Received 03/05/17

FURTHER INEQUALITIES OF HERMITE-HADAMARD TYPE FOR CONVEX FUNCTIONS AND RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. In this paper we establish several upper and lower bounds for the functions

$$\frac{1}{2}\left[J_{a+}^{\alpha}f(x)+J_{x-}^{\alpha}f(a)\right] \text{ for } a<x\leq b$$

and

$$\frac{1}{2}\left[J_{b-}^{\alpha}f(x)+J_{x+}^{\alpha}f(b)\right] \text{ for } a\leq x<b$$

in the case of Riemann-Liouville fractional integrals $J_{\cdot\pm}^{\alpha}$, for convex and h-convex functions $f:[a,b]\to\mathbb{R}$, for $\alpha>0$ and $x\in(a,b)$. Some particular cases of interest are examined. Various Hermite-Hadamard type inequalities are also provided.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality*.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [12], the recent survey paper [10] and the references therein.

Let $f:[a,b]\to\mathbb{C}$ be a complex valued *Lebesgue integrable* function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt$$

for $a<x\leq b$ and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt$$

for $a\leq x<b$, where Γ is the *Gamma function.* For $\alpha=0$, they are defined as

$$J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x) \text{ for } x\in(a,b).$$

1991 *Mathematics Subject Classification.* 26D15, 26D10, 26D07, 26A33.

Key words and phrases. Riemann-Liouville fractional integrals, Convex functions, Hermite-Hadamard inequalities.

RGMIA Res. Rep. Coll. 20 (2017), Art. 44, 12 pp.

In [25] Sarikaya et al. established the following Hermite-Hadamard type inequality for $\alpha > 0$

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

provided $f:[a,b] \rightarrow \mathbb{R}$ is a convex function.

A different version was also obtained by Sarikaya and Yildirim in [27] as follows

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}+}^{\alpha} f(b)+J_{\frac{a+b}{2}-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

provided $f:[a,b] \rightarrow \mathbb{R}$ is a convex function.

In the recent paper [11] we established the following results:

$$\begin{aligned} \frac{1}{\alpha+1}\left[\frac{1}{\alpha} f(x)+\frac{f(a)+f(b)}{2}\right] &\geq \frac{1}{2}\Gamma(\alpha)\left[\frac{J_{a+}^{\alpha} f(x)}{(x-a)^{\alpha}}+\frac{J_{b-}^{\alpha} f(x)}{(b-x)^{\alpha}}\right] \\ &\geq \int_0^1 (1-s)^{\alpha-1} f\left(sx+(1-s)\frac{a+b}{2}\right) ds \\ &\geq \frac{1}{\alpha} f\left(\frac{\alpha}{\alpha+1}\left(\frac{x}{\alpha}+\frac{a+b}{2}\right)\right) \end{aligned} \tag{1.4}$$

for any $x \in (a,b)$. In particular, we have

$$\begin{aligned} \frac{f(a)+f(b)}{2} &\geq \frac{\alpha}{\alpha+1}\left[\frac{1}{\alpha} f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \\ &\geq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)\right] \\ &\geq f\left(\frac{a+b}{2}\right) \end{aligned} \tag{1.5}$$

for any $\alpha > 0$.

We also have the dual result [11]:

$$\begin{aligned} \frac{1}{\alpha+1}\left[f(x)+\frac{1}{\alpha}\frac{f(a)+f(b)}{2}\right] &\geq \frac{1}{2}\Gamma(\alpha)\left[\frac{J_{x-}^{\alpha} f(a)}{(x-a)^{\alpha}}+\frac{J_{x+}^{\alpha} f(b)}{(b-x)^{\alpha}}\right] \\ &\geq \int_0^1 s^{\alpha-1} f\left(sx+(1-s)\frac{a+b}{2}\right) ds \\ &\geq \frac{1}{\alpha} f\left(\frac{\alpha}{\alpha+1}\left(x+\frac{1}{\alpha}\frac{a+b}{2}\right)\right) \end{aligned} \tag{1.6}$$

