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SOME INEQUALITIES OF HERMITE-HADAMARD TYPE FOR SYMMETRIZED CONVEX FUNCTIONS AND RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. In this paper we establish several upper and lower bounds for the functions

$$\frac{\Gamma(\alpha+1)}{2(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(a+b-x)\right]$$

and

$$\frac{\Gamma(\alpha+1)}{2(x-a)^{\alpha}}\left[J_{x-}^{\alpha}f(a)+J_{a+b-x+}^{\alpha}f(b)\right]$$

in the case of Riemann-Liouville fractional integrals $J_{\cdot\pm}^{\alpha}$, for several classes of symmetrized convex functions $f:[a,b]\to\mathbb{R}$, for $\alpha>0$ and $x\in(a,b)$. Some particular cases of interest are examined. Various Hermite-Hadamard type inequalities are also provided.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2},\quad a,b\in\mathbb{R},\ a\neq b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [22]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [2]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [22]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [12], the recent survey paper [10] and the references therein.

For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, which is defined by

$$\breve{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t\in[a,b].$$

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The *anti-symmetrical transform of* f on the interval $[a,b]$ is denoted by $\tilde{f}_{[a,b]}$, or simply $\tilde{f}$ and is defined by

$$\tilde{f}(t) := \frac{1}{2}\left[f(t) - f(a+b-t)\right], t \in [a,b].$$

It is obvious that for any function f we have $\breve{f} + \tilde{f} = f$.

It is easy to see that, if f is convex on $[a,b]$, then $\breve{f}$ is convex on $[a,b]$.

Consider the real numbers $a<b$ and define the function $f_0 : [a,b] \to \mathbb{R}$, $f_0(t) = t^3$. We have

$$\breve{f}_0(t) := \frac{1}{2}\left[t^3 + (a+b-t)^3\right] = \frac{3}{2}(a+b)t^2 - \frac{3}{2}(a+b)^2 t + \frac{1}{2}(a+b)^3$$

for any $t \in \mathbb{R}$.

Since the second derivative $\left(\breve{f}_0\right)''(t) = 3(a+b)$, $t \in \mathbb{R}$, then $\breve{f}_0$ is strictly convex on $[a,b]$ if $\frac{a+b}{2} > 0$ and strictly concave on $[a,b]$ if $\frac{a+b}{2} < 0$. Therefore if $a < 0 < b$ with $\frac{a+b}{2} > 0$, then we can conclude that f_0 is not convex on $[a,b]$ while $\breve{f}_0$ is convex on $[a,b]$.

We can introduce the following concept of convexity.

Definition 1. *We say that the function* $f : [a,b] \to \mathbb{R}$ *is symmetrized convex (concave) on the interval* $[a,b]$ *if the symmetrical transform* $\breve{f}$ *is convex (concave) on* $[a,b]$.

Now, if we denote by $Con[a,b]$ the closed convex cone of convex functions defined on $[a,b]$ and by $SCon[a,b]$ the class of symmetrized convex functions, then from the above remarks we can conclude that

$$Con[a,b] \subsetneq SCon[a,b]. \tag{1.2}$$

Also, if $[c,d] \subset [a,b]$ and $f \in SCon[a,b]$, then this does not imply in general that $f \in SCon[c,d]$.

The following result holds [9]:

Theorem 1. *Assume that* $f : [a,b] \to \mathbb{R}$ *is symmetrized convex on the interval* $[a,b]$. *Then for any* $x \in [a,b]$ *we have the bounds*

$$f\left(\frac{a+b}{2}\right) \le \breve{f}(x) \le \frac{f(a)+f(b)}{2}. \tag{1.3}$$

Corollary 1. *Assume that* $f : [a,b] \to \mathbb{R}$ *is symmetrized convex and integrable on the interval* $[a,b]$. *Then we have the Hermite-Hadamard inequalities*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}. \tag{1.4}$$

Corollary 2. *If* $f : [a,b] \to \mathbb{R}$ *is symmetrized convex and integrable on the interval* $[a,b]$ *and* $w : [a,b] \to [0,\infty)$ *is integrable on* $[a,b]$, *then*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt \le \int_a^b w(t)\breve{f}(t)\,dt \le \frac{f(a)+f(b)}{2}\int_a^b w(t)\,dt. \tag{1.5}$$

