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SOME HERMITE-HADAMARD TYPE INEQUALITIES

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Abstract. The aim of this paper is to present several Hermite-Hadamard type inequalities for functions whose second derivative in absolute value are convex or satisfies also other kind of convexity.

1. Introduction

The inequality of Hermite-Hadamard type has been considered very useful in mathematical analysis being extended and generalized in many directions by many authors, see [17, 6, 5, 7, 1, 10, 14, 18, 8] and the references therein.

We begin by recalling below the classical definition for the convex functions.

Definition 1. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. The function f is said to be concave on I if the inequality (1) takes place in reversed direction.

It is necessary to recall below also other kind of convexity. For other type of convexity see also [15, 13].

Definition 2. *A function $f : [a, b] \to \mathbb{R}$ is said to be quasi-convex onl $[a, b]$ if*

$$f(tx + (1-t)y) \leq \sup\{f(x), f(y)\}$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Definition 3. *A function $f : I \to \mathbb{R}$ is said to be P-convex on $[a, b]$ if it is nonnegative and for all $x, y \in I$ and $\lambda \in [9, 1]$*

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

The classical Hermite-Hadamard's inequality for convex functions is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

Moreover, if the function f is concave then the inequality (2) hold in reversed direction.

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The main aim of this paper is to establish a lemma which to order to help us to prove several new Hadamard-like inequalities for different type of convex functions using in these proves some Holder type inequalities.

2. **Main results**

The following result is a variant of the Lemma 1 from [17] and a generalization of Lemma 1 from [4].

Lemma 1. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I in $\mathbb{R}$, with $a, b \in I,\ 0 < a < b$. If $f^{''} \in L_1[a,b]$ and M is a positive constant so that $\frac{a}{b} < M$ for $l, k, n \in \mathbb{N}$ with $n \geq 2$, $l < k < n$ and $\frac{l}{k} > M$ or $(\frac{a}{b} < \frac{l}{k})$ then the following identity takes place:*

$$I(f,a,b,n,k,l) = (lb-ka)^3(I_1+I_2) + [\frac{n(b-a)}{2} - (lb-ka)]^3(I_3+I_4) =$$
$$= 2n^3\int_a^b f(x)dx + \frac{n^2(b-a)}{4}[4(lb-ka)-n(b-a)][f^{'}(\frac{(n-k)a+lb}{n}) -$$
$$-f^{'}(\frac{ka+(n-l)b}{n})] - n^3(b-a)[f(\frac{(n-k)a+lb}{n}) + f(\frac{ka+(n-l)b}{n})],$$

where

$$I_1 = \int_0^1 t^2 f^{''}\left(t\frac{(n-k)a+lb}{n} + (1-t)a\right)dt,$$
$$I_2 = \int_0^1 (t-1)^2 f^{''}\left(tb + (1-t)\frac{ka+(n-l)b}{n}\right)dt,$$
$$I_3 = \int_0^1 t^2 f^{''}\left(t\frac{ka+(n-l)b}{n} + (1-t)\frac{a+b}{2}\right)dt,$$
$$I_4 = \int_0^1 t^2 f^{''}\left(t\frac{a+b}{2} + (1-t)\frac{(n-k)a+lb}{n}\right)dt.$$

Proof. Using integration by parts for I_1, I_2, I_3, I_4 we will obtain first:

$$I_1 = \frac{n}{lb-ka}f^{'}\left(\frac{(n-k)a+lb}{n}\right) - \frac{2n}{lb-ka}\int_0^1 tf^{'}\left(t\frac{(n-k)a+lb}{n} + (1-t)a\right)dt,$$

$$I_2 = -\frac{n}{lb-ka}f^{'}\left(\frac{ka+(n-l)b}{n}\right) - \frac{2n}{lb-ka}\int_0^1 (t-1)f^{'}\left(tb + (1-t)\frac{ka+(n-l)b}{n}\right)dt,$$

$$I_3 = \frac{2n}{b(n-2l)-a(n-2k)}f^{'}\left(\frac{ka+(n-l)b}{n}\right) - \frac{4n}{b(n-2l)-a(n-2k)}\cdot$$
$$\cdot\int_0^1 tf^{'}\left(t\frac{ka+(n-l)b}{n} + (1-t)\frac{a+b}{2}\right)dt,$$