for any $x \in (a,b)$. Moreover, we have the particular inequalities

$$\begin{aligned} \frac{f(a)+f(b)}{2} &\geq \frac{\alpha}{\alpha+1}\left[f\left(\frac{a+b}{2}\right)+\frac{1}{\alpha}\frac{f(a)+f(b)}{2}\right] \\ &\geq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}-}^{\alpha} f(a)+J_{\frac{a+b}{2}+}^{\alpha} f(b)\right] \geq f\left(\frac{a+b}{2}\right), \end{aligned} \tag{1.7}$$

for any $\alpha > 0$.

The first inequality in (1.7) is improving the second inequality in (1.3).

From a different perspective we also have [11]

$$\begin{aligned}\frac{f(a)+f(b)}{2} &\geq \frac{\alpha}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{f(x)+f(a+b-x)}{2\alpha}\right] \\ &\geq \frac{1}{2}\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(a+b-x)\right] \geq f\left(\frac{a+b}{2}\right)\end{aligned} \tag{1.8}$$

for $a < x \leq b$. We observe that if we take $x = b$ in (1.8), then we get the inequality (1.2).

For other Hermite-Hadamard type inequalities for the Riemann-Liouville fractional integrals, see [1]-[6], [16]-[34] and the references therein.

Motivated by the above results, we establish in this paper several upper and lower bounds for the functions

$$\frac{1}{2}\left[J_{a+}^{\alpha}f(x)+J_{x-}^{\alpha}f(a)\right] \text{ for } a < x \leq b$$

and

$$\frac{1}{2}\left[J_{b-}^{\alpha}f(x)+J_{x+}^{\alpha}f(b)\right] \text{ for } a \leq x < b$$

in the case of convex and h-convex functions $f : [a,b] \to \mathbb{R}$ and for $\alpha > 0$. Some particular cases of interest are examined. Other Hermite-Hadamard type inequalities are also provided.

2. Main Results

We have:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function, then*

$$f\left(\frac{a+x}{2}\right) \leq \frac{1}{2}\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{x-}^{\alpha}f(a)\right] \leq \frac{f(a)+f(x)}{2} \tag{2.1}$$

for $a < x \leq b$ and

$$f\left(\frac{x+b}{2}\right) \leq \frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}\left[J_{x+}^{\alpha}f(b)+J_{b-}^{\alpha}f(x)\right] \leq \frac{f(x)+f(b)}{2} \tag{2.2}$$

for $a \leq x < b$.

In particular, for $x = b$ in (2.1) we obtain (1.2) and for $x = \frac{a+b}{2}$ we get

$$\begin{aligned}f\left(\frac{3a+b}{4}\right) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{\frac{a+b}{2}-}^{\alpha}f(a)\right] \\ &\leq \frac{f(a)+f\left(\frac{a+b}{2}\right)}{2}.\end{aligned} \tag{2.3}$$

In particular, for $x = a$ in (2.2) we re-obtain (1.2) while for $x = \frac{a+b}{2}$ we get

$$\begin{aligned}f\left(\frac{a+3b}{4}\right) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}+}^{\alpha}f(b)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right] \\ &\leq \frac{f\left(\frac{a+b}{2}\right)+f(b)}{2}.\end{aligned} \tag{2.4}$$

If we add (2.3) and (2.4) and divide by 2, then we get

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right) \\
& \leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\
& \leq \frac{2^{\alpha-2}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \\
& \times\left[J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right)+J_{\frac{a+b}{2}-}^{\alpha} f(a)+J_{\frac{a+b}{2}+}^{\alpha} f(b)+J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)\right] \\
& \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}.
\end{aligned} \tag{2.5}
$$

