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OSTROWSKI TYPE INEQUALITIES FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF ABSOLUTELY CONTINUOUS FUNCTIONS IN TERMS OF ∞-NORM

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Abstract. In this paper we establish some Ostrowski type inequalities for the Riemann-Liouville fractional integrals of absolutely continuous functions in terms of of ∞-norm. Applications for mid-point inequalities are provided as well. They generalize the know results holding for the classical Riemann integral.

1. Introduction

In 1938, A. Ostrowski [20], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1 (Ostrowski, 1938 [20]). *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.1}$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

In [14], S. S. Dragomir and S. Wang, by the use of the *Montgomery integral identity* [18, p. 565],

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b p(x,t) f'(t)\,dt, \quad x \in [a,b],$$

where $p : [a,b]^2 \to \mathbb{R}$ is given by

$$p(x,t) := \begin{cases} t-a & \text{if} \quad t \in [a,x], \\ t-b & \text{if} \quad t \in (x,b], \end{cases}$$

gave a simple proof of Ostrowski's inequality and applied it for special means (identric mean, logarithmic mean, etc.) and to the problem of estimating the error bound in approximating the Riemann integral $\int_a^b f(t)\,dt$ by one arbitrary Riemann sum (see [14], Section 3).

The following result, which is an improvement on Ostrowski's inequality, holds.

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Theorem 2 (Dragomir, 2002 [11])**.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ whose derivative $f'\in L_\infty[a,b]$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.2}$$
$$\leq \frac{1}{2(b-a)}\left[\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2\right]$$
$$\leq \begin{cases} \|f'\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a); \\ \frac{1}{2}\left[\|f'\|^p_{[a,x],\infty}+\|f'\|^p_{[x,b],\infty}\right]^{\frac{1}{p}}\left[\left(\frac{x-a}{b-a}\right)^{2q}+\left(\frac{b-x}{b-a}\right)^{2q}\right]^{\frac{1}{q}}(b-a), \\ \text{where } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,b],\infty}\right]\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^2(b-a) \end{cases}$$

for all $x\in[a,b]$, where $\|\cdot\|_{[m,n],\infty}$ denotes the usual ∞-norm on $L_\infty[m,n]$, i.e., we recall that

$$\|g\|_{[m,n],\infty}=\operatorname*{essup}_{t\in[m,n]}|g(t)|<\infty.$$

Corollary 1. *With the assumptions of Theorem 2 we have the mid-point inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{8}\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right](b-a) \tag{1.3}$$
$$\leq\frac{1}{4}\|f'\|_{[a,b],\infty}(b-a).$$

For other Ostrowski type inequalities for Lebesgue integral, see [10], [6] and the recent survey [13].

In order to extend these results for the fractional integrals we need the following preparation.

Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J^\alpha_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt$$

for $a<x\leq b$ and

$$J^\alpha_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt$$

for $a\leq x<b$, where Γ is the *Gamma function.* For $\alpha=0$, they are defined as

$$J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)\ \text{ for } x\in(a,b).$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[5], [16]-[28] and the references therein.

Motivated by the above results, in this paper we establish some Ostrowski type inequalities for the Riemann-Liouville fractional integrals of absolutely continuous functions in terms of ∞-norm. Applications for mid-point inequalities are provided as well.

2. Some Identities

We start with the following simple identities:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be a complex valued absolutely continuous function on the real interval $[a,b]$.*

(i) *For any $x\in(a,b)$ we have*

$$J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)=\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]-\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\left(\int_{t}^{x}f'(s)\,ds\right)dt+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}\left(\int_{x}^{t}f'(s)\,ds\right)dt. \tag{2.1}$$

(ii) *For any $x\in(a,b)$ we have*

$$J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)=\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]-\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(t-a)^{\alpha-1}\left(\int_{t}^{x}f'(s)\,ds\right)dt+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(b-t)^{\alpha-1}\left(\int_{x}^{t}f'(s)\,ds\right)dt. \tag{2.2}$$

(iii) *For any $x\in[a,b]$ we have*

$$\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}=\frac{1}{\Gamma(\alpha+1)}f(x)(b-a)^{\alpha}+\frac{1}{\Gamma(\alpha)}\int_{a}^{b}\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\left(\int_{x}^{t}f'(s)\,ds\right)dt. \tag{2.3}$$