$$I_4 = -\frac{2n}{b(n-2l)-a(n-2k)}f^{'}\left(\frac{(n-k)a+lb}{n}\right) - \frac{4n}{b(n-2l)-a(n-2k))}\cdot$$

$$\cdot\int_0^1 (t-1)f'\left(t\frac{a+b}{2}+(1-t)\frac{(n-k)a+lb}{n}\right)dt,$$

and then

$$I_1=\frac{n}{lb-ka}f'\left(\frac{(n-k)a+lb}{n}\right)-\frac{2n^2}{(lb-ka)^2}[f\left(\frac{(n-k)a+lb}{n}\right)-$$
$$-\int_0^1 f\left(t\frac{(n-k)a+lb}{n}+(1-t)a\right)dt],$$

$$I_2=-\frac{n}{lb-ka}f'\left(\frac{ka+(n-l)b}{n}\right)-\frac{2n^2}{(lb-ka)^2}[f\left(\frac{ka+(n-l)b}{n}\right)-$$
$$-\int_0^1 f\left(tb+(1-t)\frac{ka+(n-l)b}{n}\right)dt],$$

$$I_3=\frac{2n}{b(n-2l)-a(n-2k)}f'\left(\frac{ka+(n-l)b}{n}\right)-\frac{8n^2}{(b(n-2l)-a(n-2k))^2}[f\left(\frac{ka+(n-l)b}{n}\right)-$$
$$-\int_0^1 f\left(t\frac{ka+(n-l)b}{n}+(1-t)\frac{a+b}{2}\right)dt],$$

$$I_4=-\frac{2n}{b(n-2l)-a(n-2k)}f'\left(\frac{(n-k)a+lb}{n}\right)-\frac{8n^2}{(b(n-2l)-a(n-2k))^2}[f\left(\frac{(n-k)a+lb}{n}\right)-$$
$$-\int_0^1 f\left(t\frac{a+b}{2}+(1-t)\frac{(n-k)a+lb}{n}\right)dt].$$

By making use of the substitutions, $x=t\frac{(n-k)a+lb}{n}+(1-t)a$, $x=tb+(1-t)\frac{ka+(n-l)b}{n}$, $x=t\frac{ka+(n-l)b}{n}+(1-t)\frac{a+b}{2}$ and $x=t\frac{a+b}{2}+(1-t)\frac{(n-k)a+lb}{n}$ in the integral from the end of expressions I_1, I_2, I_3 and I_4 we get

$$I_1=\frac{n}{lb-ka}f'\left(\frac{(n-k)a+lb}{n}\right)-\frac{2n^2}{(lb-ka)^2}f\left(\frac{(n-k)a+lb}{n}\right)+\frac{2n^3}{(lb-ka)^3}\int_a^{\frac{(n-k)a+lb}{n}}f(x)dx,$$

$$I_2=-\frac{n}{lb-ka}f'\left(\frac{ka+(n-l)b}{n}\right)-\frac{2n^2}{(lb-ka)^2}f\left(\frac{ka+(n-l)b}{n}\right)+\frac{2n^3}{(lb-ka)^3}\int_{\frac{ka+(n-l)b}{n}}^{b}f(x)dx,$$

$$I_3=\frac{2n}{b(n-2l)-a(n-2k)}f'\left(\frac{ka+(n-l)b}{n}\right)-\frac{8n^2}{(b(n-2l)-a(n-2k))^2}f\left(\frac{ka+(n-l)b}{n}\right)+$$
$$+\frac{16n^3}{(b(n-2l)-a(n-2k))^3}\int_{\frac{a+b}{2}}^{\frac{ka+(n-l)b}{n}}f(x)dx$$

$$I_4=-\frac{2n}{b(n-2l)-a(n-2k)}f'\left(\frac{(n-k)a+lb}{n}\right)-\frac{8n^2}{(b(n-2l)-a(n-2k))^2}f\left(\frac{(n-k)a+lb}{n}\right)+$$
$$+\frac{16n^3}{(b(n-2l)-a(n-2k))^3}\int_{\frac{(n-k)a+lb}{n}}^{\frac{a+b}{2}}f(x)dx,$$

respectively.

We notice that our hypothesis show that the expressions, $\frac{(n-k)a+lb}{n}$ and $\frac{ka+(n-l)b}{n}$ are in (a,b). Now, we compute the expression, $I(f,a,b,n,k,l)=(lb-ka)^3(I_1+I_2)+[\frac{n(b-a)}{2}-(lb-ka)]^3(I_3+I_4)$ in order to obtain $\int_a^b f(x)dx$ and we have by calculus the desired identity.