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function, then*

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right) \\
& \leq f\left(\frac{a+x}{2}\right)\frac{x-a}{b-a}+f\left(\frac{x+b}{2}\right)\frac{b-x}{b-a} \\
& \leq \frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-a)}\left[\frac{J_{a+}^{\alpha} f(x)+J_{x-}^{\alpha} f(a)}{(x-a)^{\alpha-1}}+\frac{J_{x+}^{\alpha} f(b)+J_{b-}^{\alpha} f(x)}{(b-x)^{\alpha-1}}\right] \\
& \leq \frac{1}{2}\left[\frac{x-a}{b-a} f(a)+\frac{b-x}{b-a} f(b)+f(x)\right] \leq \frac{1}{2}[f(a)+f(b)]
\end{aligned} \tag{2.6}
$$

for any $a<x<b$.

We observe that if we take in (2.6) $x=\frac{a+b}{2}$ we recapture (2.5).
We also have the integral inequality:

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function, then*

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right) \\
& \leq \frac{1}{b-a}\int_a^b f\left(\frac{a+x}{2}\right)\frac{x-a}{b-a}dx+\frac{1}{b-a}\int_a^b f\left(\frac{x+b}{2}\right)\frac{b-x}{b-a}dx \\
& \leq \frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-a)} \\
& \times\left[\frac{1}{b-a}\int_a^b \frac{J_{a+}^{\alpha} f(x)+J_{x-}^{\alpha} f(a)}{(x-a)^{\alpha-1}}dx+\frac{1}{b-a}\int_a^b \frac{J_{x+}^{\alpha} f(b)+J_{b-}^{\alpha} f(x)}{(b-x)^{\alpha-1}}dx\right] \\
& \leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+\frac{1}{b-a}\int_a^b f(x)\,dx\right] \leq \frac{1}{2}[f(a)+f(b)].
\end{aligned} \tag{2.7}
$$

Assume that I and J are intervals in $\mathbb{R}, (0,1)\subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 1 ([32]). *Let $h:J\to[0,\infty)$ with h not identical to 0. We say that $f:I\to[0,\infty)$ is an h-convex function if for all $x,y\in I$ we have*

$$
f(tx+(1-t)y)\leq h(t)f(x)+h(1-t)f(y) \tag{2.8}
$$

for all $t \in (0,1)$.

For some results concerning this class of functions see [32], [3], [20], [26] and [30]. We can give the following examples of h-convex functions.

Definition 2 ([15])**.** *We say that* $f : I \to \mathbb{R}$ *is a Godunova-Levin function or that* f *belongs to the class* $Q(I)$ *if* f *is non-negative and for all* $x,\ y \in I$ *and* $t \in (0,1)$ *we have*

$$f(tx + (1-t)y) \le \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{2.9}$$

Definition 3 ([13])**.** *We say that a function* $f : I \to \mathbb{R}$ *belongs to the class* $P(I)$ *if it is nonnegative and for all* $x,\ y \in I$ *and* $t \in [0,1]$ *we have*

$$f(tx + (1-t)y) \le f(x) + f(y). \tag{2.10}$$

It is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions.*

We can introduce now another class of functions [9].

Definition 4. *We say that the function* $f : I \to [0,\infty)$ *is of s-Godunova-Levin type, with* $s \in [0,1]$, *if*

$$f(tx + (1-t)y) \le \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{2.11}$$

for all $t \in (0,1)$ *and* $x, y \in I$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(I)$ the class of s-Godunova-Levin functions defined on I, then we obviously have

$$P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)$$

for $0 \le s_1 \le s_2 \le 1$.

Definition 5 ([4])**.** *Let* σ *be a real number,* $\sigma \in (0,1]$. *A function* $f : [0,\infty) \to [0,\infty)$ *is said to be* σ*-convex in the second sense or Breckner* σ*-convex if*

$$f(tx + (1-t)y) \le t^{\sigma} f(x) + (1-t)^{\sigma} f(y)$$

for all $x,\ y \in [0,\infty)$ *and* $t \in [0,1]$.