Proof. Since f is absolutely continuous on $[a,b]$, then f' exists almost everywhere on $[a,b]$ and for any $x,\ y\in[a,b]$ we have

$$f(y)-f(x)=\int_{x}^{y}f'(s)\,ds.$$

(i) We have successively that

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\left(\int_{x}^{t}f'(s)\,ds\right)dt \\
&=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\left[f(t)-f(x)\right]dt \\
&=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)\,dt-f(x)\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\,dt \\
&=J_{a+}^{\alpha}f(x)-f(x)\frac{1}{\Gamma(\alpha)}\frac{(x-a)^{\alpha}}{\alpha}=J_{a+}^{\alpha}f(x)-f(x)\frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha}
\end{aligned} \tag{2.4}$$

for $a < x \le b$ and, similarly

$$\frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left(\int_x^t f'(s)\, ds \right) dt = J_{b-}^{\alpha} f(x) - f(x) \frac{1}{\Gamma(\alpha+1)} (b-x)^{\alpha} \tag{2.5}$$

for $a \le x < b$.

By adding these equalities for $x \in (a,b)$ we get the representation (2.1).

(ii) We have

$$J_{x+}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} f(t)\, dt$$

for $a \le x < b$ and

$$J_{x-}^{\alpha} f(a) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} f(t)\, dt$$

for $a < x \le b$.

Then

$$\frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} \left(\int_x^t f'(s)\, ds \right) dt = J_{x+}^{\alpha} f(b) - f(x) \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)}$$

for $a \le x < b$ and

$$\frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} \left(\int_x^t f'(s)\, ds \right) dt = J_{x-}^{\alpha} f(a) - f(x) \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}$$

for $a < x \le b$.

By adding these equalities for $x \in (a,b)$ we get the representation (2.2).

(iii) We have for any $x \in [a,b]$ that

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_a^b \frac{(b-t)^{\alpha-1} + (t-a)^{\alpha-1}}{2} \left(\int_x^t f'(s)\, ds \right) dt \\
&= \frac{1}{\Gamma(\alpha)} \int_a^b \frac{(b-t)^{\alpha-1} + (t-a)^{\alpha-1}}{2} [f(t) - f(x)]\, dt \\
&= \frac{1}{\Gamma(\alpha)} \int_a^b \frac{(b-t)^{\alpha-1} + (t-a)^{\alpha-1}}{2} f(t)\, dt \\
&- f(x) \frac{1}{\Gamma(\alpha)} \int_a^b \frac{(b-t)^{\alpha-1} + (t-a)^{\alpha-1}}{2} dt \\
&= \frac{1}{2\Gamma(\alpha)} \int_a^b \left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1} \right] f(t)\, dt - \frac{1}{\Gamma(\alpha+1)} f(x) (b-a)^{\alpha} \\
&= \frac{J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)}{2} - \frac{1}{\Gamma(\alpha+1)} f(x) (b-a)^{\alpha},
\end{aligned}$$

which proves the equality (2.3). □

Corollary 2. *With the assumption of Lemma 1, we have the particular mid-point equalities*

$$
\begin{aligned}
(2.6)\qquad & J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) \\
&= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \\
&- \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right)^{\alpha-1} \left(\int_t^{\frac{a+b}{2}} f'(s)\, ds\right) dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right)^{\alpha-1} \left(\int_{\frac{a+b}{2}}^{t} f'(s)\, ds\right) dt,
\end{aligned}
$$

$$
\begin{aligned}
(2.7)\qquad & J_{\frac{a+b}{2}+}^{\alpha} f(b) + J_{\frac{a+b}{2}-}^{\alpha} f(a) \\
&= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \\
&- \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} (t-a)^{\alpha-1} \left(\int_t^{\frac{a+b}{2}} f'(s)\, ds\right) dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^{b} (b-t)^{\alpha-1} \left(\int_{\frac{a+b}{2}}^{t} f'(s)\, ds\right) dt
\end{aligned}
$$

and

$$
\begin{aligned}
(2.8)\qquad & \frac{J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)}{2} \\
&= \frac{1}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)(b-a)^{\alpha} \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^{b} \frac{(b-t)^{\alpha-1} + (t-a)^{\alpha-1}}{2} \left(\int_{\frac{a+b}{2}}^{t} f'(s)\, ds\right) dt.
\end{aligned}
$$

