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HERMITE-HADAMARD TYPE INEQUALITIES FOR FRACTIONAL INTEGRALS

LOREDANA CIURDARIU

Abstract. Several Hermite-Hadamard type inequalities will be given in this paper for different types of convexity for fractional integrals.

1. Introduction

The inequality of Hermite-Hadamard type has been considered very useful in mathematical analysis being extended and generalized in many directions by many authors, see [22, 6, 5, 9, 1, 13, 17, 23, 11] and the references therein.

Many papers study the Riemann-Liouville fractionals integrals and give new and interesant generalizations of Hermite-Hadamard type inequalities using these kind of integrals, see for example [8, 7, 9, 10, 11, 18, 15, 17, 13, 22, 23, 24, 25, 26].

We begin by recalling below the classical definition for the convex functions.

Definition 1. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. The function f is said to be concave on I if the inequality (1) takes place in reversed direction.

It is necessary to recall below also other kind of convexity and the definition of fractionals integrals, see [8, 10, 9, 18, 19, 24]. For other type of convexity see also [20, 16].

Definition 2. *A function $f : [a, b] \to \mathbb{R}$ is said to be quasi-convex onl $[a, b]$ if*

$$f(tx + (1-t)y) \leq \sup\{f(x), f(y)\}$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$.

Definition 3. *A function $f : I \to \mathbb{R}$ is said to be P-convex on $[a, b]$ if it is nonnegative and for all $x, y \in I$ and $\lambda \in [9, 1]$*

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

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Definition 4. *A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be s-convex in the first sense on an interval I if the inequality*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t^s) f(y)$$

holds for all $x, y \in I$, $t \in [0,1]$ and for some fixed $s \in (0,1]$.

Definition 5. *A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be s-Godunova-Levin functions of second kind on an interval I if the inequality*

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y)$$

holds for all $x, y \in I$, $t \in (0,1)$ and for some fixed $s \in [0,1]$.

It is easy to see that for $s = 0$ s-Godunova-Levin functions of second kind are functions P-convex.

The classical Hermite-Hadamard's inequality for convex functions is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

Moreover, if the function f is concave then the inequality (2) hold in reversed direction.

Definition 6. *Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $\alpha \geq 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$ and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

In this paper, two new identities are established and then by making use of these equalities the author give new estimations of Hermite-Hadamard type inequalities for functions whose the n-time derivative iin absolute value of certain powers satisfies different type of convexities via Riemann-Liouville fractional integrals.

2. **Main results**

The following result is a generalization of Lemma 1 from [4] when $\alpha > n-1$ and $n \in \mathbb{N}$.

Lemma 1. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a, b \in I^0$, $0 < a < b$. Then for any $x \in [a,b]$, we have:*

$$I(f,x,a,b,\alpha,n) = (x-a)\int_0^1 t^\alpha f^{(n)}(tx+(1-t)a)dt + (b-x)\int_0^1 (1-t)^\alpha f^{(n)}(tb+(1-t)x)dt =$$

$$= \sum_{k=2}^{n} \alpha(\alpha-1)...(\alpha-k+2) f^{(n-k)}(x)\left(\frac{(-1)^{k-1}}{(x-a)^{k-1}} - \frac{1}{(b-x)^{k-1}}\right) +$$

$$+\Gamma(\alpha+1)[\frac{(-1)^n}{(x-a)^\alpha} J_{x^-}^{\alpha-n+1} f(a) + \frac{1}{(b-x)^\alpha} J_{x^+}^{\alpha-n+1} f(b)],$$

where $\alpha > n-1$.