Theorem 3. *Assume that the function* $f : [a,b] \to [0,\infty)$ *is h-symmetrized convex on the interval* $[a,b]$ *with* h *integrable on* $[0,1]$ *and* f *integrable on* $[a,b]$. *Then*

$$\begin{aligned} \frac{1}{2\alpha h\left(\frac{1}{2}\right)} f\left(\frac{a+x}{2}\right) &\le \frac{1}{2}\frac{\Gamma(\alpha)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha} f(x) + J_{x-}^{\alpha} f(a)\right] \\ &\le [f(a) + f(x)] \int_0^1 \frac{(1-u)^{\alpha-1} + u^{\alpha-1}}{2} h(u)\, du \end{aligned} \tag{2.12}$$

for $a < x \le b$ *and*

$$\begin{aligned} \frac{1}{2\alpha h\left(\frac{1}{2}\right)} f\left(\frac{x+b}{2}\right) &\le \frac{1}{2}\frac{\Gamma(\alpha)}{(b-x)^{\alpha}}\left[J_{x+}^{\alpha} f(b) + J_{b-}^{\alpha} f(x)\right] \\ &\le [f(x) + f(b)] \int_0^1 \frac{(1-u)^{\alpha-1} + u^{\alpha-1}}{2} h(u)\, du \end{aligned} \tag{2.13}$$

for $a \le x < b$.

If we take $x=b$ in (2.12) or $x=a$ in (2.13), then we get

$$\frac{1}{2\alpha h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\frac{\Gamma\left(\alpha\right)}{\left(b-a\right)^{\alpha}}\left[J_{a+}^{\alpha}f\left(b\right)+J_{b-}^{\alpha}f\left(a\right)\right]\leq\left[f\left(a\right)+f\left(b\right)\right]\int_{0}^{1}\frac{\left(1-s\right)^{\alpha-1}+s^{\alpha-1}}{2}h\left(s\right)ds, \tag{2.14}$$

that was obtained in [31].

If we take in (2.12) and (2.13) $x=\frac{a+b}{2}$, then we have

$$\frac{1}{2\alpha h\left(\frac{1}{2}\right)}f\left(\frac{3a+b}{4}\right)\leq\frac{2^{\alpha-1}\Gamma\left(\alpha\right)}{\left(b-a\right)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{\frac{a+b}{2}-}^{\alpha}f\left(a\right)\right]\leq\left[f\left(a\right)+f\left(\frac{a+b}{2}\right)\right]\int_{0}^{1}\frac{\left(1-s\right)^{\alpha-1}+s^{\alpha-1}}{2}h\left(s\right)ds \tag{2.15}$$

and

$$\frac{1}{2\alpha h\left(\frac{1}{2}\right)}f\left(\frac{a+3b}{4}\right)\leq\frac{2^{\alpha-1}\Gamma\left(\alpha\right)}{\left(b-a\right)^{\alpha}}\left[J_{\frac{a+b}{2}+}^{\alpha}f\left(b\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\leq\left[f\left(\frac{a+b}{2}\right)+f\left(b\right)\right]\int_{0}^{1}\frac{\left(1-s\right)^{\alpha-1}+s^{\alpha-1}}{2}h\left(s\right)ds. \tag{2.16}$$

If we add the inequalities (2.15) and (2.16), then we get

$$\begin{aligned}\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2\alpha h\left(\frac{1}{2}\right)}&\\ \leq\frac{2^{\alpha-1}\Gamma\left(\alpha\right)}{\left(b-a\right)^{\alpha}}&\\ \times\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{\frac{a+b}{2}-}^{\alpha}f\left(a\right)+J_{\frac{a+b}{2}+}^{\alpha}f\left(b\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]&\\ \leq\left[f\left(\frac{a+b}{2}\right)+\frac{f\left(a\right)+f\left(b\right)}{2}\right]\int_{0}^{1}\left[\left(1-s\right)^{\alpha-1}+s^{\alpha-1}\right]h\left(s\right)ds.&\end{aligned} \tag{2.17}$$

