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HERMITE-HADAMARD TYPE INEQUALITIES FOR FRACTIONAL INTEGRALS OPERATORS

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Abstract. Several Hermite-Hadamard type inequalities will be given in this paper for n-time differentiable functions whose n-time derivative in absolute value satisfy different kind of convexities via Riemann-Liouville fractional integral operators.

1. Introduction

The inequality of Hermite-Hadamard type has been considered very useful in mathematical analysis being very intensely studied, extended and generalized in many directions by many authors, see [24, 7, 6, 10, 1, 14, 18, 25, 12] and the references therein.

Many papers study the Riemann-Liouville fractionals integrals and give new and interesant generalizations of Hermite-Hadamard type inequalities using these kind of integrals, see for instance [9, 8, 10, 11, 12, 19, 16, 18, 14, 24, 25, 26, 27, 28, 21, 30].

We will begin now by recalling the classical definition for the convex functions and then the definitions for other kind of convexities.

Definition 1. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. The function f is said to be concave on I if the inequality (1) takes place in reversed direction.

It is necessary to recall below also the definition of fractionals integrals, see [9, 11, 10, 19, 20, 26] and then the definition of fractional integral operators. For other type of convexity see also [22, 17].

Definition 2. *A function $f : [a, b] \to \mathbb{R}$ is said to be quasi-convex onl $[a, b]$ if*

$$f(tx + (1-t)y) \leq \sup\{f(x), f(y)\}$$

holds for all $x, y \in [a, b]$ and $t \in [0, 1]$.

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Definition 3. *A function $f : I \to \mathbb{R}$ is said to be P-convex on $[a, b]$ if it is nonnegative and for all $x, y \in I$ and $\lambda \in [9, 1]$*

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

Definition 4. *A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be s-convex in the first sense on an interval I if the inequality*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t^s) f(y)$$

holds for all $x, y \in I$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$.

Definition 5. *A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be s-convex in the second sense on an interval I if the inequality*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$.

Definition 6. *A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be s-Godunova-Levin functions of second kind on an interval I if the inequality*

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y)$$

holds for all $x, y \in I$, $t \in (0, 1)$ and for some fixed $s \in [0, 1]$.

It is easy to see that for $s = 0$ s-Godunova-Levin functions of second kind are functions P-convex.

The classical Hermite-Hadamard's inequality for convex functions is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \tag{2}$$

Moreover, if the function f is concave then the inequality (2) hold in reversed direction.

Definition 7. *Let $f \in L[a, b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $\alpha \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$ and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

It is well-known that the beta function is defined when $a, b > 0$ by

$$R(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1dt}.$$

The following class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k+\lambda)} x^k \qquad (\rho,\ \lambda > 0;\ |x| < \mathbf{R}),$$

where the coefficients $\sigma(k),\ (k \in \mathbb{N} = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers and $\mathbf{R}$ is the set of real numbers, as in [21], was introduced in [29] and was used for giving in [3] the following left-sided and right-sided fractional integral operators from below:

$$(\mathcal{J}^{\sigma}_{\rho,\lambda,a^+;w}\varphi)(x) = \int_a^x (x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(x-t)^{\rho}]\varphi(t)dt, \quad (x > a > 0),$$

and

$$(\mathcal{J}^{\sigma}_{\rho,\lambda,b^-;w}\varphi)(x) = \int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\varphi(t)dt, \quad (0 < x < b),$$

where $\rho, \lambda > 0$, $w \in \mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exists. There are new integral inequalities for this operator, seet [21, 3, 30] and references therein.

It is important to mention that for example the classical Riemann-Liouville fractional integrals $J^{\alpha}_{a^+}$ and $J^{\alpha}_{b^-}$ of order α were obtained by setting $\lambda = \alpha,\ sigma(0) = 1$ and $w = 0$ in previous integrals.

In this paper, two new identities are given and then some applications, like Hermite-Hadamard type inequalities for functions whose the n-time derivative iin absolute value of certain powers satisfies different type of convexities via Riemann-Liouville fractional integral operators are established.