Moreover, if w *is symmetric almost everywhere on* $[a,b]$, *i.e.* $w(t) = w(a+b-t)$ *for almost every* $t \in [a,b]$, *then*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt \le \int_a^b w(t) f(t)\,dt \le \frac{f(a)+f(b)}{2}\int_a^b w(t)\,dt. \tag{1.6}$$

Remark 1. *The inequality (1.6) was obtained by L. Fejér in 1906 for convex functions f and symmetric weights w. It has been shown now that this inequality remains valid for the larger class of symmetrized convex functions f on the interval $[a,b]$.*

Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The Riemann-Liouville fractional integrals are defined for $\alpha>0$ by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)\,dt$$

for $a<x\le b$ and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)\,dt$$

for $a\le x<b$, where Γ is the *Gamma function.* For $\alpha=0$, they are defined as

$$J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)\ \text{ for } x\in(a,b).$$

In [27] Sarikaya et al. established the following Hermite-Hadamard type inequality for $\alpha>0$

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\le\frac{f(a)+f(b)}{2}\tag{1.7}$$

provided $f:[a,b]\to\mathbb{R}$ is a convex function.

A different version was also obtained by Sarikaya and Yildirim in [28] as follows

$$f\left(\frac{a+b}{2}\right)\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}+}^{\alpha}f(b)+J_{\frac{a+b}{2}-}^{\alpha}f(a)\right]\le\frac{f(a)+f(b)}{2}\tag{1.8}$$

provided $f:[a,b]\to\mathbb{R}$ is a convex function.

For other Hermite-Hadamard type inequalities for the Riemann-Liouville fractional integrals, see [1], [5], [6], [16]-[34] and the references therein.

Motivated by the above results, we establish in this paper several upper and lower bounds for the functions

$$\frac{\Gamma(\alpha+1)}{2(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(a+b-x)\right]$$

and

$$\frac{\Gamma(\alpha+1)}{2(x-a)^{\alpha}}\left[J_{x-}^{\alpha}f(a)+J_{a+b-x+}^{\alpha}f(b)\right]$$

for several classes of symmetrized convex functions $f:[a,b]\to\mathbb{R}$, for $\alpha>0$ and $x\in(a,b)$. Some particular cases of interest are examined. Other Hermite-Hadamard type inequalities are also provided.

2. Inequalities for Symmetrized Convexity

We have:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be an integrable function and $\alpha>0$, then we have the representations*

$$\frac{1}{2}\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(a+b-x)\right]=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\breve{f}(t)\,dt\tag{2.1}$$

for any $a<x\le b$ and

$$\frac{1}{2}\left[J_{a+}^{\alpha}f(a+b-x)+J_{b-}^{\alpha}f(x)\right]=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}\breve{f}(t)\,dt\tag{2.2}$$

for any $a \le x < b$.

Proof. We have for $a < x \le b$ that

$$J_{b-}^{\alpha} f(a+b-x) = \frac{1}{\Gamma(\alpha)} \int_{a+b-x}^{b} (t-a-b+x)^{\alpha-1} f(t)\, dt.$$

If we change the variable $u = a+b-t$, then we get

$$J_{b-}^{\alpha} f(a+b-x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-u)^{\alpha-1} f(a+b-u)\, du,$$

which gives

$$\begin{aligned}
&\frac{1}{2}\left[J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(a+b-x)\right] \\
&= \frac{1}{2}\left[\frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(t)\, dt + \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(a+b-t)\, dt\right] \\
&= \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} \frac{f(t)+f(a+b-t)}{2} dt
\end{aligned}$$

and the representation (2.1) is obtained.

If in (2.1) we replace x by $a+b-x$, then we get

$$\frac{1}{2}\left[J_{a+}^{\alpha} f(a+b-x) + J_{b-}^{\alpha} f(x)\right] = \frac{1}{\Gamma(\alpha)} \int_{a}^{a+b-x} (a+b-x-t)^{\alpha-1} \breve{f}(t)\, dt.$$

