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OSTROWSKI TYPE INEQUALITIES FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF ABSOLUTELY CONTINUOUS FUNCTIONS IN TERMS OF p-NORMS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we establish some Ostrowski type inequalities for the Riemann-Liouville fractional integrals of absolutely continuous functions in terms of of p-norms, with $p \geq 1$. Applications for mid-point inequalities are provided as well. They generalize the know results holding for the classical Riemann integral.

1. Introduction

In 1997, Dragomir and Wang proved the following Ostrowski type inequality [17].

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right] \|f'\|_{[a,b],1}, \tag{1.1}$$

for all $x \in [a, b]$, where $\|\cdot\|_1$ is the Lebesgue norm on $L_1[a, b]$, i.e., we recall it

$$\|g\|_{[a,b],1} := \int_a^b |g(t)|\, dt.$$

The constant $\frac{1}{2}$ is best possible.

Note that the fact that $\frac{1}{2}$ is the best constant for differentiable functions was proved in [26] and (1.1) can also be obtained from a more general result obtained by A. M. Fink in [20] choosing $n = 1$ and doing some appropriate computation. However the inequality (1.1) was not stated explicitly in [20].

From (1.1) we have the *mid-point inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{1}{2} \|f'\|_{[a,b],1}. \tag{1.2}$$

The constant $\frac{1}{2}$ is best possible.

The following result, which is an improvement on the inequality (1.1), holds.

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Theorem 2 (Dragomir, 2002 [12])**.** *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.3}$$

$$\leq \frac{x-a}{b-a}\left\|f'\right\|_{[a,x],1}+\frac{b-x}{b-a}\left\|f'\right\|_{[x,b],1}$$

$$\leq \begin{cases} \frac{1}{2}\left[\left\|f'\right\|_{[a,b],1}+\left|\left\|f'\right\|_{[a,x],1}-\left\|f'\right\|_{[x,b],1}\right|\right] \\ \left[\left(\frac{x-a}{b-a}\right)^{\beta}+\left(\frac{b-x}{b-a}\right)^{\beta}\right]^{\frac{1}{\beta}}\left(\left|\left\|f'\right\|_{[a,x],1}^{\alpha}\right|+\left\|f'\right\|_{[x,b],1}^{\alpha}\right)^{\frac{1}{\alpha}} \\ \text{where } \alpha>1 \text{ and } \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\left\|f'\right\|_{[a,b],1} \end{cases}$$

for all $x\in[a,b]$.

We observe that for $x=\frac{a+b}{2}$ we re-obtain (1.2).

In 1998, Dragomir and Wang proved the following Ostrowski type inequality [19].

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.4}$$

$$\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\left\|f'\right\|_{[a,b],p},$$

for all $x\in[a,b]$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p}:=\left(\int_a^b |g(t)|^p\,dt\right)^{1/p}.$$

Note that the inequality (1.4) can also be obtained from a more general result obtained by A. M. Fink in [20] choosing $n=1$ and doing some appropriate computation. However the inequality (1.4) was not stated explicitly in [20].

From (1.4) we get the following midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{2(q+1)^{1/q}}(b-a)^{1/q}\left\|f'\right\|_{[a,b],p}, \tag{1.5}$$

and $\frac{1}{2}$ is a best possible constant.

The following new result, which is an improvement on the inequality (1.4), holds.

Theorem 4 (Dragomir, 2013 [14]). *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. If $f' \in L_p[a,b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{1.6}$$
$$\leq \frac{1}{(q+1)^{1/q}} \left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}} \|f'\|_{[a,x],p} + \left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}} \|f'\|_{[x,b],p} \right] (b-a)^{1/q}$$
$$\leq \frac{1}{(q+1)^{1/q}} \times \begin{cases} \frac{1}{2}\left[\|f'\|_{[a,x],p} + \|f'\|_{[x,b],p} + \left|\|f'\|_{[a,x],p} - \|f'\|_{[x,b],p}\right|\right] \\ \times \left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}} + \left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}}\right](b-a)^{1/q} \\ \\ \left(\|f'\|^{\alpha}_{[a,x],p} + \|f'\|^{\alpha}_{[x,b],p}\right)^{\frac{1}{\alpha}} \left[\left(\frac{x-a}{b-a}\right)^{\frac{q+1}{q}\beta} + \left(\frac{b-x}{b-a}\right)^{\frac{q+1}{q}\beta}\right]^{\frac{1}{\beta}} (b-a)^{1/q} \\ \text{where } \alpha > 1 \text{ and } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \left[\|f'\|_{[a,x],p} + \|f'\|_{[x,b],p}\right]\left[\frac{1}{2} + \left|\frac{x - \frac{a+b}{2}}{b-a}\right|\right]^{\frac{q+1}{q}} (b-a)^{1/q} \end{cases}$$