2. **Main results**

The following result is a generalization of Lemma 1 from [5] for fractional integral operators for functions n-time differentiable.

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be an n-time differentiable mapping on (a,b) with $0 < a < b,\ \lambda > n-1, x \in (a,b)$ and $t \in [0,1]$. If $f^{(n)} \in L[a,b]$ then the following equality for generalized fractional integrals holds:*

$$\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[(x-a)^{\rho}t^{\rho}]f^{(n)}(tx+(1-t)a)dt+$$

$$+\int_0^1 (1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[(b-x)^{\rho}(1-t)^{\rho}]f^{(n)}(tb+(1-t)x)dt =$$

$$= \sum_{k=1}^{n}\{\frac{(-1)^{k-1}}{(x-a)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[(x-a)^{\rho}] - \frac{1}{(b-x)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[(b-x)^{\rho}]\}f^{(n-k)}(x)+$$

$$+\frac{(-1)^n}{(x-a)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^-;w}f\right)(a) + \frac{1}{(b-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^+;w}f\right)(b).$$

Proof. As in [21], we compute first

$$\int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[(x-a)^\rho t^\rho] f^{''}(tx+(1-t)a)dt$$

and then we will prove by induction that

$$I_1 = \int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[(x-a)^\rho t^\rho] f^{(n)}(tx+(1-t)a)dt =$$

$$= \sum_{k=1}^n \frac{(-1)^{k-1}}{(x-a)^k} f^{(n-k)}(x) \mathcal{F}^\sigma_{\rho,\lambda-k+2}[(x-a)^\rho] + \frac{(-1)^n}{(x-a)^{\lambda+1}} \left(\mathcal{J}^\sigma_{\rho,\lambda-n+1,x^-;w} f\right)(a).$$

Integrating by parts and then changing variables with $u = tx + (1-t)a$ we get

$$\int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[(x-a)^\rho t^\rho] f^{''}(tx+(1-t)a)dt =$$

$$= \mathcal{F}^\sigma_{\rho,\lambda+1}[(x-a)^\rho] \frac{f^{'}(x)}{x-a} - \frac{f(x)}{(x-a)^2} \mathcal{F}^\sigma_{\rho,\lambda}[(x-a)^\rho] +$$

$$+ \frac{1}{(x-a)^2} \int_0^1 t^{\lambda-2} \mathcal{F}^\sigma_{\rho,\lambda-1}[(x-a)^\rho t^\rho] f(tx+(1-t)a)dt$$

or

$$\int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[(x-a)^\rho t^\rho] f^{''}(tx+(1-t)a)dt =$$

$$= \mathcal{F}^\sigma_{\rho,\lambda+1}[(x-a)^\rho] \frac{f^{'}(x)}{x-a} - \frac{f(x)}{(x-a)^2} \mathcal{F}^\sigma_{\rho,\lambda}[(x-a)^\rho] +$$

$$+ \frac{1}{(x-a)^{\lambda+1}} \left(\mathcal{J}^\sigma_{\rho,\lambda-1,x^-;w} f\right)(a).$$

Analogously, by using the same method we get:

$$\int_0^1 (1-t)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[(b-x)^\rho (1-t)^\rho] f^{''}(tb+(1-t)x)dt =$$

$$= -\mathcal{F}^\sigma_{\rho,\lambda+1}[(b-x)^\rho] \frac{f^{'}(x)}{b-x} - \frac{f(x)}{(b-x)^2} \mathcal{F}^\sigma_{\rho,\lambda}[(b-x)^\rho] +$$

$$+ \frac{1}{(b-x)^2} \int_0^1 (1-t)^{\lambda-2} \mathcal{F}^\sigma_{\rho,\lambda-1}[(b-x)^\rho (1-t)^\rho] f(tb+(1-t)x)dt.$$

or by substitution $u = tb + (1-t)x$,

$$\int_0^1 (1-t)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[(b-x)^\rho (1-t)^\rho] f^{''}(tb+(1-t)x)dt =$$

$$= -\mathcal{F}^\sigma_{\rho,\lambda+1}[(b-x)^\rho] \frac{f^{'}(x)}{b-x} - \frac{f(x)}{(b-x)^2} \mathcal{F}^\sigma_{\rho,\lambda}[(b-x)^\rho] +$$

$$+ \frac{1}{(b-x)^{\lambda+1}} \left(\mathcal{J}^\sigma_{\rho,\lambda-1,x^+;w} f\right)(b).$$

