpha}ds+\left[\frac{f\left(x\right)+f\left(a+b-x\right)}{2}\right]\int_{0}^{1}\left(1-s\right)^{\alpha-1}sds\right]\\&=\frac{1}{\Gamma\left(\alpha\right)}\left(x-a\right)^{\alpha}\left[\frac{f\left(a\right)+f\left(b\right)}{2\left(\alpha+1\right)}+\frac{f\left(x\right)+f\left(a+b-x\right)}{2\alpha\left(\alpha+1\right)}\right]\\&=\frac{1}{\Gamma\left(\alpha\right)\left(\alpha+1\right)}\left(x-a\right)^{\alpha}\left[\frac{f\left(a\right)+f\left(b\right)}{2}+\frac{f\left(x\right)+f\left(a+b-x\right)}{2\alpha}\right]\\&=\frac{\alpha}{\Gamma\left(\alpha+1\right)\left(\alpha+1\right)}\left(x-a\right)^{\alpha}\left[\frac{f\left(a\right)+f\left(b\right)}{2}+\frac{f\left(x\right)+f\left(a+b-x\right)}{2\alpha}\right]\end{aligned}$$

which proves the second inequality in (3.8).

By the convexity of f we have

$$f\left(x\right)=f\left(\frac{\left(b-x\right)a+\left(x-a\right)b}{b-a}\right)\leq\frac{\left(b-x\right)f\left(a\right)+\left(x-a\right)f\left(b\right)}{b-a}$$

and

$$f\left(a+b-x\right)\leq\frac{\left(x-a\right)f\left(a\right)+\left(b-x\right)f\left(b\right)}{b-a}$$

for $a\leq x\leq b$, which by addition give

$$f\left(x\right)+f\left(a+b-x\right)\leq f\left(a\right)+f\left(b\right)$$

for $a\leq x\leq b$.

Therefore

$$\begin{aligned}\frac{f(a)+f(b)}{2}+\frac{f(x)+f(a+b-x)}{2\alpha} &\leq \frac{f(a)+f(b)}{2}+\frac{f(a)+f(b)}{2\alpha}\\ &=\frac{f(a)+f(b)}{2}\left(\frac{\alpha+1}{\alpha}\right),\end{aligned}$$

which proves the first part of (3.8).

5. Further HH-Type Inequalities

In [4] and [5] we introduced the following mapping associated to a Lebesgue integrable function $f:[a,b]\to\mathbb{R}$

$$H_f(t):=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx,\ t\in[0,1]. \tag{5.1}$$

It has been shown in the above papers that, if f is convex on $[a,b]$, then H_f is convex on $[0,1]$, H_f increases monotonically on $[0,1]$, we have the bounds

$$\begin{aligned}\inf_{t\in[0,1]} H_f(t)&=H_f(0)=f\left(\frac{a+b}{2}\right),\\ \sup_{t\in[0,1]} H_f(t)&=H_f(1)=\frac{1}{b-a}\int_a^b f(x)\,dx\end{aligned}$$

and the inequalities

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\leq\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx\\ &\leq\int_0^1 H(t)\,dt\\ &\leq\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)\,dx\right].\end{aligned} \tag{5.2}$$

By (4.1) we have for an integrable function $f:[a,b]\to\mathbb{R}$ that

$$\frac{J_{a+}^{\alpha}f(x)}{(x-a)^{\alpha}}=\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f((1-s)a+sx)\,ds \tag{5.3}$$

for $a<x\leq b$. The function

$$[a,b]\ni x\mapsto\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f((1-s)a+sx)\,ds$$

is integrable and by Fubini's theorem we have

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\frac{1}{b-a}\int_a^b\left(\int_0^1(1-s)^{\alpha-1}f((1-s)a+sx)\,ds\right)dx\\ &=\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left(\frac{1}{b-a}\int_a^b f((1-s)a+sx)\,dx\right)ds.\end{aligned}$$

By (5.3) we then have

$$\begin{aligned}(5.4)\qquad &\frac{1}{b-a}\int_a^b \frac{J_{a+}^{\alpha}f(x)}{(x-a)^{\alpha}}dx\\ &=\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left(\frac{1}{b-a}\int_a^b f((1-s)a+sx)\,dx\right)ds.\end{aligned}$$