for all $x \in [a,b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. In particular

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{(b-a)^{1/q}}{2^{1+1/q}(q+1)^{1/q}} \left[\|f'\|_{[a,\frac{a+b}{2}],p} + \|f'\|_{[\frac{a+b}{2},b],p}\right].$$

For other Ostrowski type inequalities for Lebesgue integral, see [10], [6] and the recent survey [15].

In order to extend these results for the fractional integrals we need the following preparation.

Let $f : [a,b] \to \mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha > 0$ by

$$J^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt$$

for $a < x \leq b$ and

$$J^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt$$

for $a \leq x < b$, where Γ is the *Gamma function*. For $\alpha = 0$, they are defined as

$$J^{0}_{a+} f(x) = J^{0}_{b-} f(x) = f(x) \text{ for } x \in (a,b).$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[5], [21]-[34] and the references therein.

Motivated by the above results, in this paper we establish some Ostrowski type inequalities for the Riemann-Liouville fractional integrals of absolutely continuous functions in terms of p-norms with $p \geq 1$. Applications for mid-point inequalities are provided as well.

2. Inequalities for 1-Norm

We use the following identities of interest that have been established in the recent paper [16]:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be a complex valued absolutely continuous function on the real interval $[a,b]$.*

(i) *For any $x\in(a,b)$ we have*

$$J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)=\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]-\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\left(\int_{t}^{x}f'(s)\,ds\right)dt+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}\left(\int_{x}^{t}f'(s)\,ds\right)dt. \tag{2.1}$$

(ii) *For any $x\in(a,b)$ we have*

$$J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)=\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]-\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(t-a)^{\alpha-1}\left(\int_{t}^{x}f'(s)\,ds\right)dt+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(b-t)^{\alpha-1}\left(\int_{x}^{t}f'(s)\,ds\right)dt. \tag{2.2}$$

The following particular case is of interest:

Corollary 1. *With the assumption of Lemma 1, we have the particular mid-point equalities*

$$J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)-\frac{1}{\Gamma(\alpha)}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^{\alpha-1}\left(\int_{t}^{\frac{a+b}{2}}f'(s)\,ds\right)dt+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)^{\alpha-1}\left(\int_{\frac{a+b}{2}}^{t}f'(s)\,ds\right)dt, \tag{2.3}$$

and

$$J_{\frac{a+b}{2}+}^{\alpha}f(b)+J_{\frac{a+b}{2}-}^{\alpha}f(a)=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)-\frac{1}{\Gamma(\alpha)}\int_{a}^{\frac{a+b}{2}}(t-a)^{\alpha-1}\left(\int_{t}^{\frac{a+b}{2}}f'(s)\,ds\right)dt+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha-1}\left(\int_{\frac{a+b}{2}}^{t}f'(s)\,ds\right)dt. \tag{2.4}$$

The following results providing bounds in terms of the Lebesgue 1-norm holds:

Theorem 5. *Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on $[a, b]$.*

(i) *For any $x \in (a, b)$ we have*

$$
\begin{aligned}
(2.5)\quad & \left| J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) - \frac{f(x)}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x (x-t)^{\alpha-1} \|f'\|_{[t,x],1}\, dt + \int_x^b (t-x)^{\alpha-1} \|f'\|_{[x,t],1}\, dt \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[\|f'\|_{[a,x],1} (x-a)^{\alpha} + \|f'\|_{[x,b],1} (b-x)^{\alpha} \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \\
&\times \begin{cases} \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{\alpha} \|f'\|_{[a,b],1}; \\[2ex] \left[(x-a)^{m\alpha} + (b-x)^{m\alpha} \right]^{1/m} \left[\|f'\|_{[a,x],1}^{n} + \|f'\|_{[x,b],1}^{n} \right]^{1/n} \\ \text{with } m,\ n > 1,\ \frac{1}{m} + \frac{1}{n} = 1 \\[2ex] \left[\frac{1}{2} \|f'\|_{[a,b],1} + \frac{1}{2} \left| \|f'\|_{[a,x],1} - \|f'\|_{[x,b],1} \right| \right] \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right]. \end{cases}
\end{aligned}
$$

(ii) *For any $x \in (a, b)$ we have*

$$
\begin{aligned}
(2.6)\quad & \left| J_{x+}^{\alpha} f(b) + J_{x-}^{\alpha} f(a) - \frac{f(x)}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x (t-a)^{\alpha-1} \|f'\|_{[t,x],1}\, dt + \int_x^b (b-t)^{\alpha-1} \|f'\|_{[x,t],1}\, dt \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[\|f'\|_{[a,x],1} (x-a)^{\alpha} + \|f'\|_{[x,b],1} (b-x)^{\alpha} \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \\
&\times \begin{cases} \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{\alpha} \|f'\|_{[a,b],1}; \\[2ex] \left[(x-a)^{m\alpha} + (b-x)^{m\alpha} \right]^{1/m} \left[\|f'\|_{[a,x],1}^{n} + \|f'\|_{[x,b],1}^{n} \right]^{1/n} \\ \text{with } m,\ n > 1,\ \frac{1}{m} + \frac{1}{n} = 1 \\[2ex] \left[\frac{1}{2} \|f'\|_{[a,b],1} + \frac{1}{2} \left| \|f'\|_{[a,x],1} - \|f'\|_{[x,b],1} \right| \right] \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right]. \end{cases}
\end{aligned}
$$

Proof. (i) Using (2.1) we have

$$\begin{aligned}
&\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right|\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}(x-t)^{\alpha-1}\left|\int_{t}^{x}f'(s)\,ds\right|dt+\int_{x}^{b}(t-x)^{\alpha-1}\left|\int_{x}^{t}f'(s)\,ds\right|dt\right]\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}(x-t)^{\alpha-1}\int_{t}^{x}|f'(s)\,ds|\,dt+\int_{x}^{b}(t-x)^{\alpha-1}\int_{x}^{t}|f'(s)\,ds|\,dt\right]\\
&=\frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}(x-t)^{\alpha-1}\|f'\|_{[t,x],1}\,dt+\int_{x}^{b}(t-x)^{\alpha-1}\|f'\|_{[x,t],1}\,dt\right]\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\|f'\|_{[a,x],1}\int_{a}^{x}(x-t)^{\alpha-1}\,dt+\|f'\|_{[x,b],1}\int_{x}^{b}(t-x)^{\alpha-1}\,dt\right]\\
&=\frac{1}{\Gamma(\alpha+1)}\left[\|f'\|_{[a,x],1}(x-a)^{\alpha}+\|f'\|_{[x,b],1}(b-x)^{\alpha}\right],
\end{aligned}$$

which proves the first two inequalities in (2.5).

Now, by making use of the elementary Hölder type inequalities for positive real numbers c, d, u, $v\geq 0$

$$uc+vd\leq\begin{cases}\max\{u,v\}(c+d);\\[2mm](u^{m}+v^{m})^{1/m}(c^{n}+d^{n})^{1/n}\ \text{ with } m,\ n>1,\ \frac{1}{m}+\frac{1}{n}=1\end{cases}$$