Therefore by induction we have,

$$I_2 = -\sum_{k=1}^n f^{(n-k)}(x) \frac{1}{(b-x)^k} \mathcal{F}^\sigma_{\rho,\lambda-k+2}[(b-x)^\rho] + \frac{1}{(b-x)^{\lambda+1}} \left(\mathcal{J}^\sigma_{\rho,\lambda-n+1,x^+;w} f\right)(b).$$

Now summing I_1 and I_2 we obtain the desired equality.

∎

Using this lemma we obtain the following result for n-time differentiable functions whose absolute value is convex via fractional integral operator.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be an n-time differentiable mapping on (a,b) with $0<a<b$, $\lambda>n-1, x\in(a,b)$ and $t\in[0,1]$. If $f^{(n)}\in L[a,b]$ and $|f^{(n)}|$ is convex on (a,b) then the following inequality for generalized fractional integral operators takes place:*

$$|\sum_{k=1}^{n}(-1)^{k-1}f^{(n-k)}(x)\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[(x-a)^{\rho}]}{(x-a)^{k}}-\frac{\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[(b-x)^{\rho}]}{(b-x)^{k}}\}+$$
$$+\frac{(-1)^{n}}{(x-a)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^{-};w}f\right)(a)+\frac{1}{(b-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^{+};w}f\right)(b)|\leq$$
$$\leq\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]\left(\frac{|f^{(n)}(x)|}{\lambda+2}+\frac{|f^{(n)}(a)|}{(\lambda+1)(\lambda+2)}\right)+$$
$$+\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]\left(\frac{|f^{(n)}(x)|}{\lambda+2}+\frac{|f^{(n)}(b)|}{(\lambda+1)(\lambda+2)}\right).$$

Proof. Using the properties of modulus, Lemma 1 and that $|f^{(n)}|$ is convex function we get:

$$|\sum_{k=1}^{n}(-1)^{k-1}f^{(n-k)}(x)\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[(x-a)^{\rho}]}{(x-a)^{k}}-\frac{\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[(b-x)^{\rho}]}{(b-x)^{k}}\}+$$
$$+\frac{(-1)^{n}}{(x-a)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^{-};w}f\right)(a)+\frac{1}{(b-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^{+};w}f\right)(b)|=$$
$$=|I_1+I_2|\leq\int_0^1 t^{\lambda}|\mathcal{F}^{\sigma}_{\rho,\lambda+1}[(x-a)^{\rho}t^{\rho}]f^{(n)}(tx+(1-t)a)|dt+$$
$$+\int_0^1(1-t)^{\lambda}|\mathcal{F}^{\sigma}_{\rho,\lambda+1}[(b-x)^{\rho}(1-t)^{\rho}]f^{(n)}(tb+(1-t)x)|dt\leq$$
$$\leq\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\left(|f^{(n)}(x)|\int_0^1 t^{\lambda+1}dt+|f^{(n)}(a)|\int_0^1 t^{\lambda}(1-t)dt\right)+$$
$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^{k}(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\left(|f^{(n)}(b)|\int_0^1(1-t)^{\lambda}tdt+|f^{(n)}(x)|\int_0^1(1-t)^{\lambda+1}dt\right).$$

From here by easily calculus we get the desired inequality.