If we change the variable $u = a+b-t$, then we have

$$\begin{aligned}
\int_{a}^{a+b-x} (a+b-x-t)^{\alpha-1} \breve{f}(t)\, dt &= \int_{x}^{b} (u-x)^{\alpha-1} \breve{f}(a+b-u)\, du \\
&= \int_{x}^{b} (u-x)^{\alpha-1} \breve{f}(u)\, du,
\end{aligned}$$

which proves the representation (2.2). □

Corollary 3. *With the assumptions of Lemma 1 we have the representations*

$$\begin{aligned}
&\frac{1}{2}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \\
&= \frac{1}{\Gamma(\alpha)} \int_{a}^{b} (b-t)^{\alpha-1} \breve{f}(t)\, dt = \frac{1}{\Gamma(\alpha)} \int_{a}^{b} (t-a)^{\alpha-1} \breve{f}(t)\, dt \\
&= \frac{1}{\Gamma(\alpha)} \int_{a}^{b} \frac{(b-t)^{\alpha-1} + (t-a)^{\alpha-1}}{2} \breve{f}(t)\, dt
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&\frac{1}{2}\left[J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)\right] \\
&= \frac{1}{\Gamma(\alpha)} \int_{a}^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right)^{\alpha-1} \breve{f}(t)\, dt \\
&= \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right)^{\alpha-1} \breve{f}(t)\, dt.
\end{aligned} \tag{2.4}$$

Theorem 2. *Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized convex and integrable on the interval $[a,b]$. Then we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(x-a)^{\alpha}}\left[J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(a+b-x)\right] \le \frac{f(a)+f(b)}{2} \tag{2.5}$$

for any $a < x \le b$ and

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-x)^{\alpha}}\left[J_{a+}^{\alpha} f(a+b-x) + J_{b-}^{\alpha} f(x)\right] \le f\left(\frac{a+b}{2}\right) \tag{2.6}$$

for any $a \le x < b$.

Proof. From (1.3) we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\,dt &\le \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\breve{f}(t)\,dt \\ &\le \frac{f(a)+f(b)}{2}\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\,dt \end{aligned} \tag{2.7}$$

for any $a < x \le b$.

Since

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\,dt = \frac{1}{\alpha\Gamma(\alpha)}(x-a)^{\alpha} = \frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha}$$

for any $a < x \le b$, then by the representation (2.1) and (2.7) we get the desired result (2.5).

The inequality (2.6) follows in a similar way. □

Corollary 4. *With the assumption of Theorem 2 we have*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\left[\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{2}\right] \\ &\le \frac{\Gamma(\alpha+1)}{2}\left[\left(J_{a+}^{\alpha} f\right)^{\smile}(x) + \left(J_{b-}^{\alpha} f\right)^{\smile}(a+b-x)\right] \\ &\le \frac{f(a)+f(b)}{2}\left[\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{2}\right] \end{aligned} \tag{2.8}$$

for any $a < x < b$.

The proof follows by (2.5) and (2.6) by addition.

Corollary 5. *With the assumption of Theorem 2 we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+1}}\int_a^b \frac{J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x)}{2}dx \le \frac{f(a)+f(b)}{2} \tag{2.9}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+1}}\int_a^b \frac{\left(J_{a+}^{\alpha} f\right)^{\smile}(x) + \left(J_{b-}^{\alpha} f\right)^{\smile} f(x)}{2}dx \\ &\le \frac{f(a)+f(b)}{2}, \end{aligned} \tag{2.10}$$

where $\left(J_{a+}^{\alpha} f\right)^{\smile}$ and $\left(J_{b-}^{\alpha} f\right)^{\smile}$ are the symmetrical transforms of $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$, respectively.

Proof. From (2.5) we have

$$f\left(\frac{a+b}{2}\right)(x-a)^{\alpha} \leq \frac{\Gamma(\alpha+1)}{2}\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(a+b-x)\right] \leq \frac{f(a)+f(b)}{2}(x-a)^{\alpha} \tag{2.11}$$

for any $a<x\leq b$.

Taking the integral mean over x on $[a,b]$ and taking into account that

$$\frac{1}{b-a}\int_a^b J_{b-}^{\alpha}f(a+b-x)\,dx = \frac{1}{b-a}\int_a^b J_{b-}^{\alpha}f(x)\,dx$$

and

$$\frac{1}{b-a}\int_a^b (x-a)^{\alpha}\,dx = \frac{(b-a)^{\alpha}}{\alpha+1},$$

then from (2.11) we get (2.9).