∎

We will use this result in order to obtain below several Hermite-Hadamard type inequalities which extend some Hermite-Hadamard type inequalities..

Theorem 1. *Let $f:I\to\mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I in $\mathbb{R}$, with $a,b\in I$, $0<a<b$. If $f^{''}\in L[a,b]$ for $l,k,n\in\mathbb{N}$ with $n\geq 2$ and $l<k<n$. If $|f^{''}|$ is a convex function, M is a positive constant so that $\frac{a}{b}<M$ and $\frac{l}{k}>M$ or $(\frac{a}{b}<\frac{l}{k})$ then the following inequality takes place:*

$$|I(f,a,b,n,k,l)|\leq\{\frac{(lb-ka)^3}{4}+\frac{|\frac{n(b-a)}{2}-(lb-ka)|^3}{4}\}[|f^{''}(\frac{(n-k)a+lb}{n})|+$$

$$+|f^{''}(\frac{ka+(n-l)b}{n})|]+\frac{(lb-ka)^3}{12}[|f^{''}(a)|+|f^{''}(b)|]+\frac{|\frac{n(b-a)}{2}-(lb-ka)|^3}{6}|f^{''}(\frac{a+b}{2})|.$$

Proof. From Lemma 1 we have

$$|I(f,a,b,n,k,l)|=|(lb-ka)^3(I_1+I_2)+[\frac{n(b-a)}{2}-(lb-ka)]^3(I_3+I_4)|\leq$$

$$\leq(lb-ka)^3\{\int_0^1 t^2|f^{''}\left(t\frac{(n-k)a+lb}{n}+(1-t)a\right)|dt+$$

$$+\int_0^1(t-1)^2|f^{''}\left(tb+(1-t)\frac{ka+(n-l)b}{n}\right)|dt\}+$$

$$+|\frac{n(b-a)}{2}-(lb-ka)|^3\{\int_0^1 t^2|f^{''}\left(t\frac{ka+(n-l)b}{n}+(1-t)\frac{a+b}{2}\right)|dt+$$

$$+\int_0^1 t^2|f^{''}\left(t\frac{a+b}{2}+(1-t)\frac{(n-k)a+lb}{n}\right)|dt\}$$

and use the convexity of $|f^{''}|$ on $[a,b]$ we obtain,

$$|I(f,a,b,n,k,l)|\leq(lb-ka)^3\{\int_0^1 t^2[t|f^{''}(\frac{(n-k)a+lb}{n})|+(1-t)|f^{''}(a)|]dt+$$

$$+\int_0^1(t-1)^2[t|f^{''}(b)|+(1-t)|f^{''}(\frac{ka+(n-l)b}{n})|]dt\}+$$

$$+|\frac{n(b-a)}{2}-(lb-ka)|^3\{\int_0^1 t^2[t|f^{''}(\frac{ka+(n-l)b}{n})|+(1-t)|f^{''}(\frac{a+b}{2})|]dt+$$

$$+\int_0^1 t^2[t|f^{''}(\frac{a+b}{2})|+(1-t)|f^{''}(\frac{(n-k)a+lb}{n})|]dt\}.$$

By calculus we get the desired inequality.

∎

Theorem 2. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I in $\mathbb{R}$, with $a, b \in I$, $a < b$. Let $l, k, n \in \mathbb{N}$ with $n \geq 2$ and $l < k < n$. If $f'' \in L[a, b]$, and if $|f''|^q$ is a convex function for some fixed $q > 1$ and M is a positive constant so that $\frac{a}{b} < M$ and $\frac{l}{k} > M$ or $(\frac{a}{b} < \frac{l}{k})$ then the following inequality takes place:*

$$|I(f,a,b,n,k,l)| \leq \frac{(lb-ka)^3}{2^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\{C(p,l_1)[|f''(\frac{(n-k)a+lb}{n})|^q + |f''(a)|^q\}]^{\frac{1}{q}}+$$

$$+C(p,l_2)[|f''(b)|^q + |f''(\frac{ka+(n-l)b}{n})|^q]^{\frac{1}{q}}\} + \frac{|\frac{n(b-a)}{2} - (lb-ka)|^3}{2(2p+1)^{\frac{1}{p}}}\{C(p,l_3)\cdot$$

$$\cdot[|f''(\frac{ka+(n-l)b}{n})|^q + |f''(\frac{a+b}{2})|^q\}]^{\frac{1}{q}} + C(p,l_4)[|f''(\frac{a+b}{2})|^q + |f''(\frac{(n-k)a+lb}{n})|^q]^{\frac{1}{q}}\}\},$$

where $C(p, l_i), (i = 1, 2, 3, 4)$ are defined as in Theorem 1.4 from [17]*, see also* [4].