Proof. By integration by parts and calculus we have,

$$\int_0^1 t^\alpha f^{''}(tx+(1-t)a)dt = \frac{f^{'}(x)}{x-a} - \frac{\alpha f(x)}{(x-a)^2} + \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha+1}} J_{x^-}^{\alpha-1} f(a)$$

and from here by induction we get:

$$I_1 = \int_0^1 t^\alpha f^{(n)}(tx+(1-t)a)dt = \sum_{k=2}^{n} (-1)^{k-1} \frac{\alpha(\alpha-1)...(\alpha-k+2)}{(x-a)^k} f^{(n-k)}(x) + \frac{f^{(n-1)}(x)}{x-a} +$$

$$+(-1)^n \frac{\Gamma(\alpha+1)}{(x-a)^{\alpha+1}} J_{x^-}^{\alpha-n+1} f(a).$$

Similarly, for $I_2 = \int_0^1 (1-t)^\alpha f^{(n)}(tb+(1-t)x)dt$ we will obtain:

$$I_2 = -\sum_{k=2}^{n} \frac{\alpha(\alpha-1)...(\alpha-k+2)}{(b-x)^k} f^{(n-k)}(x) - \frac{f^{(n-1)}(x)}{b-x} + \frac{\Gamma(\alpha+1)}{(b-x)^{\alpha+1}} J_{x^+}^{\alpha-n+1} f(b).$$

Now, multiplying I_1 by $x-a$ and I_2 by $b-x$ and adding the resulting identities we have the desired result.

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Remark 1. *Under conditions of Lemma 1, for $n = 2$, we obtain the following equality:*

$$I(f,x,a,b,\alpha,2) = (x-a)\int_0^1 t^\alpha f^{''}(tx+(1-t)a)dt + (b-x)\int_0^1 (1-t)^\alpha f^{''}(tb+(1-t)x)dt =$$

$$= \Gamma(\alpha+1)[\frac{1}{(x-a)^\alpha} J_{x^-}^{\alpha-1} f(a) + \frac{1}{(b-x)^\alpha} J_{x^+}^{\alpha-1} f(b)] - \frac{b-a}{(x-a)(b-x)} f(x)$$

Theorem 1. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and t $f^{(n)} \in L[a,b]$ with $a, b \in I^0$, $0 < a < b$. If $|f^{(n)}|^q$ is convex on $[a,b]$ for some fixed $q \geq 1$, where $\frac{1}{p} + \frac{1}{q} = 1$ then the following inequality takes place:*

$$|I(f,x,a,b,\alpha,n)| = |\sum_{k=2}^{n} \alpha(\alpha-1)...(\alpha-k+2) f^{(n-k)}(x)\left(\frac{(-1)^{k-1}}{(x-a)^{k-1}} - \frac{1}{(b-x)^{k-1}}\right) +$$

$$+\Gamma(\alpha+1)[\frac{(-1)^n}{(x-a)^\alpha}J_{x^-}^{\alpha-n+1}f(a)+\frac{1}{(b-x)^\alpha}J_{x^+}^{\alpha-n+1}f(b)]|\leq$$

$$\leq\frac{1}{2^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\{(x-a)\left(|f^{(n)}(x)|^q+|f^{(n)}(a)|^q\right)^{\frac{1}{q}}+(b-x)\left(|f^{(n)}(b)|^q+|f^{(n)}(x)|\right)^{\frac{1}{q}}\}$$

Proof. From Lemma 1 using the property of the modulus and the power mean inequality we obtain:

$$|I(f,x,a,b,\alpha,n)|\leq(x-a)\left(\int_0^1 t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f^{(n)}(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}+$$

$$+(b-x)\left(\int_0^1(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f^{(n)}(tb+(1-t)x)|^q dt\right)^{\frac{1}{q}}.$$

If we take into account that $|f^{(n)}|^q$ is convex we obtain the inequality from below:

$$|I(f,x,a,b,\alpha,n)|\leq\frac{1}{2^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\{(x-a)\left(\int_0^1[t|f^{(n)}(x)|^q+(1-t)|f^{(n)}(a)|^q]dt\right)^{\frac{1}{q}}+$$

$$+(b-x)\left(\int_0^1[t|f^{(n)}(b)|^q+(1-t)|f^{(n)}(x)|^q]dt\right)^{\frac{1}{q}}\}$$

which by calculus leads to desired inequality.