Remark 1. *If we use the change of variable $s = (1-u)x + ut$, $u \in [0,1]$, then*

$$\int_x^t f'(s)\, ds = (t-x) \int_0^1 f'((1-u)x + ut)\, du.$$

From (2.1) we have

$$
\begin{aligned}
(2.9)\qquad J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) &= \frac{f(x)}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha}\right] \\
&- \frac{1}{\Gamma(\alpha)} \int_a^{x} (x-t)^{\alpha} \left(\int_0^1 f'((1-u)x + ut)\, du\right) dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_x^{b} (t-x)^{\alpha} \left(\int_0^1 f'((1-u)x + ut)\, du\right) dt
\end{aligned}
$$

for any $x \in (a,b)$.

From (2.2) we have

$$J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a) = \frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right] - \frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1}(x-t)\left(\int_0^1 f'((1-u)x+ut)\,du\right)dt + \frac{1}{\Gamma(\alpha)}\int_x^b (b-t)^{\alpha-1}(t-x)\left(\int_0^1 f'((1-u)x+ut)\,du\right)dt. \tag{2.10}$$

for any $x\in(a,b)$.

For any $x\in[a,b]$ *we have*

$$\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2} = \frac{1}{\Gamma(\alpha+1)}f(x)(b-a)^{\alpha} + \frac{1}{\Gamma(\alpha)}\int_a^b \frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}(t-x) \times\left(\int_0^1 f'((1-u)x+ut)\,du\right)dt. \tag{2.11}$$

If we take in these identities $x=\frac{a+b}{2}$, *we can get the corresponding mid-point equalities. The details are omitted.*

3. Inequalities in Terms of ∞-Norm

We consider the following Lebesgue norms for complex valued functions $g:[a,b]\to\mathbb{C}$

$$\|g\|_{[a,b],\infty}:=\operatorname*{essup}_{t\in[a,b]}|g(t)| \text{ if } g\in L_{\infty}[a,b]$$

and

$$\|g\|_{[a,b],p}:=\left(\int_a^b |g(t)|^p\,dt\right)^{1/p},\text{ if } g\in L_p[a,b],\ p\geq 1.$$

Consider the *Beta function*

$$B(\alpha,\beta)=\int_0^1 s^{\alpha-1}(1-s)^{\beta-1}\,ds,\ \alpha,\ \beta>0.$$

We have the following result for absolutely continuous functions with essentially bounded derivatives.

Theorem 3. *Let* $f:[a,b]\to\mathbb{C}$ *be an absolutely continuous function on* $[a,b]$ *and assume that* $f'\in L_{\infty}[a,b]$.

(i) *For any $x \in (a,b)$ we have*

$$
\begin{aligned}
&\left| J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) - \frac{f(x)}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x (x-t)^{\alpha} \|f'\|_{[t,x],\infty}\, dt + \int_x^b (t-x)^{\alpha} \|f'\|_{[x,t],\infty}\, dt \right] \\
&\leq \frac{1}{(\alpha+1)\Gamma(\alpha)} \left[\|f'\|_{[a,x],\infty} (x-a)^{\alpha+1} + \|f'\|_{[x,b],\infty} (b-x)^{\alpha+1} \right] \\
&\leq \frac{1}{(\alpha+1)\Gamma(\alpha)} \\
&\times \begin{cases} \|f'\|_{[a,b],\infty} \left[(x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right]; \\ \left(\|f'\|_{[a,x],\infty}^{p} + \|f'\|_{[x,b],\infty}^{p} \right)^{1/p} \left[(x-a)^{(\alpha+1)q} + (b-x)^{(\alpha+1)q} \right]^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\|f'\|_{[a,x],\infty} + \|f'\|_{[x,b],\infty} \right) \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{\alpha+1}. \end{cases}
\end{aligned} \tag{3.1}
$$