The inequality (2.10) follows in a similar way from (2.8). □

Remark 2. *If we either take $x=b$ in (2.5) or $x=a$ in (2.6), then we get*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{2.12}$$

that holds for f symmetrized convex and integrable on the interval $[a,b]$. This extends the inequality (1.7) to the larger class of symmetrized convex and integrable functions on the interval $[a,b]$.

If we take $x=\frac{a+b}{2}$ in either (2.5) or (2.6), then we get

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right] \leq f\left(\frac{a+b}{2}\right), \tag{2.13}$$

for f symmetrized convex and integrable on the interval $[a,b]$.

The following lemma holds:

Lemma 2. *Let $f:[a,b]\to\mathbb{C}$ be an integrable function and $\alpha>0$, then we have the representations*

$$\frac{1}{2}\left[J_{x-}^{\alpha}f(a)+J_{a+b-x+}^{\alpha}f(b)\right] = \frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1}\breve{f}(t)\,dt \tag{2.14}$$

for any $a<x\leq b$ and

$$\frac{1}{2}\left[J_{a+b-x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)\right] = \frac{1}{\Gamma(\alpha)}\int_x^b (b-t)^{\alpha-1}\breve{f}(t)\,dt \tag{2.15}$$

for any $a\leq x<b$.

Proof. Using the definitions of Riemann-Liouville fractional integrals we have

$$J_{x-}^{\alpha}f(a) = \frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1}f(t)\,dt$$

and

$$J_{a+b-x+}^{\alpha}f(b) = \frac{1}{\Gamma(\alpha)}\int_{a+b-x}^b (b-t)^{\alpha-1}f(t)\,dt$$

for any $a < x \leq b$.

If we change the variable in the second integral to $u = a + b - t$, then we get

$$\int_{a+b-x}^{b} (b-t)^{\alpha-1} f(t)\,dt = \int_{a}^{x} (u-a)^{\alpha-1} f(a+b-u)\,du.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\left[J_{x-}^{\alpha} f(a) + J_{a+b-x+}^{\alpha} f(b)\right] \\
&= \frac{1}{2}\left[\frac{1}{\Gamma(\alpha)}\int_{a}^{x} (t-a)^{\alpha-1} f(t)\,dt + \frac{1}{\Gamma(\alpha)}\int_{a}^{x} (t-a)^{\alpha-1} f(a+b-t)\,dt\right] \\
&= \frac{1}{\Gamma(\alpha)}\int_{a}^{x} (t-a)^{\alpha-1} \frac{f(t) + f(a+b-t)}{2} dt = \frac{1}{\Gamma(\alpha)}\int_{a}^{x} (t-a)^{\alpha-1} \breve{f}(t)\,dt
\end{aligned}$$

and the representation (2.14) is obtained.

By replacing x with $a + b - x$ in (2.14) we have

$$\frac{1}{2}\left[J_{a+b-x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b)\right] = \frac{1}{\Gamma(\alpha)}\int_{a}^{a+b-x} (t-a)^{\alpha-1} \breve{f}(t)\,dt.$$

If we change the variable to $u = a + b - t$, then we get

$$\begin{aligned}
\int_{a}^{a+b-x} (t-a)^{\alpha-1} \breve{f}(t)\,dt &= \int_{x}^{b} (b-u)^{\alpha-1} \breve{f}(a+b-u)\,du \\
&= \int_{x}^{b} (b-u)^{\alpha-1} \breve{f}(u)\,du,
\end{aligned}$$

which proves the representation (2.15). □

Corollary 6. *With the assumptions of Lemma 2 we have the representations*

$$\begin{aligned}
\frac{1}{2}\left[J_{\frac{a+b}{2}-}^{\alpha} f(a) + J_{\frac{a+b}{2}+}^{\alpha} f(b)\right] &= \frac{1}{\Gamma(\alpha)}\int_{a}^{\frac{a+b}{2}} (t-a)^{\alpha-1} \breve{f}(t)\,dt \\
&= \frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b} (b-t)^{\alpha-1} \breve{f}(t)\,dt \\
&= \frac{1}{\Gamma(\alpha)}\int_{a}^{b} K(t)\,\breve{f}(t)\,dt
\end{aligned} \tag{2.16}$$

where

$$K(t) := \frac{1}{2}\begin{cases} (t-a)^{\alpha-1} & \text{if } a < t < \frac{a+b}{2}, \\ (b-t)^{\alpha-1} & \text{if } \frac{a+b}{2} \leq t < b. \end{cases}$$