∎

Theorem 2. *Let $n\in\mathbb{N}^*$ and $f:I\subset\mathbb{R}\to\mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)}\in L[a,b]$ with $a,b\in I^0$, $0<a<b$. If $|f^{(n)}|^q$ is quasi-convex on $[a,b]$ for some fixed $q\geq 1$, where $\frac{1}{p}+\frac{1}{q}=1$ then the following inequality holds:*

$$|I(f,x,a,b,\alpha,n)|=|\sum_{k=2}^{n}\alpha(\alpha-1)...(\alpha-k+2)f^{(n-k)}(x)\left(\frac{(-1)^{k-1}}{(x-a)^{k-1}}-\frac{1}{(b-x)^{k-1}}\right)+$$

$$+\Gamma(\alpha+1)[\frac{(-1)^n}{(x-a)^\alpha}J_{x^-}^{\alpha-n+1}f(a)+\frac{1}{(b-x)^\alpha}J_{x^+}^{\alpha-n+1}f(b)]|\leq$$

$$\leq\frac{1}{(\alpha p+1)^{\frac{1}{p}}}\{(x-a)\sup\{|f^{(n)}(x)|,|f^{(n)}(a)|\}+(b-x)\sup\{|f^{(n)}(b)|,|f^{(n)}(x)|\}\}$$

Proof. We will use the property of the modulus, the power mean inequality and then the definition of quasi-convex functions like before. ∎

Next result is also a generalization of Lemma 4 from [3].

Lemma 2. *Let $n\in\mathbb{N}^*$, $n\geq 2$ and $f:I\subset\mathbb{R}\to\mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)}\in L[a,b]$ with $a,b\in I^0$, $0<a<b$, $x\in[a,b]$, $\lambda\in(0,1)$. Then the following identity holds:*

$$\mathcal{I}(f,x,a,b,\lambda,\alpha,n)=(1-\lambda)(x-a)\int_0^1 t^\alpha f^{(n)}(t(\lambda a+(1-\lambda)x)+(1-t)a)dt+$$

$$+\lambda(x-a)\int_0^1(1-t)^\alpha f^{(n)}(tx+(1-t)(\lambda a+(1-\lambda)x))dt+$$

$$+(1-\lambda)(b-x)\int_0^1 t^\alpha f^{(n)}(t(\lambda x+(1-\lambda)b)+(1-t)x)dt+$$

$$+\lambda(b-x)\int_0^1 (1-t)^\alpha f^{(n)}(tb+(1-t)(\lambda x+(1-\lambda)b))dt=$$

$$=\sum_{k=2}^{n}\alpha(\alpha-1)...(\alpha-k+2)[(\frac{(-1)^{k-1}}{(1-\lambda)^{k-1}}-\frac{1}{\lambda^{k-1}})(\frac{f^{(n-k)}(\lambda a+(1-\lambda)x)}{(x-a)^{k-1}}+$$

$$+\frac{f^{(n-k)}(\lambda x+(1-\lambda)b)}{(b-x)^{k-1}})]+\Gamma(\alpha+1)\{\frac{(-1)^n}{(1-\lambda)^\alpha(x-a)^\alpha}J^{\alpha-n+1}_{(\lambda a+(1-\lambda)x)^-}f(a)+$$

$$+\frac{1}{\lambda^\alpha(b-x)^\alpha}J^{\alpha-n+1}_{(\lambda x+(1-\lambda)b)^+}f(b)+\frac{1}{\lambda^\alpha(x-a)^\alpha}J^{\alpha-n+1}_{(\lambda a+(1-\lambda)x)^+}f(x)+$$

$$+\frac{(-1)^n}{(1-\lambda)^\alpha(b-x)^\alpha}J^{\alpha-n+1}_{(\lambda x+(1-\lambda)b)^-}f(x)\},$$

where $\alpha>n-1$.