we have

$$\begin{aligned}
&\|f'\|_{[a,x],1}(x-a)^{\alpha}+\|f'\|_{[x,b],1}(b-x)^{\alpha}\\
&\leq\begin{cases}\max\{(x-a)^{\alpha},(b-x)^{\alpha}\}\left[\|f'\|_{[a,x],1}+\|f'\|_{[x,b],1}\right]\\[2mm]
\left[(x-a)^{m\alpha}+(b-x)^{m\alpha}\right]^{1/m}\left[\|f'\|_{[a,x],1}^{n}+\|f'\|_{[x,b],1}^{n}\right]^{1/n}\\
\text{with } m,\ n>1,\ \frac{1}{m}+\frac{1}{n}=1\\[2mm]
\max\left\{\|f'\|_{[a,x],1},\|f'\|_{[x,b],1}\right\}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\end{cases}\\
&=\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}\|f'\|_{[a,b],1}\\[2mm]
\left[(x-a)^{m\alpha}+(b-x)^{m\alpha}\right]^{1/m}\left[\|f'\|_{[a,x],1}^{n}+\|f'\|_{[x,b],1}^{n}\right]^{1/n}\\
\text{with } m,\ n>1,\ \frac{1}{m}+\frac{1}{n}=1\\[2mm]
\left[\frac{1}{2}\|f'\|_{[a,b],1}+\frac{1}{2}\left|\|f'\|_{[a,x],1}-\|f'\|_{[x,b],1}\right|\right]\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right],\end{cases}
\end{aligned}$$

which proves the last part of (2.5).

(ii) Using (2.6), we have

$$
\begin{aligned}
&\left|J_{x+}^{\alpha} f(b)+J_{x-}^{\alpha} f(a)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}(t-a)^{\alpha-1}\left|\int_{t}^{x} f'(s)\, ds\right| dt+\int_{x}^{b}(b-t)^{\alpha-1}\left|\int_{x}^{t} f'(s)\, ds\right| dt\right] \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}(t-a)^{\alpha-1} \int_{t}^{x}\left|f'(s)\, ds\right| dt+\int_{x}^{b}(b-t)^{\alpha-1} \int_{x}^{t}\left|f'(s)\, ds\right| dt\right] \\
&=\frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}(t-a)^{\alpha-1}\left\|f'\right\|_{[t, x], 1} dt+\int_{x}^{b}(b-t)^{\alpha-1}\left\|f'\right\|_{[x, t], 1} dt\right] \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\left\|f'\right\|_{[a, x], 1} \int_{a}^{x}(t-a)^{\alpha-1} dt+\left\|f'\right\|_{[x, b], 1} \int_{x}^{b}(b-t)^{\alpha-1} dt\right] \\
&=\frac{1}{\Gamma(\alpha+1)}\left[\left\|f'\right\|_{[a, x], 1}(x-a)^{\alpha}+\left\|f'\right\|_{[x, b], 1}(b-x)^{\alpha}\right].
\end{aligned}
$$

The proof follows now along the lines of (i) and we omit the details. □

Corollary 2. *With the assumptions of Theorem 5 we have*

$$
\begin{aligned}
&\left|J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)-\frac{1}{2^{\alpha-1} \Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^{\alpha-1}\left\|f'\right\|_{\left[t, \frac{a+b}{2}\right], 1} dt \\
&+\frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)^{\alpha-1}\left\|f'\right\|_{\left[\frac{a+b}{2}, t\right], 1} dt \\
&\leq \frac{1}{2^{\alpha} \Gamma(\alpha+1)}(b-a)^{\alpha}\left\|f'\right\|_{[a, b], 1}
\end{aligned} \tag{2.7}
$$

and

$$
\begin{aligned}
&\left|J_{\frac{a+b}{2}+}^{\alpha} f(b)+J_{\frac{a+b}{2}-}^{\alpha} f(a)-\frac{1}{2^{\alpha-1} \Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{a}^{\frac{a+b}{2}}(t-a)^{\alpha-1}\left\|f'\right\|_{\left[t, \frac{a+b}{2}\right], 1} dt \\
&+\frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha-1}\left\|f'\right\|_{\left[\frac{a+b}{2}, t\right], 1} dt \\
&\leq \frac{1}{2^{\alpha} \Gamma(\alpha+1)}(b-a)^{\alpha}\left\|f'\right\|_{[a, b], 1}.
\end{aligned} \tag{2.8}
$$