We have the following Hermite-hadamard type inequalities as well:

Theorem 3. *Assume that $f : [a, b] \to \mathbb{R}$ is symmetrized convex and integrable on the interval $[a, b]$. Then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(x-a)^{\alpha}}\left[J_{x-}^{\alpha} f(a) + J_{a+b-x+}^{\alpha} f(b)\right] \leq \frac{f(a) + f(b)}{2} \tag{2.17}$$

for any $a < x \leq b$ and

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-x)^{\alpha}}\left[J_{a+b-x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b)\right] \leq \frac{f(a) + f(b)}{2} \tag{2.18}$$

for any $a \leq x < b$.

Proof. From (1.3) we have

$$f\left(\frac{a+b}{2}\right)\frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1}\,dt \le \frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1}\,\breve{f}(t)\,dt \le \frac{f(a)+f(b)}{2}\frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1}\,dt \tag{2.19}$$

for any $a < x \le b$.

Since

$$\frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1}\,dt = \frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha},$$

then by representation (2.14) and inequality (2.19) we get (2.17).

The inequality (2.18) follows in a similar way. □

Remark 3. *If we take $x = \frac{a+b}{2}$ in either (2.17) or (2.18), then we get the inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}+}f(b) + J^{\alpha}_{\frac{a+b}{2}-}f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{2.20}$$

that holds for f symmetrized convex and integrable on the interval $[a,b]$. This extends the inequality (1.8) to the larger class of symmetrized convex and integrable functions on the interval $[a,b]$.

Corollary 7. *With the assumption of Theorem 3 we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+1}}\int_a^b \frac{J^{\alpha}_{x-}f(a) + J^{\alpha}_{x+}f(b)}{2}dx \le \frac{f(a)+f(b)}{2}. \tag{2.21}$$

3. Inequalities for Wright-quasi-convex Functions

A real function f defined on some nonempty interval I of real line $\mathbb{R}$ is called *quasi-convex* on, and we write $f \in QC(I)$ if

$$f(tx + (1-t)y) \le \max\{f(x), f(y)\}$$

holds for all x, $y \in I$ and $t \in [0,1]$.

Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex. For example, consider the function $f_0(x) = \ln x$, $x > 0$. This function is quasi-convex. However f_0 is not convex functions.

The function $f : I \to \mathbb{R}$ is *Jensen-* or *J-quasi-convex* if [11]

$$f\left(\frac{x+y}{2}\right) \le \max\{f(x), f(y)\}$$

for all x, $y \in I$. We denote $f \in JQC(I)$.

For $I \subset \mathbb{R}$, the mapping $f : I \to \mathbb{R}$ is *Wright-quasi-convex* [11] if, for all x, $y \in I$ and $t \in [0,1]$, one has the inequality

$$\frac{1}{2}[f(tx + (1-t)y) + f((1-t)x + ty)] \le \max\{f(x), f(y)\}. \tag{3.1}$$

We denote $f \in WQC(I)$.

It has been shown in [11] that the following strict inclusions hold

$$QC(I) \subsetneqq WQC(I) \subsetneqq JQC(I) \tag{3.2}$$

and if a, $b \in I$ with $a < b$ and f is Wright-quasi-convex on I and integrable on $[a, b]$, then the following Hermite-Hadamard type inequality holds

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \max\{f(a), f(b)\}. \tag{3.3}$$

Theorem 4. *Assume that $f : [a, b] \to \mathbb{R}$ is Wright-quasi-convex and integrable on the interval $[a, b]$. Then we have*

$$\frac{\Gamma(\alpha+1)}{2(x-a)^{\alpha}}\left[J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(a+b-x)\right] \le \max\{f(a), f(b)\} \tag{3.4}$$

for any $a < x \le b$.