Proof. By integration by parts and then using the substitution $u=t(\lambda a+(1-\lambda)x)+(1-t)a$ we get

$$\int_0^1 t^\alpha f^{''}(t(\lambda a+(1-\lambda)x)+(1-t)a))dt=\frac{f^{'}(\lambda a+(1-\lambda)x)}{(1-\lambda)(x-a)}-\frac{\alpha f(\lambda a+(1-\lambda)x)}{(1-\lambda)^2(x-a)^2}+$$

$$+\frac{\Gamma(\alpha+1)}{(1-\lambda)^{\alpha+1}(x-a)^{\alpha+1}}J^{\alpha-1}_{(\lambda a+(1-\lambda)x)^-}f(a).$$

Then we check easily by induction that

$$I_1=\int_0^1 t^\alpha f^{(n)}(t(\lambda a+(1-\lambda)x)+(1-t)a))dt=$$

$$=\sum_{k=2}^{n}(-1)^{k-1}\frac{\alpha(\alpha-1)...(\alpha-k+2)}{(1-\lambda)^k(x-a)^k}f^{(n-k)}(\lambda a+(1-\lambda)x)+\frac{f^{(n-1)}(\lambda a+(1-\lambda)x)}{(1-\lambda)(x-a)}+$$

$$+\frac{(-1)^n\Gamma(\alpha+1)}{(1-\lambda)^{\alpha+1}(x-a)^{\alpha+1}}J^{\alpha-n+1}_{(\lambda a+(1-\lambda)x)^-}f(a).$$

Analogously we obtain

$$I_3=\int_0^1 t^\alpha f^{(n)}(t(\lambda x+(1-\lambda)b)+(1-t)x))dt=$$

$$=\sum_{k=2}^{n}(-1)^{k-1}\frac{\alpha(\alpha-1)...(\alpha-k+2)}{(1-\lambda)^k(b-x)^k}f^{(n-k)}(\lambda x+(1-\lambda)b)+\frac{f^{(n-1)}(\lambda x+(1-\lambda)b)}{(1-\lambda)(b-x)}+$$

$$+\frac{(-1)^n\Gamma(\alpha+1)}{(1-\lambda)^{\alpha+1}(b-x)^{\alpha+1}}J^{\alpha-n+1}_{(\lambda x+(1-\lambda)b)^-}f(x).,$$

$$I_2=\int_0^1 (1-t)^\alpha f^{(n)}(tx+(1-t)(\lambda a+(1-\lambda)x))dt=$$

$$=-\sum_{k=2}^{n}\frac{\alpha(\alpha-1)...(\alpha-k+2)}{\lambda^k(x-a)^k}f^{(n-k)}(\lambda a+(1-\lambda)x)-\frac{f^{(n-1)}(\lambda a+(1-\lambda)x)}{\lambda(x-a)}+$$

$$+\frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}(x-a)^{\alpha+1}}J^{\alpha-n+1}_{(\lambda a+(1-\lambda)x)^+}f(x).$$

and

$$I_4 = \int_0^1 (1-t)^\alpha f^{(n)}(tb + (1-t)(\lambda x + (1-\lambda)b))dt =$$

$$= \sum_{k=2}^{n} \frac{\alpha(\alpha-1)...(\alpha-k+2)}{\lambda^k (b-x)^k} f^{(n-k)}(\lambda x + (1-\lambda)b) - \frac{f^{(n-1)}(\lambda x + (1-\lambda)b)}{\lambda(b-x)} +$$

$$+ \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}(b-x)^{\alpha+1}} J^{\alpha-n+1}_{(\lambda x+(1-\lambda)b)^+} f(b).$$

Multiplying now I_1 by $(1-\lambda)(x-a)$, I_2 by $\lambda(x-a)$, $I-3$ by $(1-\lambda)(b-x)$ and I_4 by $\lambda b - x)$ and summing then these expressions we find by calculus the desired equality.