(ii) *For any $x \in (a,b)$ we have*

$$
\begin{aligned}
&\left| J_{x+}^{\alpha} f(b) + J_{x-}^{\alpha} f(a) - \frac{f(x)}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x (t-a)^{\alpha-1} (x-t) \|f'\|_{[t,x],\infty}\, dt \right. \\
&\left. + \int_x^b (b-t)^{\alpha-1} (t-x) \|f'\|_{[x,t],\infty}\, dt \right] \\
&\leq \frac{1}{\Gamma(\alpha+2)} \left[\|f'\|_{[a,x],\infty} (x-a)^{\alpha+1} + \|f'\|_{[x,b],\infty} (b-x)^{\alpha+1} \right] \\
&\leq \frac{1}{\Gamma(\alpha+2)} \\
&\times \begin{cases} \|f'\|_{[a,b],\infty} \left[(x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right]; \\ \left(\|f'\|_{[a,x],\infty}^{p} + \|f'\|_{[x,b],\infty}^{p} \right)^{1/p} \left[(x-a)^{(\alpha+1)q} + (b-x)^{(\alpha+1)q} \right]^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\|f'\|_{[a,x],\infty} + \|f'\|_{[x,b],\infty} \right) \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{\alpha+1}. \end{cases}
\end{aligned} \tag{3.2}
$$

(iii) *For any* $x\in[a,b]$ *we have*

$$\left|\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}-\frac{1}{\Gamma(\alpha+1)}f(x)(b-a)^{\alpha}\right| \tag{3.3}$$
$$\le\frac{1}{\Gamma(\alpha)}\int_a^x\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}(x-t)\|f'\|_{[t,x],\infty}dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_x^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}(t-x)\|f'\|_{[x,t],\infty}dt$$
$$\le\frac{1}{2\Gamma(\alpha)}\left[A(x,\alpha)\|f'\|_{[a,x],\infty}+C(x,\alpha)\|f'\|_{[x,b],\infty}\right]$$
$$\le\frac{1}{\Gamma(\alpha+2)}\left[(b-x)^{\alpha+1}+(x-a)^{\alpha+1}+\frac{1}{2}(\alpha-1)(b-a)^{\alpha+1}\right]\|f'\|_{[a,b],\infty}$$

where

$$A(x,\alpha):=\frac{1}{\alpha(\alpha+1)}\left[(b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right]+(b-a)^{\alpha}\left(\frac{b-a}{\alpha+1}-\frac{b-x}{\alpha}\right)$$

and

$$C(x,\alpha):=\frac{1}{\alpha(\alpha+1)}\left[(b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right]+(b-a)^{\alpha}\left(\frac{b-a}{\alpha+1}-\frac{x-a}{\alpha}\right).$$

Proof. (i) From (2.1) we have for $x\in(a,b)$ that

$$\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \tag{3.4}$$
$$\le\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}\left|\int_t^x f'(s)\,ds\right|dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}\left|\int_x^t f'(s)\,ds\right|dt$$
$$\le\frac{1}{\Gamma(\alpha)}\left[\int_a^x(x-t)^{\alpha}\|f'\|_{[t,x],\infty}dt+\int_x^b(t-x)^{\alpha}\|f'\|_{[x,t],\infty}dt\right]$$
$$\le\frac{1}{\Gamma(\alpha)}\left[\|f'\|_{[a,x],\infty}\int_a^x(x-t)^{\alpha}dt+\|f'\|_{[x,b],\infty}\int_x^b(t-x)^{\alpha}dt\right]$$
$$=\frac{1}{(\alpha+1)\Gamma(\alpha)}\left[\|f'\|_{[a,x],\infty}(x-a)^{\alpha+1}+\|f'\|_{[x,b],\infty}(b-x)^{\alpha+1}\right],$$

which proves the first two inequalities in (3.1).

Now, by making use of the elementary Hölder type inequalities for positive real numbers $c,\ d,\ m,\ n\ge0$

$$mc+nd\le\begin{cases}\max\{m,n\}(c+d);\\[2ex](m^p+n^p)^{1/p}(c^q+d^q)^{1/q}\ \text{ with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1\end{cases}$$