If $f:[a,b]\to[0,\infty)$ is of P-type and integrable, namely $h\left(t\right)=1,\ t\in[0,1]$, then by Theorem 3 we have for $\alpha>0$ that

$$\frac{1}{2}f\left(\frac{a+x}{2}\right)\leq\frac{1}{2}\frac{\Gamma\left(\alpha+1\right)}{\left(x-a\right)^{\alpha}}\left[J_{a+}^{\alpha}f\left(x\right)+J_{x-}^{\alpha}f\left(a\right)\right]\leq f\left(a\right)+f\left(x\right) \tag{2.18}$$

for $a<x\leq b$ and

$$\frac{1}{2}f\left(\frac{x+b}{2}\right)\leq\frac{1}{2}\frac{\Gamma\left(\alpha+1\right)}{\left(b-x\right)^{\alpha}}\left[J_{x+}^{\alpha}f\left(b\right)+J_{b-}^{\alpha}f\left(x\right)\right]\leq f\left(x\right)+f\left(b\right) \tag{2.19}$$

for $a\leq x<b$.

If $f:[a,b]\to[0,\infty)$ is integrable and of s-Godunova-Levin type, with $s\in(0,1)$, namely $h(t)=\frac{1}{t^s}$, $t\in[0,1]$, then by Theorem 3 we have for $\alpha>s$ that

$$\begin{aligned}\frac{1}{\alpha 2^{s+1}}f\left(\frac{a+x}{2}\right)&\le\frac{1}{2}\frac{\Gamma(\alpha)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{x-}^{\alpha}f(a)\right]\\&\le\frac{f(a)+f(x)}{2}\left[B(1-s,\alpha)+\frac{1}{\alpha-s}\right]\end{aligned}\tag{2.20}$$

for $a<x\le b$ and

$$\begin{aligned}\frac{1}{\alpha 2^{s+1}}f\left(\frac{x+b}{2}\right)&\le\frac{1}{2}\frac{\Gamma(\alpha)}{(b-x)^{\alpha}}\left[J_{x+}^{\alpha}f(b)+J_{b-}^{\alpha}f(x)\right]\\&\le\frac{f(x)+f(b)}{2}\left[B(1-s,\alpha)+\frac{1}{\alpha-s}\right]\end{aligned}\tag{2.21}$$

for $a\le x<b$, where B is *Beta function*, i.e.

$$B(u,v):=\int_0^1 t^{u-1}(1-t)^{v-1}\,dt \text{ and } u,\ v>0.$$

If $f:[a,b]\subset[0,\infty)\to[0,\infty)$ is integrable and Breckner σ-convex with $\sigma\in(0,1]$ then by Theorem 3 we have for $\alpha>0$ that

$$\begin{aligned}\frac{1}{\alpha 2^{1-\sigma}}f\left(\frac{a+x}{2}\right)&\le\frac{1}{2}\frac{\Gamma(\alpha)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{x-}^{\alpha}f(a)\right]\\&\le\frac{f(a)+f(x)}{2}\left[B(1+\sigma,\alpha)+\frac{1}{\alpha+\sigma}\right]\end{aligned}\tag{2.22}$$

for $a<x\le b$ and

$$\begin{aligned}\frac{1}{\alpha 2^{1-\sigma}}f\left(\frac{x+b}{2}\right)&\le\frac{1}{2}\frac{\Gamma(\alpha)}{(b-x)^{\alpha}}\left[J_{x+}^{\alpha}f(b)+J_{b-}^{\alpha}f(x)\right]\\&\le\frac{f(x)+f(b)}{2}\left[B(1+\sigma,\alpha)+\frac{1}{\alpha+\sigma}\right]\end{aligned}\tag{2.23}$$

for $a\le x<b$.