Proof. Using Lemma 1 and the well-known Holder integral inequality (1), see [17], we have

$$|I(f,a,b,n.k,l)| \leq (lb-ka)^3\{C(p,l_1)\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(t\frac{(n-k)a+lb}{n} + (1-t)a)|^q dt\right)^{\frac{1}{q}} +$$

$$+C(p,l_2)\left(\int_0^1 (t-1)^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(tb + (1-t)\frac{ka+(n-l)b}{n})|^q dt\right)^{\frac{1}{q}}\}+$$

$$+|\frac{n(b-a)}{2} - (lb-ka)|^3\{C(p,l_3)\left(\int_0^1 t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(t\frac{ka+(n-l)b}{n} + (1-t)\frac{a+b}{2})|^q dt\right)^{\frac{1}{q}} +$$

$$+C(p,l_4)\left(\int_0^1 (t-1)^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(t\frac{a+b}{2} + (1-t)\frac{(n-k)a+lb}{n})|^q dt\right)^{\frac{1}{q}}\}.$$

By definition of the convexity of $|f''|^q$ we have:

$$|I(f,a,b,n.k,l)| \leq (lb-ka)^3\{C(p,l_1)\frac{1}{(2p+1)^{\frac{1}{p}}}\left[\int_0^1 (t|f''(\frac{(n-k)a+lb}{n})|^q + (1-t)|f''(a)|^q)dt\right]^{\frac{1}{q}} +$$

$$+C(p,l_2))\frac{1}{(2p+1)^{\frac{1}{p}}}\left[\int_0^1 (t|f''(b)|^q + (1-t)|f(\frac{ka+(n-l)b}{n})|^q)dt\right]^{\frac{1}{q}}\}+$$

$$+|\frac{n(b-a)}{2} - (lb-ka)|^3\{C(p,l_3))\frac{1}{(2p+1)^{\frac{1}{p}}}\left[\int_0^1 (t|f''(\frac{ka+(n-l)b}{n})|^q + (1-t)|f(\frac{a+b}{2})|^q)dt\right]^{\frac{1}{q}} +$$

$$+C(p,l_4))\frac{1}{(2p+1)^{\frac{1}{p}}}\left[\int_0^1 (t|f''(\frac{a+b}{2})|^q + (1-t)|f(\frac{(n-k)a+lb}{n})|^q)dt\right]^{\frac{1}{q}}\}$$

and by calculus we get the inequality from Theorem 2.

∎

Theorem 3. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I in $\mathbb{R}$, with $a, b \in I,\ a < b$. Let $l, k, n \in \mathbb{N}$ with $n \geq 2$ and $l < k < n$. If $f^{''} \in L[a, b]$, and if $|f^{''}|^q$ is a convex function for some fixed $q \geq 1$ and M is a positive constant so that $\frac{a}{b} < M$ and $\frac{l}{k} > M$ or $(\frac{a}{b} < \frac{l}{k})$ then the following inequality takes place:*

$$|I(f,a,b,n,k,l)| \leq \frac{1}{3^{\frac{1}{p}}4^{\frac{1}{q}}}\{(lb-ka)^3[(|f^{''}(\frac{(n-k)a+lb}{n})|^q + \frac{1}{3}|f^{''}(a)|^q)^{\frac{1}{q}} +$$

$$+(\frac{1}{3}|f^{''}(b)|^q + |f^{''}(\frac{ka+(n-l)b}{n})|^q)^{\frac{1}{q}}] + |\frac{n(b-a)}{2} - (lb-ka)|^3 \cdot$$

$$\cdot[(|f^{''}(\frac{ka+(n-l)b}{n})|^q + \frac{1}{3}|f^{''}(\frac{a+b}{2})|^q\})^{\frac{1}{q}} + (\frac{1}{3}|f^{''}(\frac{a+b}{2})|^q + |f^{''}(\frac{(n-k)a+lb}{n})|^q)^{\frac{1}{q}}]\},$$

where $C(p, l_i), (i = 1, 2, 3, 4)$ are defined as in Theorem 1.4 from [17], *see also* [4].