we have

$$\begin{aligned}
&\left[\|f'\|_{[a,x],\infty}(x-a)^{\alpha+1}+\|f'\|_{[x,b],\infty}(b-x)^{\alpha+1}\right]\\
&\leq\begin{cases}
\max\left\{\|f'\|_{[a,x],\infty},\|f'\|_{[x,b],\infty}\right\}\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right];\\[2ex]
\left(\|f'\|^p_{[a,x],\infty}+\|f'\|^p_{[x,b],\infty}\right)^{1/p}\left[(x-a)^{(\alpha+1)q}+(b-x)^{(\alpha+1)q}\right]^{1/q}\\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]
\left(\|f'\|_{[a,x],\infty}+\|f'\|_{[x,b],\infty}\right)\max\left\{(x-a)^{\alpha+1},(b-x)^{\alpha+1}\right\};
\end{cases}\\
&=\begin{cases}
\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\|f'\|_{[a,b],\infty};\\[2ex]
\left(\|f'\|^p_{[a,x],\infty}+\|f'\|^p_{[x,b],\infty}\right)^{1/p}\left[(x-a)^{(\alpha+1)q}+(b-x)^{(\alpha+1)q}\right]^{1/q}\\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]
\left(\|f'\|_{[a,x],\infty}+\|f'\|_{[x,b],\infty}\right)\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha+1},
\end{cases}
\end{aligned}$$

which proves the last part of (3.1).

(ii) Using the equality (2.2) we have

$$\begin{aligned}
&\left|J^{\alpha}_{x+}f(b)+J^{\alpha}_{x-}f(a)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \qquad (3.5)\\
&\leq\frac{1}{\Gamma(\alpha)}\int_a^x(t-a)^{\alpha-1}(x-t)\|f'\|_{[t,x],\infty}dt\\
&+\frac{1}{\Gamma(\alpha)}\int_x^b(b-t)^{\alpha-1}(t-x)\|f'\|_{[x,t],\infty}dt\\
&\leq\frac{1}{\Gamma(\alpha)}\|f'\|_{[a,x],\infty}\int_a^x(t-a)^{\alpha-1}(x-t)\,dt\\
&+\frac{1}{\Gamma(\alpha)}\|f'\|_{[x,b],\infty}\int_x^b(b-t)^{\alpha-1}(t-x)\,dt.
\end{aligned}$$

We observe that, by using the change of variable $t=(1-s)a+sb$ we have for α, $\beta>0$ that

$$\begin{aligned}
\int_a^b(t-a)^{\alpha-1}(b-t)^{\beta-1}dt&=(b-a)^{\alpha+\beta-1}\int_0^1s^{\alpha-1}(1-s)^{\beta-1}ds\\
&=(b-a)^{\alpha+\beta-1}B(\alpha,\beta),
\end{aligned}$$

where $B(\cdot,\cdot)$ is Beta function.

Therefore

$$\begin{aligned}
\int_a^x(t-a)^{\alpha-1}(x-t)\,dt&=(x-a)^{\alpha+1}B(\alpha,2)=(x-a)^{\alpha+1}\int_0^1s^{\alpha-1}(1-s)\,ds\\
&=(x-a)^{\alpha+1}\left(\frac{1}{\alpha}-\frac{1}{\alpha+1}\right)=\frac{1}{\alpha(\alpha+1)}(x-a)^{\alpha+1}
\end{aligned}$$

and, similarly

$$\int_x^b (b-t)^{\alpha-1}(t-x)\,dt = \frac{1}{\alpha(\alpha+1)}(b-x)^{\alpha+1}.$$

Using (3.5) we get (3.2).

(iii) Using the representation (2.3) we have for $x\in[a,b]$ that

$$\begin{aligned}&\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}-\frac{1}{\Gamma(\alpha+1)}f(x)(b-a)^{\alpha}\\&=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\left(\int_x^t f'(s)\,ds\right)dt\\&+\frac{1}{\Gamma(\alpha)}\int_x^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\left(\int_x^t f'(s)\,ds\right)dt.\end{aligned}$$

If we take the modulus in this equality, we get by the triangle inequality and the properties of integral that

$$\begin{aligned}&\left|\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}-\frac{1}{\Gamma(\alpha+1)}f(x)(b-a)^{\alpha}\right|\\&\leq\frac{1}{\Gamma(\alpha)}\int_a^x\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\left|\int_x^t f'(s)\,ds\right|dt\\&+\frac{1}{\Gamma(\alpha)}\int_x^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\left|\int_x^t f'(s)\,ds\right|dt\\&\leq\frac{1}{\Gamma(\alpha)}\int_a^x\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}(x-t)\|f'\|_{[t,x],\infty}\,dt\\&+\frac{1}{\Gamma(\alpha)}\int_x^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}(t-x)\|f'\|_{[x,t],\infty}\,dt,\end{aligned}\tag{3.6}$$

which proves the first inequality in (3.3).