Finally, we note that if $f:[a,b]\to[0,\infty)$ is *quasi-convex*, namely

$$f(tx+(1-t)y)\le\max\{f(x),f(y)\}\ \text{ for all } x,y\in I \text{ and } t\in[0,1]$$

then

$$\frac{1}{2}\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{x-}^{\alpha}f(a)\right]\le\max\{f(a),f(x)\},\ \text{ for } a<x\le b\tag{2.24}$$

and

$$\frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha}}\left[J_{x+}^{\alpha}f(b)+J_{b-}^{\alpha}f(x)\right]\le\max\{f(x),f(b)\}\ \text{ or } a\le x<b.\tag{2.25}$$

Similar results may be obtained by writing the inequalities (2.17) for these examples of h-convex functions. The details are omitted.

3. Proofs

In 1906, Fejér [14], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Lemma 1 (Fejér's Inequality)**.** *Consider the integral* $\int_a^b h(x)\,g(x)\,dx$*, where* h *is a convex function in the interval* (a,b) *and* g *is a positive function in the same interval such that*

$$g(a+t)=g(b-t),\ \ 0\le t\le \frac{1}{2}(b-a),$$

i.e., g *is symmetric. Under those conditions the following inequalities are valid:*

$$h\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dt\le \int_a^b h(t)\,g(t)\,dx\le \frac{h(a)+h(b)}{2}\int_a^b g(t)\,dt. \tag{3.1}$$

If h *is concave on* (a,b)*, then the inequalities reverse in (3.1).*

Clearly, for $g(x)\equiv 1$ on $[a,b]$ we get (1.1).

Using the definition of Riemann-Liouville fractional integrals we have

$$J_{x-}^{\alpha}f(a)=\frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1}f(t)\,dt$$

and then

$$\frac{1}{2}\left[J_{a+}^{\alpha}f(x)+J_{x-}^{\alpha}f(a)\right]=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{(x-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}f(t)\,dt$$

for $a<x\le b$, which is an identity of interest in itself.

Observe that the function $g_{x,a}:[a,x]\to[0,\infty)$ defined by

$$g_{x,a}(t)=\frac{1}{2\Gamma(\alpha)}\left[(x-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]$$

is symmetric on the interval $[a,x]$ for any $a<x\le b$.

Applying (3.1) on the interval $[a,x]$, we get

$$\begin{aligned} &h\left(\frac{a+x}{2}\right)\frac{1}{\Gamma(\alpha)}\int_a^x\frac{(x-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}dt \\ &\le \frac{1}{\Gamma(\alpha)}\int_a^x\frac{(x-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}f(t)\,dt \\ &\le \frac{h(a)+h(x)}{2}\frac{1}{\Gamma(\alpha)}\int_a^x\frac{(x-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}dt \end{aligned} \tag{3.2}$$

for any $a<x\le b$.

Since

$$\int_a^x\frac{(x-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}dt=\frac{1}{\alpha}(x-a)^{\alpha},$$

then by (3.2) we get

$$\begin{aligned} h\left(\frac{a+x}{2}\right)\frac{1}{\Gamma(\alpha)}\frac{1}{\alpha}(x-a)^{\alpha} &\le \frac{1}{2}\left[J_{a+}^{\alpha}f(x)+J_{x-}^{\alpha}f(a)\right] \\ &\le \frac{h(a)+h(x)}{2}\frac{1}{\Gamma(\alpha)}\frac{1}{\alpha}(x-a)^{\alpha}, \end{aligned}$$

which proves (2.1).

We have

$$J_{x+}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} f(t)\, dt$$

and then

$$\frac{1}{2}\left[J_{b-}^{\alpha} f(x) + J_{x+}^{\alpha} f(b)\right] = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{(b-t)^{\alpha-1} + (t-x)^{\alpha-1}}{2} f(t)\, dt \tag{3.3}$$

for $a \leq x < b$, which is an identity of interest in itself.