For $\alpha = 1$ in (2.5) we obtain

$$\left|\int_a^b f(t)\,dt - (b-a) f(x)\right| \tag{2.9}$$
$$\leq \int_a^x \|f'\|_{[t,x],1}\,dt + \int_x^b \|f'\|_{[x,t],1}\,dt$$
$$\leq \left[\|f'\|_{[a,x],1}(x-a) + \|f'\|_{[x,b],1}(b-x)\right]$$
$$\leq \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \|f'\|_{[a,b],1}; \\ \\ \left[(x-a)^m + (b-x)^m\right]^{1/m} \left[\|f'\|_{[a,x],1}^n + \|f'\|_{[x,b],1}^n\right]^{1/n} \\ \text{with } m,\ n > 1,\ \frac{1}{m} + \frac{1}{n} = 1 \\ \\ \left[\frac{1}{2}\|f'\|_{[a,b],1} + \frac{1}{2}\left|\|f'\|_{[a,x],1} - \|f'\|_{[x,b],1}\right|\right](b-a), \end{cases}$$

for any $x \in (a,b)$, which improves (1.3).

3. Inequalities for p-Norms

Consider the *Beta function*

$$B(\alpha,\beta) = \int_0^1 s^{\alpha-1}(1-s)^{\beta-1}\,ds,\ \ \alpha,\ \beta > 0.$$

The case of p-norms is as follows:

Theorem 6. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$ with $f' \in L_p[a,b]$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) *For any $x \in (a,b)$ we have*

$$\left|J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) - \frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha} + (b-x)^{\alpha}\right]\right| \tag{3.1}$$
$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x (x-t)^{\alpha+1/q-1}\|f'\|_{[t,x],p}\,dt + \int_x^b (t-x)^{\alpha+1/q-1}\|f'\|_{[x,t],p}\,dt\right]$$
$$\leq \frac{1}{(\alpha+1/q)\Gamma(\alpha)}\left[\|f'\|_{[a,x],p}(x-a)^{\alpha+1/q} + \|f'\|_{[x,b],p}(b-x)^{\alpha+1/q}\right]$$
$$\leq \frac{1}{(\alpha+1/q)\Gamma(\alpha)}$$
$$\times \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{\alpha+1/q}\left[\|f'\|_{[a,x],p} + \|f'\|_{[x,b],p}\right]; \\ \\ \left[(x-a)^{m(\alpha+1/q)} + (b-x)^{m(\alpha+1/q)}\right]^{1/m}\left[\|f'\|_{[a,x],p}^n + \|f'\|_{[x,b],p}^n\right]^{1/n} \\ \textit{with } m,\ n > 1,\ \frac{1}{m} + \frac{1}{n} = 1 \\ \\ \left[\max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,b],p}\right\}\right]\left[(x-a)^{\alpha+1/q} + (b-x)^{\alpha+1/q}\right]. \end{cases}$$

(ii) *For any $x \in (a,b)$ we have*

$$
\begin{aligned}
(3.2)\quad & \left| J_{x+}^{\alpha} f(b) + J_{x-}^{\alpha} f(a) - \frac{f(x)}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} (x-t)^{1/q} \left\| f' \right\|_{[t,x],p} dt \\
& + \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} (t-x)^{1/q} \left\| f' \right\|_{[x,t],p} dt \\
& \leq \frac{\Gamma(1+1/q)}{\Gamma(\alpha+1+1/q)} \left[(x-a)^{\alpha+1/q} \left\| f' \right\|_{[a,x],p} + (b-x)^{\alpha+1/q} \left\| f' \right\|_{[x,b],p} \right] \\
& \leq \frac{\Gamma(1+1/q)}{\Gamma(\alpha+1+1/q)} \\
& \times \begin{cases} \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{\alpha+1/q} \left[\left\| f' \right\|_{[a,x],p} + \left\| f' \right\|_{[x,b],p} \right]; \\ \\ \left[(x-a)^{m(\alpha+1/q)} + (b-x)^{m(\alpha+1/q)} \right]^{1/m} \left[\left\| f' \right\|_{[a,x],p}^{n} + \left\| f' \right\|_{[x,b],p}^{n} \right]^{1/n} \\ \text{with } m,\ n>1,\ \frac{1}{m}+\frac{1}{n}=1 \\ \\ \left[\max\left\{ \left\| f' \right\|_{[a,x],p}, \left\| f' \right\|_{[x,b],p} \right\} \right] \left[(x-a)^{\alpha+1/q} + (b-x)^{\alpha+1/q} \right]. \end{cases}
\end{aligned}
$$