We also have

$$\begin{aligned}&\int_a^x\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}(x-t)\|f'\|_{[t,x],\infty}\,dt\\&\leq\frac{1}{2}\|f'\|_{[a,x],\infty}\int_a^x\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right](x-t)\,dt\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}&\int_x^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}(t-x)\|f'\|_{[x,t],\infty}\,dt\\&\leq\frac{1}{2}\|f'\|_{[x,b],\infty}\int_x^b\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right](t-x)\,dt.\end{aligned}\tag{3.8}$$

We have

$$\int_a^x\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right](x-t)\,dt=\int_a^x(b-t)^{\alpha-1}(x-t)\,dt+\int_a^x(t-a)^{\alpha-1}(x-t)\,dt.$$

If we change the variable $u=b-t$, then $dt=-du$, $x-t=x+u-b$ and

$$\begin{aligned}
&\int_a^x (b-t)^{\alpha-1}(x-t)\,dt \\
&= -\int_{b-a}^{b-x} u^{\alpha-1}(x-b+u)\,du \\
&= \int_{b-a}^{b-x} u^{\alpha-1}(b-x)\,du - \int_{b-a}^{b-x} u^{\alpha}du \\
&= (b-x)\frac{(b-x)^{\alpha}-(b-a)^{\alpha}}{\alpha} - \frac{(b-x)^{\alpha+1}-(b-a)^{\alpha+1}}{\alpha+1}.
\end{aligned}$$

Also, we have

$$\int_a^x (t-a)^{\alpha-1}(x-t)\,dt = \frac{1}{\alpha(\alpha+1)}(x-a)^{\alpha+1}.$$

Then

$$\begin{aligned}
&(b-x)\frac{(b-x)^{\alpha}-(b-a)^{\alpha}}{\alpha} - \frac{(b-x)^{\alpha+1}-(b-a)^{\alpha+1}}{\alpha+1} \\
&+\frac{1}{\alpha(\alpha+1)}(x-a)^{\alpha+1} \\
&= \frac{(b-x)^{\alpha+1}-(b-x)(b-a)^{\alpha}}{\alpha} - \frac{(b-x)^{\alpha+1}-(b-a)^{\alpha+1}}{\alpha+1} \\
&+\frac{1}{\alpha(\alpha+1)}(x-a)^{\alpha+1} \\
&= \frac{(b-x)^{\alpha+1}}{\alpha} - \frac{(b-x)^{\alpha+1}}{\alpha+1} + \frac{(b-a)^{\alpha+1}}{\alpha+1} - \frac{(b-x)(b-a)^{\alpha}}{\alpha} \\
&+\frac{1}{\alpha(\alpha+1)}(x-a)^{\alpha+1} \\
&= \frac{1}{\alpha(\alpha+1)}\left[(b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right] + (b-a)^{\alpha}\left(\frac{b-a}{\alpha+1}-\frac{b-x}{\alpha}\right) = A(x,\alpha).
\end{aligned}$$

Also

$$\int_x^b \left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right](t-x)\,dt = \int_x^b (b-t)^{\alpha-1}(t-x)\,dt + \int_x^b (t-a)^{\alpha-1}(t-x)\,dt.$$

If we change the variable $u=t-a$, then $du=dt$, $t-x=u+a-x$ and

$$\begin{aligned}
&\int_x^b (t-a)^{\alpha-1}(t-x)\,dt \\
&= \int_{x-a}^{b-a} u^{\alpha-1}(u+a-x)\,du \\
&= \int_{x-a}^{b-a} u^{\alpha}du - (x-a)\int_{x-a}^{b-a} u^{\alpha-1}du \\
&= \frac{(b-a)^{\alpha+1}-(x-a)^{\alpha+1}}{\alpha+1} - (x-a)\frac{(b-a)^{\alpha}-(x-a)^{\alpha}}{\alpha} \\
&= (b-a)^{\alpha}\left[\frac{b-a}{\alpha+1}-\frac{x-a}{\alpha}\right] + \frac{1}{\alpha(\alpha+1)}(x-a)^{\alpha+1}.
\end{aligned}$$

Since

$$\int_x^b (b-t)^{\alpha-1}(t-x)\,dt = \frac{1}{\alpha(\alpha+1)}(b-x)^{\alpha+1},$$

then

$$(b-a)^{\alpha}\left(\frac{b-a}{\alpha+1}-\frac{x-a}{\alpha}\right)+\frac{1}{\alpha(\alpha+1)}(x-a)^{\alpha+1}+\frac{1}{\alpha(\alpha+1)}(b-x)^{\alpha+1}=C(x,\alpha)$$

and the second inequality in (3.3) is proved.