By (4.3) we also have

$$\begin{aligned}(5.5)\qquad &\frac{1}{b-a}\int_a^b \frac{J_{b-}^{\alpha}f(x)}{(b-x)^{\alpha}}dx\\ &=\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left(\frac{1}{b-a}\int_a^b f(sx+(1-s)b)\,dx\right)ds.\end{aligned}$$

Proposition 1. *Let f be a convex function on the interval $[a,b]$. If $\alpha>0$, then we have*

$$\begin{aligned}(5.6)\qquad &\frac{1}{2}\Gamma(\alpha+1)\left[\frac{1}{b-a}\int_a^b \frac{J_{a+}^{\alpha}f(x)}{(x-a)^{\alpha}}dx+\frac{1}{b-a}\int_a^b \frac{J_{b-}^{\alpha}f(x)}{(b-x)^{\alpha}}dx\right]\\ &\geq \alpha\int_0^1 (1-s)^{\alpha-1}H_f(s)\,ds\\ &\geq \frac{1}{b-a}\int_a^b f\left(\frac{\alpha}{\alpha+1}\left(\frac{x}{\alpha}+\frac{a+b}{2}\right)\right)dx\geq f\left(\frac{a+b}{2}\right)\end{aligned}$$

and

$$\begin{aligned}(5.7)\qquad &\frac{f(a)+f(b)}{2}\\ &\geq \frac{\alpha}{\alpha+1}\left[\frac{1}{\alpha}\frac{1}{b-a}\int_a^b f(x)\,dx+\frac{f(a)+f(b)}{2}\right]\\ &\geq \frac{1}{2}\Gamma(\alpha+1)\left[\frac{1}{b-a}\int_a^b \frac{J_{a+}^{\alpha}f(x)}{(x-a)^{\alpha}}dx+\frac{1}{b-a}\int_a^b \frac{J_{b-}^{\alpha}f(x)}{(b-x)^{\alpha}}dx\right]\end{aligned}$$

for any $x\in(a,b)$.

The proofs of the first and second inequality in (5.6) follow by (3.4). The last part follows by Jensen's inequality.

The first inequality in (5.7) is obvious by the fact that $\frac{f(a)+f(b)}{2}\geq\frac{1}{b-a}\int_a^b f(x)\,dx$, while the second inequality follows by (3.2).

By using Theorem 3 we have:

Proposition 2. *Let f be a convex function on the interval $[a,b]$. If $\alpha>0$, then we have*

$$\begin{aligned}(5.8)\qquad &\frac{1}{2}\Gamma(\alpha+1)\left[\frac{1}{b-a}\int_a^b \frac{J_{x-}^{\alpha}f(a)}{(x-a)^{\alpha}}dx+\frac{1}{b-a}\int_a^b \frac{J_{x+}^{\alpha}f(b)}{(b-x)^{\alpha}}dx\right]\\ &\geq \alpha\int_0^1 s^{\alpha-1}H_f(s)\,ds\\ &\geq \frac{1}{b-a}\int_a^b f\left(\frac{\alpha}{\alpha+1}\left(x+\frac{1}{\alpha}\frac{a+b}{2}\right)\right)dx\geq f\left(\frac{a+b}{2}\right)\end{aligned}$$

and

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} \\
&\geq \frac{\alpha}{\alpha+1}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+\frac{1}{\alpha}\frac{f(a)+f(b)}{2}\right] \\
&\geq \frac{1}{2}\Gamma(\alpha+1)\left[\frac{1}{b-a}\int_a^b \frac{J_{x-}^{\alpha}f(a)}{(x-a)^{\alpha}}dx+\frac{1}{b-a}\int_a^b \frac{J_{x+}^{\alpha}f(b)}{(b-x)^{\alpha}}dx\right].
\end{aligned} \tag{5.9}$$

From (3.7) and (3.8) we have

$$\frac{1}{2}\Gamma(\alpha+1)\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(a+b-x)\right]\geq f\left(\frac{a+b}{2}\right)(x-a)^{\alpha} \tag{5.10}$$

and

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}(x-a)^{\alpha} \\
&\geq \frac{\alpha}{\alpha+1}\left[\frac{f(a)+f(b)}{2}(x-a)^{\alpha}+\frac{f(x)+f(a+b-x)}{2\alpha}(x-a)^{\alpha}\right] \\
&\geq \frac{1}{2}\Gamma(\alpha+1)\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(a+b-x)\right]
\end{aligned} \tag{5.11}$$

for any $x\in(a,b)$.