∎

BY this lemma we also obtain the following result for n-time differentiable functions whose absolute value is s-convex in the second sense via fractional integral operator.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be an n-time differentiable mapping on (a,b) with $0<a<b$, $\lambda>n-1, x\in(a,b)$, $s\in(0,1]$ and $t\in[0,1]$. If $f^{(n)}\in L[a,b]$ and $|f^{(n)}|$ is s-convex in the second sense on (a,b) then the following inequality for generalized fractional integral operators takes place:*

$$|\sum_{k=1}^{n}(-1)^{k-1}f^{(n-k)}(x)\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[(x-a)^{\rho}]}{(x-a)^{k}}-\frac{\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[(b-x)^{\rho}]}{(b-x)^{k}}\}+$$

$$+\frac{(-1)^n}{(x-a)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^-;w}f\right)(a)+\frac{1}{(b-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^+;w}f\right)(b)|\leq$$

$$\leq\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}]\left(\frac{|f^{(n)}(x)|}{\lambda+s+1}+|f^{(n)}(a)|B(\lambda+1,s+1)\right)+$$

$$+\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}]\left(\frac{|f^{(n)}(x)|}{\lambda+s+1}+|f^{(n)}(b)|B(\lambda+1,s+1)\right).$$

Proof. We use the same method as in Theorem 1, but this time we apply the definition of s-convex function in the second sense. ∎

Next result is a generalization of Lemma 4 from [4] for fractional integral operators for functions n-time differentiable.

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be an n-time differentiable mapping on (a,b) with $0<a<b$, $\lambda>n-1, x\in(a,b)$ and $t,\ r\in[0,1]$. If $f^{(n)}\in L[a,b]$ then the following equality for generalized fractional integrals holds:*

$$\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[(1-r)^{\rho}(x-a)^{\rho}t^{\rho}]f^{(n)}(t(ra+(1-r)x)+(1-t)a)dt+$$

$$+\int_0^1(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[r^{\rho}(x-a)^{\rho}(1-t)^{\rho}]f^{(n)}(tx+(1-t)(ra+(1-r)x))dt+$$

$$+\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[(1-r)^{\rho}(b-x)^{\rho}t^{\rho}]f^{(n)}(tb+(1-t)(rx+(1-r)b))dt+$$

$$+\int_0^1(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[r^{\rho}(b-x)^{\rho}(1-t)^{\rho}]f^{(n)}(tb+(1-t)(rx+(1-r)b))dt=$$

$$=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{(1-r)^k}\{\frac{f^{(n-k)}(ra+(1-r)x)}{(x-a)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[(1-r)^{\rho}(x-a)^{\rho}]+$$

$$+\frac{f^{(n-k)}(rx+(1-r)b)}{(b-x)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[(1-r)^{\rho}(b-x)^{\rho}]\}-$$

$$-\sum_{k=1}^{n}\frac{1}{r^k}\{\frac{f^{(n-k)}(ra+(1-r)x)}{(x-a)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[r^{\rho}(x-a)^{\rho}]+$$

$$+\frac{f^{(n-k)}(rx+(1-r)b)}{(b-x)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[r^{\rho}(b-x)^{\rho}]\}+$$

$$+\frac{(-1)^n}{(1-r)^{\lambda+1}(x-a)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(ra+(1-r)x)^-;w}f)(a)+$$

$$+\frac{1}{r^{\lambda+1}(x-a)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(ra+(1-r)x)^+;w}f)(x)+$$

$$+\frac{(-1)^n}{(1-r)^{\lambda+1}(b-x)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(rx+(1-r)b)^-;w}f)(x)+$$

$$+\frac{1}{r^{\lambda+1}(b-x)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(rx+(1-r)b)^+;w}f)(b).$$

Proof. We denote

$$I_1=\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[(1-r)^{\rho}(x-a)^{\rho}t^{\rho}]f^{(n)}(t(ra+(1-r)x)+(1-t)a)dt,$$

$$I_2=\int_0^1 (1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[r^{\rho}(x-a)^{\rho}(1-t)^{\rho}]f^{(n)}(tx+(1-t)(ra+(1-r)x))dt,$$

$$I_3=\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[(1-r)^{\rho}(b-x)^{\rho}t^{\rho}]f^{(n)}(tb+(1-t)(rx+(1-r)b))dt$$

and

$$I_4=\int_0^1 (1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[r^{\rho}(b-x)^{\rho}(1-t)^{\rho}]f^{(n)}(tb+(1-t)(rx+(1-r)b))dt.$$