Proof. We use Lemma 1 and the well-known power mean inequality and then the convexity of $|f^{''}|^q$, as in [17], obtaining:

$$|I(f,a,b,n.k,l)| \leq (lb-ka)^3 \frac{1}{3^{\frac{1}{p}}}[(\frac{1}{4}|f^{''}(\frac{(n-k)a+lb}{n})|^q + \frac{1}{12}|f^{''}(a)|^q)^{\frac{1}{q}} + (\frac{1}{12}|f^{''}(b)|^q +$$

$$+\frac{1}{4}|f^{''}(\frac{ka+(n-l)b}{n}))|^q)^{\frac{1}{q}}] + [\frac{n(b-a)}{2} - (lb-ka)]^3 \frac{1}{3^{\frac{1}{p}}}[(\frac{1}{4}|f^{''}(\frac{ka+(n-l)b}{n})|^q +$$

$$+\frac{1}{12}|f^{''}(\frac{a+b}{2})|^q)^{\frac{1}{q}} + (\frac{1}{12}|f^{''}(\frac{a+b}{2})|^q + \frac{1}{4}|f^{''}(\frac{(n-k)a+lb}{n})|^q)^{\frac{1}{q}}]$$

which leads to desired inequality.

∎

Theorem 4. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I in $\mathbb{R}$, with $a, b \in I,\ a < b$. Let $l, k, n \in \mathbb{N}$ with $n \geq 2$ and $l < k < n$. If $f^{''} \in L[a, b]$, and if $|f^{''}|^q$ is a concave function on $[a, b]$ for some fixed $q > 1$ and M is a positive constant so that $\frac{a}{b} < M$ and $\frac{l}{k} > M$ or $(\frac{a}{b} < \frac{l}{k})$ then the following inequality takes place:*

$$|I(f,a,b,n,k,l)| \leq \left(\frac{q-1}{3q-1}\right)^{\frac{1}{p}}\{(lb-ka)^3[C(p,l_1)|f^{''}(\frac{(2n-k)a+lb}{2n})| +$$

$$+C(p,l_2)|f^{''}(\frac{(2n-l)b+ka}{2n})|] + |\frac{n(b-a)}{2} - (lb-ka)|^3[C(p,l_3)|f^{''}(\frac{(n+2k)a+(3n-2l)b}{4n})| +$$

$$+C(p,l_4)|f^{''}(\frac{(3n-2k)a+(n+2l)b}{4n})|]\}.$$

Proof. We use again Lemma 1, the power mean inequality and the Holder integral inequality , as in [17] Theorem 2.4 , and we will have:

$$|I(f,a,b,n,k,l)| \leq (lb-ka)^3[C(p,l_1)(\int_0^1 t^{2p}dt)^{\frac{1}{p}}(\int_0^1 |f^{''}(\frac{(n-k)a+lb}{n} + (1-t)a)|^q dt)^{\frac{1}{q}} +$$

$$+C(p,l_2)(\int_0^1 (t-1)^{2p}dt)^{\frac{1}{p}}(\int_0^1 |f^{''}(tb+(1-t)\frac{ka+(n-l)b}{n})|^q dt)^{\frac{1}{q}} +$$

$$+|\frac{n(b-a)}{2}-(lb-ka)|^3[C(p,l_3)(\int_0^1 t^{2p}dt)^{\frac{1}{p}}(\int_0^1 |f^{''}(t\frac{ka+(n-l)b}{n}+(1-t)\frac{a+b}{2})|^q dt)^{\frac{1}{q}}+$$

$$+C(p,l_4)(\int_0^1 (t-1)^{2p}dt)^{\frac{1}{p}}(\int_0^1 |f^{''}(t\frac{a+b}{2}+(1-t)\frac{(n-k)a+lb}{n})|^q dt)^{\frac{1}{q}}].$$