∎

Theorem 3. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a, b \in I^0,\ 0 < a < b,\ \lambda \in (0,1),\ x \in [a,b]$. If $|f^{(n)}|^q$ is convex on $[a,b]$ for some fixed $q \geq 1$, where $\frac{1}{p} + \frac{1}{q} = 1$ then the following inequality holds:*

$$\mathcal{I}(f, x, a, b, \lambda, \alpha, n) \leq \frac{1}{2^{\frac{1}{q}}(\alpha p + 1)^{\frac{1}{p}}} \{(1-\lambda)(x-a)(|f^{(n)}(\lambda a + (1-\lambda)x)|^q + |f^{(n)}(a)|^q)^{\frac{1}{q}} +$$

$$+\lambda(x-a)(|f^{(n)}(x)|^q + |f^{(n)}(\lambda a + (1-\lambda)x)|^q)^{\frac{1}{q}} +$$

$$+(1-\lambda)(b-x)(|f^{(n)}(\lambda x + (1-\lambda)b)|^q + |f^{(n)}(x)|^q)^{\frac{1}{q}} +$$

$$+\lambda(b-x)(|f^{(n)}(b)|^q + |f^{(n)}(\lambda x + (1-\lambda)b)|^q)^{\frac{1}{q}}\},$$

where $\alpha > n-1$.

Proof. We use the power mean inequality and the definition of convex functions as in previous theorem. ∎

Theorem 4. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a, b \in I^0,\ 0 < a < b,\ \lambda \in (0,1),\ x \in [a,b]$. If $|f^{(n)}|^q$ is convex on $[a,b]$ then the following inequality holds:*

$$\mathcal{I}(f, x, a, b, \lambda, \alpha, n) \leq \frac{1}{(\alpha+2)^{\frac{1}{q}}} \{(1-\lambda)(x-a)(|f^{(n)}(\lambda a + (1-\lambda)x)|^q + \frac{1}{\alpha+1}|f^{(n)}(a)|^q)^{\frac{1}{q}} +$$

$$+\lambda(x-a)(\frac{1}{\alpha+1}|f^{(n)}(x)|^q + |f^{(n)}(\lambda a + (1-\lambda)x)|^q)^{\frac{1}{q}} +$$

$$+(1-\lambda)(b-x)(|f^{(n)}(\lambda x + (1-\lambda)b)|^q + \frac{1}{\alpha+1}|f^{(n)}(x)|^q)^{\frac{1}{q}} +$$

$$+\lambda(b-x)(\frac{1}{\alpha+1}|f^{(n)}(b)|^q + |f^{(n)}(\lambda x + (1-\lambda)b)|^q)^{\frac{1}{q}}\},$$

where $\alpha > n-1$.

Proof. In this case we will use the Holder's inequality and then the definition of convex functions. ∎

Theorem 5. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a, b \in I^0$, $0 < a < b$, $\lambda \in (0,1)$, $x \in [a,b]$. If $|f^{(n)}|^q$ is P-convex on $[a,b]$ then the following inequality holds:*

$$\mathcal{I}(f,x,a,b,\lambda,\alpha,n) \leq \frac{1}{(\alpha p+1)^{\frac{1}{p}}}\{(1-\lambda)(x-a)(|f^{(n)}(\lambda a+(1-\lambda)x)|^q+|f^{(n)}(a)|^q)^{\frac{1}{q}}+$$
$$+\lambda(x-a)(|f^{(n)}(x)|^q+|f^{(n)}(\lambda a+(1-\lambda)x)|^q)^{\frac{1}{q}}+$$
$$+(1-\lambda)(b-x)(|f^{(n)}(\lambda x+(1-\lambda)b)|^q+|f^{(n)}(x)|^q)^{\frac{1}{q}}+$$
$$+\lambda(b-x)(|f^{(n)}(b)|^q+|f^{(n)}(\lambda x+(1-\lambda)b)|^q)^{\frac{1}{q}}\},$$

where $\alpha > n-1$.