As in Lemma 1 we prove by induction that

$$I_1=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{(1-r)^k(x-a)^k}f^{(n-k)}(ra+(1-r)x)\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[(1-r)^{\rho}(x-a)^{\rho}]+$$
$$+\frac{(-1)^n}{(1-r)^{\lambda+1}(x-a)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(ra+(1-r)x)^-;w}f)(a)$$

and then similarly we can find $I_2,\ I_3$ and I_4. Therefore we have:

$$I_2=-\sum_{k=1}^{n}\frac{1}{r^k(x-a)^k}f^{(n-k)}(ra+(1-r)x)\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[r^{\rho}(x-a)^{\rho}]+$$
$$+\frac{(-1)^n}{r^{\lambda+1}(x-a)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(ra+(1-r)x)^+;w}f)(x)$$

Summing now I_1, I_2 I_3 and I_4 we find the desired equality.
∎

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be an n-time differentiable mapping on (a,b) with $0<a<b$, $\lambda>n-1, x\in(a,b)$ and $t,r\in[0,1]$. If $f^{(n)}\in L[a,b]$ and $|f^{(n)}|$ is convex on (a,b) then the following inequality for generalized fractional integral operators takes place:*

$$|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{(1-r)^k}\{\frac{f^{(n-k)}(ra+(1-r)x)}{(x-a)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[(1-r)^{\rho}(x-a)^{\rho}]+$$
$$+\frac{f^{(n-k)}(rx+(1-r)b)}{(b-x)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[(1-r)^{\rho}(b-x)^{\rho}]\}-$$
$$-\sum_{k=1}^{n}\frac{1}{r^k}\{\frac{f^{(n-k)}(ra+(1-r)x)}{(x-a)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[r^{\rho}(x-a)^{\rho}]+$$
$$+\frac{f^{(n-k)}(rx+(1-r)b)}{(b-x)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[r^{\rho}(b-x)^{\rho}]\}+$$
$$+\frac{(-1)^n}{(1-r)^{\lambda+1}(x-a)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(ra+(1-r)x)^-;w}f)(a)+$$
$$+\frac{1}{r^{\lambda+1}(x-a)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(ra+(1-r)x)^+;w}f)(x)+$$
$$+\frac{(-1)^n}{(1-r)^{\lambda+1}(b-x)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(rx+(1-r)b)^-;w}f)(x)+$$

$$+\frac{1}{r^{\lambda+1}(b-x)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(rx+(1-r)b)^{+};w}f)(b)|\leq$$
$$\leq \mathcal{F}^{\sigma}_{\rho,\lambda+1}[(1-r)^{\rho}(x-a)^{\rho}w]\left(\frac{|f^{(n)}(ra+(1-r)x)|}{\lambda+2}+\frac{|f^{(n)}(a)|}{(\lambda+1)(\lambda+2)}\right)+$$
$$+\mathcal{F}^{\sigma}_{\rho,\lambda+1}[r^{\rho}(x-a)^{\rho}w]\left(\frac{|f^{(n)}(x)|}{(\lambda+2)(\lambda+1)}+\frac{|f^{(n)}(ra+(1-r)x)|}{\lambda+2}\right)+$$
$$+\mathcal{F}^{\sigma}_{\rho,\lambda+1}[(1-r)^{\rho}(b-x)^{\rho}w]\left(\frac{|f^{(n)}(rx+(1-r)b)|}{\lambda+2}+\frac{|f^{(n)}(x)|}{(\lambda+1)(\lambda+2)}\right)+$$
$$+\mathcal{F}^{\sigma}_{\rho,\lambda+1}[r^{\rho}(b-x)^{\rho}w]\left(\frac{|f^{(n)}(b)|}{(\lambda+2)(\lambda+1)}+\frac{|f^{(n)}(rx+(1-r)b)|}{\lambda+2}\right).$$

Proof. We use the same method as in Theorem 1, we shall apply Lemma 2 and the definition of the convex functions.

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