We take into account that $|f^{''}|^q$ is concave on $[a,b]$ and substracting $x=\frac{(n-k)a+lb}{n}+(1-t)a$ and then using the inequality

$$\frac{1}{b-a}\int_a^b f(x)dx \leq f\left(\frac{a+b}{2}\right)$$

we get

$$\int_0^1 |f^{''}\left(t\frac{(n-k)a+lb}{n}+(1-t)a\right)|^q dt=\frac{n}{lb-ka}\int_a^{\frac{(n-k)a+lb}{n}} f(x)dx\leq$$

$$\leq |f^{''}(\frac{(2n-k)a+lb}{2n})|^q,$$

$$\int_0^1 |f^{''}\left(tb+(1-t)\frac{ka+(n-l)b}{n}\right)|^q dt\leq |f^{''}(\frac{(2n-l)b+ka}{2n})|^q,$$

$$\int_0^1 |f^{''}\left(t\frac{ka+(n-l)b}{n}+(1-t)\frac{a+b}{2}\right)|^q dt\leq |f^{''}(\frac{(n+2k)a+(3n-2l)b}{4n})|^q$$

and

$$\int_0^1 |f^{''}\left(t\frac{a+b}{2}+(1-t)\frac{(n-k)a+lb}{n}\right)|^q dt\leq |f^{''}(\frac{(3n-2k)a+(n+2l)b}{4n})|^q.$$

From these inequalities we obtain the desired inequality.

∎

Now, we write for $(\alpha,m)-$ convex functions, Theorem 2 from above.

Theorem 5. *Let $f: I\subset[0,b^*]\to\mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I so that $f^{''}\in L[a,b]$ with $a,b\in I,\ 0<a<b,\ b^*>0$. Let $l,k,n\in\mathbb{N}$ with $n\geq 2$ and $l<k<n$. If $f^{''}\in L[a,b]$, and if $|f^{''}|^q$ is $(\alpha,m)-$ convex function for $(\alpha,m)\in[0,1]\times[0,1]$ and $p>1$ and M is a positive constant so that $\frac{a}{b}<M$ and $\frac{l}{k}>M$ or ($\frac{a}{b}<\frac{l}{k}$) then the following inequality holds:*

$$|I(f,a,b,n,k,l)|\leq\frac{(lb-ka)^3}{(2p+1)^{\frac{1}{p}}(\alpha+1)^{\frac{1}{q}}}[C(p,l_1)(|f^{''}(\frac{(n-k)a+lb}{n})|^q+m\alpha|f^{''}(\frac{a}{m})|^q)^{\frac{1}{q}}+$$

$$+C(p,l_2)(m|f^{''}(\frac{b}{m})|^q+\alpha|f^{''}(\frac{ka+(n-l)b}{n})|^q)^{\frac{1}{q}}]+$$

$$+\frac{[\frac{n(b-a)}{2}-(lb-ka)]^3}{(2p+1)^{\frac{1}{p}}(\alpha+1)^{\frac{1}{q}}}[C(p,l_3)(|f^{''}(\frac{ka+(n-l)b}{n})|^q+m\alpha|f^{''}(\frac{a+b}{2m})|^q)^{\frac{1}{q}}+$$

$$+C(p,l_4)m|f^{''}(\frac{a+b}{2m})|^q+\alpha|f^{''}(\frac{(n-k)a+lb}{n})|^q)^{\frac{1}{q}})].$$

Proof. We use Lemma 1 and then Theorem 2, see [4] and [3], we get

$$|I(f,a,b,n,k,l)| \leq (lb-ka)^3[\int_0^1 t^2|f''\left(t\frac{(n-k)a+lb}{n}+(1-t)a\right)|dt+$$

$$+\int_0^1 (t-1)^2|f''\left(tb+(1-t)\frac{ka+(n-l)b}{n}\right)|dt]+[\frac{n(b-a)}{2}-(lb-ka)]^3\cdot$$

$$\cdot[\int_0^1 t^2|f''\left(t\frac{ka+(n-l)b}{n}+(1-t)\frac{a+b}{2}\right)|dt+$$

$$+\int_0^1 (t-1)^2|f''\left(t\frac{a+b}{2}+(1-t)\frac{(n-k)a+lb}{n}\right)|dt]$$

and

$$|I(f,a,b,n,k,l)| \leq \frac{(lb-ka)^3}{(2p+1)^{\frac{1}{p}}}[C(p,l_1)(\int_0^1 |f''\left(t\frac{(n-k)a+lb}{n}+(1-t)a\right)|^q dt)^{\frac{1}{q}}+$$