Using Fejér's inequality for the symmetrical function

$$g_{b,x}(t) = \frac{(b-t)^{\alpha-1} + (t-x)^{\alpha-1}}{2}$$

on $[x, b]$, we get in a similar way the inequality (2.2).

If we multiply (2.1) by $\frac{x-a}{b-a}$ and (2.2) by $\frac{b-x}{b-a}$ we have

$$\begin{aligned} f\left(\frac{a+x}{2}\right)\frac{x-a}{b-a} &\leq \frac{1}{2}\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha-1}(b-a)}\left[J_{a+}^{\alpha} f(x) + J_{x-}^{\alpha} f(a)\right] \\ &\leq \frac{f(a)+f(x)}{2}\frac{x-a}{b-a} \end{aligned}$$

for $a < x \leq b$ and

$$\begin{aligned} f\left(\frac{x+b}{2}\right)\frac{b-x}{b-a} &\leq \frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-x)^{\alpha-1}(b-a)}\left[J_{x+}^{\alpha} f(b) + J_{b-}^{\alpha} f(x)\right] \\ &\leq \frac{f(x)+f(b)}{2}\frac{b-x}{b-a} \end{aligned}$$

for $a \leq x < b$.

If we add these inequalities, we get

$$\begin{aligned} &f\left(\frac{a+x}{2}\right)\frac{x-a}{b-a} + f\left(\frac{x+b}{2}\right)\frac{b-x}{b-a} \\ &\leq \frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-a)}\left[\frac{J_{a+}^{\alpha} f(x) + J_{x-}^{\alpha} f(a)}{(x-a)^{\alpha-1}} + \frac{J_{x+}^{\alpha} f(b) + J_{b-}^{\alpha} f(x)}{(b-x)^{\alpha-1}}\right] \\ &\leq \frac{f(a)+f(x)}{2}\frac{x-a}{b-a} + \frac{f(x)+f(b)}{2}\frac{b-x}{b-a} \\ &= \frac{1}{2}\left[\frac{x-a}{b-a} f(a) + \frac{b-x}{b-a} f(b) + f(x)\right] \end{aligned} \tag{3.4}$$

for $a < x < b$. This proves the second and third inequalities in (2.6).

By the convexity of f we also have

$$\begin{aligned} &f\left(\frac{a+x}{2}\right)\frac{x-a}{b-a} + f\left(\frac{x+b}{2}\right)\frac{b-x}{b-a} \\ &\geq f\left(\frac{a+x}{2}\frac{x-a}{b-a} + \frac{x+b}{2}\frac{b-x}{b-a}\right) = f\left(\frac{a+b}{2}\right) \end{aligned}$$

for $a \leq x \leq b$. This proves the first inequality in (2.6).

By the convexity of f we have

$$f(x) = f\left(\frac{x-a}{b-a} b + \frac{b-x}{b-a} a\right) \leq \frac{x-a}{b-a} f(b) + \frac{b-x}{b-a} f(a)$$

for $a \leq x \leq b$.

Therefore

$$\begin{aligned}
&\frac{1}{2}\left[\frac{x-a}{b-a}f(a)+\frac{b-x}{b-a}f(b)+f(x)\right] \\
&\leq \frac{1}{2}\left[\frac{x-a}{b-a}f(a)+\frac{b-x}{b-a}f(b)+\frac{x-a}{b-a}f(b)+\frac{b-x}{b-a}f(a)\right] \\
&= \frac{1}{2}\left[f(a)+f(b)\right],
\end{aligned}$$

which proves the last part of (2.6).