By taking the integral mean in (5.10) we have

$$\begin{aligned}
&\frac{1}{2}\Gamma(\alpha+1)\left[\frac{1}{b-a}\int_a^b J_{a+}^{\alpha}f(x)\,dx+\frac{1}{b-a}\int_a^b J_{b-}^{\alpha}f(a+b-x)\,dx\right] \\
&\geq f\left(\frac{a+b}{2}\right)\frac{(b-a)^{\alpha}}{\alpha+1}
\end{aligned} \tag{5.12}$$

and since

$$\frac{1}{b-a}\int_a^b J_{b-}^{\alpha}f(a+b-x)\,dx=\frac{1}{b-a}\int_a^b J_{b-}^{\alpha}f(x)\,dx,$$

then by (5.12) we have

$$\frac{1}{2}\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha}}\left[\frac{1}{b-a}\int_a^b J_{a+}^{\alpha}f(x)\,dx+\frac{1}{b-a}\int_a^b J_{b-}^{\alpha}f(x)\,dx\right]\geq f\left(\frac{a+b}{2}\right). \tag{5.13}$$

By taking the integral mean in (5.11) we have

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}\frac{(b-a)^{\alpha}}{\alpha+1} \\
&\geq \frac{\alpha}{\alpha+1}\left[\frac{f(a)+f(b)}{2}\frac{(b-a)^{\alpha}}{\alpha+1}\right. \\
&\left.+\frac{1}{\alpha(b-a)}\int_a^b \frac{f(x)+f(a+b-x)}{2}(x-a)^{\alpha}\,dx\right] \\
&\geq \frac{1}{2}\Gamma(\alpha+1)\left[\frac{1}{b-a}\int_a^b J_{a+}^{\alpha}f(x)\,dx+\frac{1}{b-a}\int_a^b J_{b-}^{\alpha}f(x)\,dx\right].
\end{aligned} \tag{5.14}$$

Since $\int_a^b f(a+b-x)(x-a)^\alpha dx = \int_a^b f(x)(b-x)^\alpha dx$ then

$$\int_a^b \frac{f(x)+f(a+b-x)}{2}(x-a)^\alpha dx = \int_a^b \frac{(x-a)^\alpha + (b-x)^\alpha}{2} f(x)\,dx$$

and by (5.14) we get

$$\begin{aligned} &\frac{f(a)+f(b)}{2} \\ &\geq \frac{\alpha}{(b-a)^\alpha}\left[\frac{f(a)+f(b)}{2}\frac{(b-a)^\alpha}{\alpha+1}\right. \\ &\left.+\frac{1}{\alpha(b-a)}\int_a^b \frac{(x-a)^\alpha+(b-x)^\alpha}{2} f(x)\,dx\right] \\ &\geq \frac{1}{2}\frac{\Gamma(\alpha+2)}{(b-a)^\alpha}\left[\frac{1}{b-a}\int_a^b J_{a+}^\alpha f(x)\,dx + \frac{1}{b-a}\int_a^b J_{b-}^\alpha f(x)\,dx\right]. \end{aligned} \tag{5.15}$$

We finally have:

Proposition 3. *Let f be a convex function on the interval $[a,b]$. If $\alpha > 0$, then we have*

$$\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+1}}\int_a^b \frac{J_{a+}^\alpha f(x) + J_{b-}^\alpha f(x)}{2}dx \geq f\left(\frac{a+b}{2}\right) \tag{5.16}$$

and

$$\begin{aligned} \frac{f(a)+f(b)}{2} &\geq \frac{\alpha}{(b-a)^\alpha}\left[\frac{f(a)+f(b)}{2}\frac{(b-a)^\alpha}{\alpha+1}\right. \\ &\left.+\frac{1}{\alpha(b-a)}\int_a^b \frac{(x-a)^\alpha+(b-x)^\alpha}{2} f(x)\,dx\right] \\ &\geq \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+1}}\int_a^b \frac{J_{a+}^\alpha f(x)+J_{b-}^\alpha f(x)}{2}dx. \end{aligned} \tag{5.17}$$

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