In particular, we have

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \le \max\{f(a), f(b)\} \tag{3.5}$$

and

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)\right] \le \max\{f(a), f(b)\}. \tag{3.6}$$

Proof. By (3.1) for $x = a$, $y = b$ and $t = \frac{s-a}{b-a} \in [0, 1]$ with $s \in [a, b]$, we have

$$\breve{f}(s) = \frac{1}{2}[f(a+b-s) + f(s)] \le \max\{f(a), f(b)\}. \tag{3.7}$$

By the equality (2.1) we have

$$\begin{aligned}\frac{1}{2}\left[J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(a+b-x)\right] &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}\breve{f}(s)\,ds \\ &\le \max\{f(a), f(b)\}\frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}\,ds \\ &= \max\{f(a), f(b)\}\frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)},\end{aligned}$$

which is equivalent to (3.4). □

Remark 4. *The inequality (3.5) was obtained in 2013 by Özdemir and Yildiz for the case of integrable quasi-convex functions.*

Corollary 8. *With the assumption of Theorem 4 we have*

$$\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+1}}\int_a^b \frac{J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x)}{2}dx \le \max\{f(a), f(b)\}. \tag{3.8}$$

We also have:

Theorem 5. *Assume that $f : [a, b] \to \mathbb{R}$ is Wright-quasi-convex and integrable on the interval $[a, b]$. Then we have*

$$\frac{\Gamma(\alpha+1)}{2(x-a)^{\alpha}}\left[J_{x-}^{\alpha} f(a) + J_{a+b-x+}^{\alpha} f(b)\right] \le \max\{f(a), f(b)\} \tag{3.9}$$

for any $a < x \le b$.

In particular, we have

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}+}^{\alpha} f(b) + J_{\frac{a+b}{2}-}^{\alpha} f(a)\right] \le \max\{f(a), f(b)\}. \tag{3.10}$$

The proof follows in a similar way by using the representation (2.14).

Corollary 9. *With the assumption of Theorem 5 we have*

$$\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha+1}}\int_a^b \frac{J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)}{2}dx \le \max\{f(a),f(b)\}. \tag{3.11}$$

4. Inequalities for Symmetrized h-Convexity

Assume that I and J are intervals in $\mathbb{R}, (0,1)\subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 2 ([32])**.** *Let $h:J\to[0,\infty)$ with h not identical to 0. We say that $f:I\to[0,\infty)$ is an h-convex function if for all $x,y\in I$ we have*

$$f(tx+(1-t)y)\le h(t)f(x)+h(1-t)f(y) \tag{4.1}$$

for all $t\in(0,1)$.

For some results concerning this class of functions see [32], [3], [20], [26] and [31].
We can give the following examples of h-convex functions.

Definition 3 ([15])**.** *We say that $f:I\to\mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x,y\in I$ and $t\in(0,1)$ we have*

$$f(tx+(1-t)y)\le\frac{1}{t}f(x)+\frac{1}{1-t}f(y). \tag{4.2}$$

Definition 4 ([13])**.** *We say that a function $f:I\to\mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x,y\in I$ and $t\in[0,1]$ we have*

$$f(tx+(1-t)y)\le f(x)+f(y). \tag{4.3}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions.*
We can introduce now another class of functions [9].

Definition 5. *We say that the function $f:I\to[0,\infty)$ is of s-Godunova-Levin type, with $s\in[0,1]$, if*

$$f(tx+(1-t)y)\le\frac{1}{t^s}f(x)+\frac{1}{(1-t)^s}f(y), \tag{4.4}$$

for all $t\in(0,1)$ and $x,y\in I$.

We observe that for $s=0$ we obtain the class of P-functions while for $s=1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(I)$ the class of s-Godunova-Levin functions defined on I, then we obviously have

$$P(I)=Q_0(I)\subseteq Q_{s_1}(I)\subseteq Q_{s_2}(I)\subseteq Q_1(I)=Q(I)$$

for $0\le s_1\le s_2\le 1$.

Definition 6 ([4])**.** *Let s be a real number, $s\in(0,1]$. A function $f:[0,\infty)\to[0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx+(1-t)y)\le t^sf(x)+(1-t)^sf(y)$$

for all $x,y\in[0,\infty)$ and $t\in[0,1]$.

Following [9], we can introduce the following concept generalizing the notion of h-convexity.