If we change the variable $t=(1-s)a+sx$, with $a<x\leq b$, then $dt=(x-a)\,ds$, $x-t=(1-s)(x-a)$, $t-a=s(x-a)$

$$\begin{aligned}
&\frac{1}{2}\frac{\Gamma(\alpha)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{x-}^{\alpha}f(a)\right] \\
&= \int_0^1 \frac{(1-s)^{\alpha-1}+s^{\alpha-1}}{2} f((1-s)a+sx)\,ds
\end{aligned} \tag{3.5}$$

and, by replacing s with $1-s$ we also have

$$\begin{aligned}
&\frac{1}{2}\frac{\Gamma(\alpha)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{x-}^{\alpha}f(a)\right] \\
&= \int_0^1 \frac{(1-s)^{\alpha-1}+s^{\alpha-1}}{2} f(sa+(1-s)x)\,ds
\end{aligned}$$

giving

$$\begin{aligned}
&\frac{1}{2}\frac{\Gamma(\alpha)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{x-}^{\alpha}f(a)\right] \\
&= \int_0^1 \frac{(1-s)^{\alpha-1}+s^{\alpha-1}}{2}\,\frac{f((1-s)a+sx)+f(sa+(1-s)x)}{2}\,ds
\end{aligned} \tag{3.6}$$

for $a<x\leq b$.

Now, if we assume that f is h-convex on $[a,b]$, then

$$f(sa+(1-s)x)\leq h(s)f(a)+h(1-s)f(x),$$

which implies, by (3.5), that

$$\begin{aligned}
&\frac{1}{2}\frac{\Gamma(\alpha)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{x-}^{\alpha}f(a)\right] \\
&\leq \int_0^1 \frac{(1-s)^{\alpha-1}+s^{\alpha-1}}{2}\left[h(s)f(a)+h(1-s)f(x)\right]ds \\
&= f(a)\int_0^1 \frac{(1-s)^{\alpha-1}+s^{\alpha-1}}{2}h(s)\,ds \\
&+ f(x)\int_0^1 \frac{(1-s)^{\alpha-1}+s^{\alpha-1}}{2}h(1-s)\,ds.
\end{aligned} \tag{3.7}$$

Since

$$\int_0^1 \frac{(1-s)^{\alpha-1}+s^{\alpha-1}}{2}h(1-s)\,ds=\int_0^1 \frac{(1-s)^{\alpha-1}+s^{\alpha-1}}{2}h(s)\,ds,$$

then by (3.7) we get the second inequality in (2.12).

By the fact that f is h-convex on $[a,b]$, we have

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+x}{2}\right)\leq\frac{f\left(\left(1-s\right)a+sx\right)+f\left(sa+\left(1-s\right)x\right)}{2}$$

for $s\in[0,1]$. Then we get

$$\begin{aligned}(3.8)\qquad &\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+x}{2}\right)\frac{\left(1-s\right)^{\alpha-1}+s^{\alpha-1}}{2}\\ &\leq\frac{\left(1-s\right)^{\alpha-1}+s^{\alpha-1}}{2}\frac{f\left(\left(1-s\right)a+sx\right)+f\left(sa+\left(1-s\right)x\right)}{2}\end{aligned}$$

for $s\in[0,1]$.

If we integrate the inequality (3.8) over $s\in[0,1]$ and use (3.6) we get

$$\begin{aligned}(3.9)\qquad &\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+x}{2}\right)\int_0^1\frac{\left(1-s\right)^{\alpha-1}+s^{\alpha-1}}{2}dt\\ &\leq\frac{1}{2}\frac{\Gamma\left(\alpha\right)}{\left(x-a\right)^{\alpha}}\left[J_{a+}^{\alpha}f\left(x\right)+J_{x-}^{\alpha}f\left(a\right)\right].\end{aligned}$$

Since

$$\int_0^1\frac{\left(1-s\right)^{\alpha-1}+s^{\alpha-1}}{2}dt=\frac{1}{\alpha},$$

hence by (3.9) we obtain the first inequality in (2.12).

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