$$+C(p,l_2)(\int_0^1 |f''\left(tb+(1-t)\frac{ka+(n-l)b}{n}\right)|^q dt)^{\frac{1}{q}}]+[\frac{n(b-a)}{2}-(lb-ka)]^3\cdot$$

$$\cdot[C(p,l_3)(\int_0^1 |f''\left(t\frac{ka+(n-l)b}{n}+(1-t)\frac{a+b}{2}\right)|^q dt)^{\frac{1}{q}}+$$

$$+C(p,l_4)(\int_0^1 |f''\left(t\frac{a+b}{2}+(1-t)\frac{(n-k)a+lb}{n}\right)|^q dt)^{\frac{1}{q}}].$$

By definition of the $(\alpha,m)-$ convexity we have below the following inequality:

$$|I(f,a,b,n,k,l)| \leq$$

$$\leq \frac{(lb-ka)^3}{(2p+1)^{\frac{1}{p}}}\{C(p,l_1)\left(\int_0^1[t^\alpha|f''\left(\frac{(n-k)a+lb}{n}\right)|^q+m(1-t^\alpha)|f''\left(\frac{a}{m}\right)|^q]dt\right)^{\frac{1}{q}}+$$

$$+C(p,l_2)\left(\int_0^1[mt^\alpha|f''\left(\frac{b}{m}\right)|^q+(1-t^\alpha)|f''\left(\frac{ka+(n-l)b}{n}\right)|^q]dt\right)^{\frac{1}{q}}\}+$$

$$+\{\frac{n(b-a)}{2}-(lb-ka)]^3\cdot[C(p,l_3)\left(\int_0^1[t^\alpha|f''(\frac{ka+(n-l)b}{n})|^q+m(1-t^\alpha)|f''(\frac{a+b}{2m})|^q]dt\right)^{\frac{1}{q}}+$$

$$+C(p,l_4)\left(\int_0^1[mt^\alpha|f''\left(\frac{a+b}{2m}\right)|^q+(1-t^\alpha)|f''\left(\frac{(n-k)a+lb}{n}\right)|^q]dt\right)^{\frac{1}{q}}\}.$$

From here, by calculus, we obtain the desired inequality.

∎

The following result is a generalization of Theorem 12 from [4] for P-convex functions.

Proposition 1. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on the interior I^0 of an interval I in $\mathbb{R}$, with $a, b \in I$, $a < b$. Let $l, k, n \in \mathbb{N}$ with $n \geq 2$ and $l < k < n$. If $f^{''} \in L^1[a, b]$, and if $|f^{''}|^q$ is a P-convex function on I for some fixed $p > 1$ and M is a positive constant so that $\frac{a}{b} < M$ and $\frac{l}{k} > M$ or $(\frac{a}{b} < \frac{l}{k})$ then the following inequality takes place:*

$$|I(f,a,b,n,k,l)| \leq \frac{(lb-ka)^3}{(2p+1)^{\frac{1}{p}}}[C(p,l_1)(|f^{''}(\frac{(n-k)a+lb}{n})|^q + |f^{''}(a)|^q)^{\frac{1}{q}} +$$

$$+C(p,l_2)(|f^{''}(b)|^q + |f^{''}(\frac{ka+(n-l)b}{n})|^q)^{\frac{1}{q}}] +$$

$$+\frac{[\frac{n(b-a)}{2}-(lb-ka)]^3}{(2p+1)^{\frac{1}{p}}}[C(p,l_3)(|f^{''}(\frac{ka+(n-l)b}{n})|^q + |f^{''}(\frac{a+b}{2})|^q)^{\frac{1}{q}} +$$

$$+C(p,l_4)(|f^{''}(\frac{a+b}{2})|^q + |f^{''}(\frac{(n-k)a+lb}{n})|^q)^{\frac{1}{q}})].$$

Next result is a generalization of Theorem 7 from [4] for quasi-convex functions.