Observe that

$$\begin{aligned}
&\frac{1}{2}\left[A(x,\alpha)+C(x,\alpha)\right]\\
&=\frac{1}{\alpha(\alpha+1)}\left[(b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right]\\
&+\frac{1}{2}\left[(b-a)^{\alpha}\left(\frac{b-a}{\alpha+1}-\frac{b-x}{\alpha}\right)+(b-a)^{\alpha}\left(\frac{b-a}{\alpha+1}-\frac{x-a}{\alpha}\right)\right]\\
&=\frac{1}{\alpha(\alpha+1)}\left[(b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right]\\
&+\frac{1}{2}(b-a)^{\alpha}\left(\frac{b-a}{\alpha+1}-\frac{b-x}{\alpha}+\frac{b-a}{\alpha+1}-\frac{x-a}{\alpha}\right)\\
&=\frac{1}{\alpha(\alpha+1)}\left[(b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right]+\frac{1}{2}(b-a)^{\alpha}\left(2\frac{b-a}{\alpha+1}-\frac{b-a}{\alpha}\right)\\
&=\frac{1}{\alpha(\alpha+1)}\left[(b-x)^{\alpha+1}+(x-a)^{\alpha+1}+\frac{1}{2}(\alpha-1)(b-a)^{\alpha+1}\right],
\end{aligned}$$

which proves the last part of (3.3). □

Corollary 3. *With the assumptions of Theorem 3, we have the midpoint inequalities*

$$\begin{aligned}
(3.9)\quad &\left|J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)\right|\\
&\le\frac{1}{\Gamma(\alpha)}\left[\int_a^{\frac{a+b}{2}}(x-t)^{\alpha}\|f'\|_{\left[t,\frac{a+b}{2}\right],\infty}dt+\int_{\frac{a+b}{2}}^{b}(t-x)^{\alpha}\|f'\|_{\left[\frac{a+b}{2},t\right],\infty}dt\right]\\
&\le\frac{1}{2^{\alpha+1}(\alpha+1)\Gamma(\alpha)}(b-a)^{\alpha+1}\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right]\\
&\le\frac{1}{2^{\alpha}(\alpha+1)\Gamma(\alpha)}(b-a)^{\alpha+1}\|f'\|_{[a,b],\infty},
\end{aligned}$$

$$\begin{aligned}
&\left|J^{\alpha}_{\frac{a+b}{2}+}f(b)+J^{\alpha}_{\frac{a+b}{2}-}f(a)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^{\frac{a+b}{2}}(t-a)^{\alpha-1}\left(\frac{a+b}{2}-t\right)\|f'\|_{\left[t,\frac{a+b}{2}\right],\infty}dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha-1}\left(t-\frac{a+b}{2}\right)\|f'\|_{\left[\frac{a+b}{2},t\right],\infty}dt\right] \\
&\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right] \\
&\leq \frac{1}{2^{\alpha}\Gamma(\alpha+2)}(b-a)^{\alpha+1}\|f'\|_{[a,b],\infty}
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&\left|\frac{J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)}{2}-\frac{1}{\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha}\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\left(\frac{a+b}{2}-t\right)\|f'\|_{\left[t,\frac{a+b}{2}\right],\infty}dt \\
&+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\left(t-\frac{a+b}{2}\right)\|f'\|_{\left[\frac{a+b}{2},t\right],\infty}dt \\
&\leq \frac{1}{2\Gamma(\alpha+2)}(b-a)^{\alpha+1}\left[\frac{1}{2^{\alpha}}+\frac{1}{2}(\alpha-1)\right] \\
&\times\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right] \\
&\leq \frac{1}{\Gamma(\alpha+2)}\left[\frac{1}{2^{\alpha}}+\frac{1}{2}(\alpha-1)\right](b-a)^{\alpha+1}\|f'\|_{[a,b],\infty}.
\end{aligned} \tag{3.11}$$

Remark 2. *For $\alpha = 1$ in either (3.1) or (3.2) we recapture the Ostrowski inequality for Riemann integral (1.2).*

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