Definition 7. *Assume that h is as in Definition 2. We say that the function $f:[a,b]\to[0,\infty)$ is h-symmetrized convex (concave) on the interval $[a,b]$ if the symmetrical transform $\breve{f}$ is h-convex (concave) on $[a,b]$.*

Now, if we denote by $Con_h[a,b]$ the closed convex cone of h-convex functions defined on $[a,b]$ and by $SCon_h[a,b]$ the class of h-symmetrized convex, then, as in the previous section, we can conclude in general that

$$Con_h[a,b]\subsetneqq SCon_h[a,b].$$

We have the following results as well [9]:

Theorem 6. *Assume that h is as in Definition 2. If the function $f:[a,b]\to[0,\infty)$ is h-symmetrized convex on the interval $[a,b]$, then we have the bounds*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\le\breve{f}(x)\le\left[h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right]\frac{f(a)+f(b)}{2}\tag{4.5}$$

for any $x\in[a,b]$.

Corollary 10. *Assume that the function $f:[a,b]\to[0,\infty)$ is h-symmetrized convex on the interval $[a,b]$ with h integrable on $[0,1]$ and f integrable on $[a,b]$. If $w:[a,b]\to[0,\infty)$ is integrable on $[a,b]$, then*

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt&\le\int_a^b w(t)\,\breve{f}(t)\,dt\\&\le[f(a)+f(b)]\int_a^b h\left(\frac{t-a}{b-a}\right)\breve{w}(t)\,dt.\end{aligned}\tag{4.6}$$

Moreover, if w is symmetric almost everywhere on $[a,b]$, then

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt&\le\int_a^b w(t)\,f(t)\,dt\\&\le[f(a)+f(b)]\int_a^b h\left(\frac{t-a}{b-a}\right)w(t)\,dt.\end{aligned}\tag{4.7}$$

We observe that if the function $f:[a,b]\to[0,\infty)$ is h-symmetrized convex on $[a,b]$, then by (4.5) we have

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\le\breve{f}(t)\le\left[h\left(\frac{b-t}{b-a}\right)+h\left(\frac{t-a}{b-a}\right)\right]\frac{f(a)+f(b)}{2}\tag{4.8}$$

for any $t\in[a,b]$.

Using the representation (2.1) we have by (4.8) that

$$\begin{aligned}\frac{1}{2}\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(a+b-x)\right]&\ge\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}\,dt\\&=\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha}\end{aligned}\tag{4.9}$$

giving that

$$\frac{1}{2}\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(a+b-x)\right]\ge\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\tag{4.10}$$

for any $a < x \leq b$, where $f : [a,b] \to [0,\infty)$ is integrable and h-symmetrized convex on $[a,b]$.

Integrating over $x \in [a,b]$ in the inequality (4.9), we get

$$\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha}} \int_a^b \frac{J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x)}{2} dx \geq \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right). \tag{4.11}$$

For the same class of functions we also have

$$\begin{aligned} &\frac{1}{2}\left[J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(a+b-x)\right] \\ &\leq \frac{f(a)+f(b)}{2} \frac{1}{\Gamma(\alpha)} \int_a^x \left[h\left(\frac{b-t}{b-a}\right) + h\left(\frac{t-a}{b-a}\right)\right] (x-t)^{\alpha-1} dt \end{aligned} \tag{4.12}$$

for any $a < x \leq b$.

If we consider the change of variable $t = (1-s)a + sx$ with $s \in [0,1]$, then $dt = (x-a)\,ds$, $\frac{b-t}{b-a} = 1 - \frac{x-a}{b-a}s$, $\frac{t-a}{b-a} = \frac{x-a}{b-a}s$ and $x - t = (1-s)(x-a)$ and by (4.12) we have

$$\begin{aligned} &\frac{1}{2}\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}\left[J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(a+b-x)\right] \\ &\leq \alpha\frac{f(a)+f(b)}{2} \int_0^1 \left[h\left(1 - \frac{x-a}{b-a}s\right) + h\left(\frac{x-a}{b-a}s\right)\right] (1-s)^{\alpha-1} ds \end{aligned} \tag{4.13}$$

for any $a < x \leq b$.

From (4.10) and (4.13) for $x = b$ we get

$$\begin{aligned} &\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \\ &\leq \alpha\frac{f(a)+f(b)}{2} \int_0^1 \left[h(1-s) + h(s)\right] (1-s)^{\alpha-1} ds \end{aligned} \tag{4.14}$$

that was obtained in [30].