Proposition 2. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a twice differentiable function on the interior I^0 so that $f^{''} \in L^1[a, b]$, where $a, b \in I$, $a < b$. Let $l, k, n \in \mathbb{N}$ with $n \geq 2$ and $l < k < n$ so that $\frac{a}{b} < M$ and $\frac{l}{k} > M$ or $(\frac{a}{b} < \frac{l}{k})$. If $|f^{''}|^q$ is quasi-convex on $[a, b]$ for $p > 1$ then we have:*

$$|I(f,a,b,n,k,l)| \leq \frac{(lb-ka)^3}{(2p+1)^{\frac{1}{p}}}[C(p,l_1)\sup(|f^{''}(\frac{(n-k)a+lb}{n})|, |f^{''}(a)|) +$$

$$+C(p,l_2)\sup(|f^{''}(b)|, |f^{''}(\frac{ka+(n-l)b}{n})|)] +$$

$$+\frac{[\frac{n(b-a)}{2}-(lb-ka)]^3}{(2p+1)^{\frac{1}{p}}}[C(p,l_3)\sup(|f^{''}(\frac{ka+(n-l)b}{n})|, |f^{''}(\frac{a+b}{2})|) +$$

$$+C(p,l_4)\sup(|f^{''}(\frac{a+b}{2})|, |f^{''}(\frac{(n-k)a+lb}{n})|)].$$

References

[1] Alomari, M., Darus, M., Kirmaci, U. S., Some inequalities of Hermite-Hadamard ty6pe for s-convex functions, *Acta Mathematica Scientia*, (2011) 31 B(4), 1643-1652.

[2] Alomari, M., Darus, M., Kirmaci, U. S., Refinements ofHadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to specail means, *Computers and Mathematica with Applications*, **59** (2010) 225-232.

[3] Changjian, Z., Bencze, M., On Holder's inequality and its applications, *Creative Math.* ,**18** (1) (2009), 10-16.

[4] Ciurdariu, L., On some Hermite-Hadamard type inequalities for functions whose power of absolute value of derivatives are $(\alpha, m)-$ convex, *Int. J. of Math. Anal.*, **6**(**48**) (2012), 2361-2383.

[5] Dragomir, S. S., Pearce, C. E. M., Selected topic on Hermite-Hadamard inequalities and applications, *Melbourne and Adelaide* December, (2001).

[6] Dragomir, S. S., Fitzpatrick, S., The Hadamard's inequality for s-convex functionsin the second sense, *Demonstratio Math.* , **32 (4)** (1999), 687-696.

[7] Iscan Imdat, Generalizations of different type integral inequalities for s-convex functions via fractional integrals, *Appl. Anal.,* (2013) 1-17.

[8] Iscan, Imdat, Kunt, M., Yazici, N., Gozutok, Tuncay, K., New general integral inequalities for Lipschitzian functions via Riemann-Liouville fractional integrals and applications, *Joirnal of Inequalities and Special Functions*, **7** 4, (2016), 1-12.

[9] Kasvurmaci, H., Avci, M., Ozdemir, M. E., New inequalities of Hermite-Hadamard type for convex functions with applications, arXiv:1006.1593v1[math.CA].

[10] Kirmaci, U. S., Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. Comput.*, **147 (1)** (2014),137-146.

[11] Kirmaci, U. S., Klaricic, K., Bakula, Ozdemir, M. E., Pecaric, J., Hadamard-type inequalities for s-convex functions, *Appl. Math. Comput.*, **193 (1)** 2007, 26-35.

[12] Latif, M. A., Dragomir, S. S., New inequalities of Hermite-Hadanard type for functions whose derivatives in absolute value are convex with applications, *Acta Univ. Matthiae Belii, Series Math.*, (2013), 24-39.

[13] Mihesan, V. G., A generalization of the convexity, Seminar of Functional Equations, *Approx. and Convex,* Cluj-Napoca, Romania (1993).

[14] Set, E., New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second via fractional integrals, *Comput. Math. Appl.* **(2010)** Art ID:531976, 7 pages.

[15] Toader, Gh., On a generalization of the convexity, *Mathematica*, 30 (53) (1988), 83-87.

[16] Tunc, M., On some new inequalities for convex functions, *Turk. J. Math.*, **35** (2011) , 1-7.

[17] Park, J., New Inequalities of Hermite-Hadamard-like Type for the Functions whose Second Derivatives in Absolute Value are Convex, *Int. Journal of Math. Analysis*, **8**, 16 (2014), 777–791.

[18] Park, J., Hermite-Hadamard-like type inequalities for n-times differentiable functions which are m-convex and s-convex in the second sense, *Int. Journal of Math. Analysis*, **6** (2014), 25, 1187-1200.

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