Proof. In this case we will use the power mean inequality and then the definition of P-convex functions. ∎

Theorem 6. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a, b \in I^0$, $0 < a < b$, $\lambda \in (0,1)$, $x \in [a,b]$. If $|f^{(n)}|^q$ is quasi-convex on $[a,b]$ then the following inequality holds:*

$$\mathcal{I}(f,x,a,b,\lambda,\alpha,n) \leq \frac{1}{(\alpha p+1)^{\frac{1}{p}}}\{(1-\lambda)(x-a)\sup\{|f^{(n)}(\lambda a+(1-\lambda)x)|,|f^{(n)}(a)|\}+$$
$$+\lambda(x-a)\sup\{|f^{(n)}(x)|,|f^{(n)}(\lambda a+(1-\lambda)x)|\}+$$
$$+(1-\lambda)(b-x)\sup\{|f^{(n)}(\lambda x+(1-\lambda)b)|,|f^{(n)}(x)|\}+$$
$$+\lambda(b-x)\sup\{|f^{(n)}(b)|,|f^{(n)}(\lambda x+(1-\lambda)b)|\}\},$$

where $\alpha > n-1$.

Proof. In this case we will use the power mean inequality and then the definition of quasi-convex functions. ∎

Theorem 7. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a, b \in I^0$, $0 < a < b$, $\lambda \in (0,1)$, $x \in [a,b]$. If $|f^{(n)}|^q$ is s-convex in the first sense on $[a,b]$ and $\alpha > n-1$. then the following inequality takes place:*

$$\mathcal{I}(f,x,a,b,\lambda,\alpha,n) \leq \frac{1}{(\alpha p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\{(1-\lambda)(x-a)(|f^{(n)}(\lambda a+(1-\lambda)x)|^q+|f^{(n)}(a)|^q)^{\frac{1}{q}}+$$
$$+\lambda(x-a)(|f^{(n)}(x)|^q+|f^{(n)}(\lambda a+(1-\lambda)x)|^q)^{\frac{1}{q}}+$$
$$+(1-\lambda)(b-x)(|f^{(n)}(\lambda x+(1-\lambda)b)|^q+|f^{(n)}(x)|^q)^{\frac{1}{q}}+$$
$$+\lambda(b-x)(|f^{(n)}(b)|^q+|f^{(n)}(\lambda x+(1-\lambda)b)|^q)^{\frac{1}{q}}\}.$$

Proof. In this case we will use the power mean inequality and then the definition of s-convex functions in the first sense. ∎

Theorem 8. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o nonnegative function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a, b]$ with $a, b \in I^0$, $0 < a < b$, $\lambda \in (0, 1)$, $x \in [a, b]$. If $|f^{(n)}|^q$ is s-Godunova-Levin functiono of second kind on $[a, b]$ and $\alpha > n - 1$. then the following inequality takes place:*

$$\mathcal{I}(f, x, a, b, \lambda, \alpha, n) \leq \frac{1}{(\alpha p + 1)^{\frac{1}{p}} (1 - s)^{\frac{1}{q}}} \{(1-\lambda)(x-a)(|f^{(n)}(\lambda a+(1-\lambda)x)|^q+|f^{(n)}(a)|^q)^{\frac{1}{q}}+$$

$$+\lambda(x - a)(|f^{(n)}(x)|^q + |f^{(n)}(\lambda a + (1 - \lambda)x)|^q)^{\frac{1}{q}}+$$

$$+(1 - \lambda)(b - x)(|f^{(n)}(\lambda x + (1 - \lambda)b)|^q + |f^{(n)}(x)|^q)^{\frac{1}{q}}+$$

$$+\lambda(b - x)(|f^{(n)}(b)|^q + |f^{(n)}(\lambda x + (1 - \lambda)b)|^q)^{\frac{1}{q}}\}.$$

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Department of Mathematics, "Politehnica" University of Timisoara, P-ta. Victoriei, No.2, 300006-Timisoara