From (4.10) and (4.13) for $x = \frac{a+b}{2}$ we get

$$\begin{aligned} &\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \\ &\leq 2^{\alpha-1}\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)\right] \\ &\leq \alpha\frac{f(a)+f(b)}{2} \int_0^1 \left[h\left(1 - \frac{1}{2}s\right) + h\left(\frac{1}{2}s\right)\right] (1-s)^{\alpha-1} ds. \end{aligned} \tag{4.15}$$

By taking the integral mean in (4.12) we get

$$\begin{aligned} &\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha}} \int_a^b \frac{J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x)}{2} dx \\ &\leq \alpha(\alpha+1)\frac{f(a)+f(b)}{2} \\ &\times \frac{1}{(b-a)^{\alpha}} \int_a^b \left(\int_a^x \left[h\left(\frac{b-t}{b-a}\right) + h\left(\frac{t-a}{b-a}\right)\right] (x-t)^{\alpha-1} dt\right) dx. \end{aligned} \tag{4.16}$$

From (4.8) and the representation (2.14) we have

$$\begin{aligned}\frac{1}{2}\left[J_{x-}^{\alpha}f(a)+J_{a+b-x+}^{\alpha}f(b)\right] &\geq \frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\frac{1}{\Gamma(\alpha)}\int_a^x(t-a)^{\alpha-1}dt\\ &= \frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha},\end{aligned}$$

which gives

$$\frac{1}{2}\Gamma(\alpha+1)\left[J_{x-}^{\alpha}f(a)+J_{a+b-x+}^{\alpha}f(b)\right]\geq\frac{1}{2h\left(\frac{1}{2}\right)}(x-a)^{\alpha}f\left(\frac{a+b}{2}\right) \tag{4.17}$$

for any $a<x\leq b$, where $f:[a,b]\to[0,\infty)$ is integrable and h-symmetrized convex on $[a,b]$.

If we take the integral in 4.17, then we get

$$\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha}}\int_a^b\frac{J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)}{2}dx\geq\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right). \tag{4.18}$$

For the same class of functions we also have

$$\begin{aligned}&\frac{1}{2}\left[J_{x-}^{\alpha}f(a)+J_{a+b-x+}^{\alpha}f(b)\right]\\ &\leq\frac{f(a)+f(b)}{2}\frac{1}{\Gamma(\alpha)}\int_a^x\left[h\left(\frac{b-t}{b-a}\right)+h\left(\frac{t-a}{b-a}\right)\right](t-a)^{\alpha-1}dt\end{aligned} \tag{4.19}$$

for any $a<x\leq b$.

If we change the variable $t=(1-s)a+sb$, then $dt=(b-a)ds$, $t-a=s(b-a)$, $\frac{t-a}{b-a}=s$ and $\frac{b-t}{b-a}=1-s$. By (4.19) we have

$$\begin{aligned}&\frac{1}{2}\Gamma(\alpha+1)\left[J_{x-}^{\alpha}f(a)+J_{a+b-x+}^{\alpha}f(b)\right]\\ &\leq\alpha\frac{f(a)+f(b)}{2}(b-a)^{\alpha}\int_0^{\frac{x-a}{b-a}}[h(1-s)+h(s)]s^{\alpha-1}dt\end{aligned} \tag{4.20}$$

for any $a<x\leq b$.

From (4.17) and (4.20) for $x=\frac{a+b}{2}$ we get

$$\begin{aligned}\frac{1}{2^{\alpha+1}h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)&\leq\frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}-}^{\alpha}f(a)+J_{\frac{a+b}{2}+}^{\alpha}f(b)\right]\\ &\leq\alpha\frac{f(a)+f(b)}{2}\int_0^{\frac{1}{2}}[h(1-s)+h(s)]s^{\alpha-1}dt.\end{aligned} \tag{4.21}$$

Finally, if we take the integral in (4.20), then we get

$$\begin{aligned}&\frac{\Gamma(\alpha+2)}{(b-a)^{\alpha}}\int_a^b\frac{J_{x-}^{\alpha}f(a)+J_{a+b-x+}^{\alpha}f(b)}{2}dx\\ &\leq\alpha(\alpha+1)\frac{f(a)+f(b)}{2}\int_a^b\left(\int_0^{\frac{x-a}{b-a}}[h(1-s)+h(s)]s^{\alpha-1}dt\right)dx.\end{aligned} \tag{4.22}$$

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