Proof. (i) Using the representation (2.1) and the Hölder's integral inequality we have

$$
\begin{aligned}
& \left| J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) - \frac{f(x)}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x (t-a)^{\alpha-1} \left| \int_t^x f'(s)\, ds \right| dt + \int_x^b (b-t)^{\alpha-1} \left| \int_x^t f'(s)\, ds \right| dt \right] \\
& \leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} (x-t)^{1/q} \left(\int_t^x \left| f'(s) \right|^p ds \right)^{1/p} dt \\
& + \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} (t-x)^{1/q} \left(\int_x^t \left| f'(s) \right|^p ds \right)^{1/p} dt \\
& = \frac{1}{\Gamma(\alpha)} \left[\int_a^x (x-t)^{\alpha+1/q-1} \left\| f' \right\|_{[t,x],p} dt + \int_x^b (t-x)^{\alpha+1/q-1} \left\| f' \right\|_{[x,t],p} dt \right] \\
& \leq \frac{1}{\Gamma(\alpha)} \left[\left\| f' \right\|_{[a,x],p} \int_a^x (x-t)^{\alpha+1/q-1} dt + \left\| f' \right\|_{[x,b],p} \int_x^b (t-x)^{\alpha+1/q-1} dt \right] \\
& = \frac{1}{(\alpha+1/q)\Gamma(\alpha)} \left[\left\| f' \right\|_{[a,x],p} (x-a)^{\alpha+1/q} + \left\| f' \right\|_{[x,b],p} (b-x)^{\alpha+1/q} \right],
\end{aligned}
$$

for any $x \in (a,b)$. This proves (3.1).

(ii) Using (2.2) we have, by Hölder's integral inequality, that

$$
\begin{aligned}
&\left| J_{x+}^{\alpha} f(b) + J_{x-}^{\alpha} f(a) - \frac{f(x)}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \right| \qquad (3.3) \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x (t-a)^{\alpha-1} \left| \int_t^x f'(s)\, ds \right| dt + \int_x^b (b-t)^{\alpha-1} \left| \int_x^t f'(s)\, ds \right| dt \right] \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} (x-t)^{1/q} \left(\int_t^x |f'(s)|^p\, ds \right)^{1/p} dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} (t-x)^{1/q} \left(\int_x^t |f'(s)|^p\, ds \right)^{1/p} dt \\
&= \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} (x-t)^{1/q} \|f'\|_{[t,x],p}\, dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} (t-x)^{1/q} \|f'\|_{[x,t],p}\, dt \\
&\leq \frac{1}{\Gamma(\alpha)} \|f'\|_{[a,x],p} \int_a^x (t-a)^{\alpha-1} (x-t)^{1/q}\, dt \\
&+ \frac{1}{\Gamma(\alpha)} \|f'\|_{[x,b],p} \int_x^b (b-t)^{\alpha-1} (t-x)^{1/q}\, dt.
\end{aligned}
$$

We observe that, by using the change of variable $t = (1-s)a + sb$ we have for α, $\beta > 0$ that

$$
\begin{aligned}
\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1}\, dt &= (b-a)^{\alpha+\beta-1} \int_0^1 s^{\alpha-1} (1-s)^{\beta-1}\, ds \\
&= (b-a)^{\alpha+\beta-1} B(\alpha, \beta),
\end{aligned}
$$

where $B(\cdot, \cdot)$ is Beta function.

Therefore,

$$
\int_a^x (t-a)^{\alpha-1} (x-t)^{1/q}\, dt = (x-a)^{\alpha+1/q} B(\alpha, 1+1/q),
$$

$$
\int_x^b (b-t)^{\alpha-1} (t-x)^{1/q}\, dt = (b-x)^{\alpha+1/q} B(\alpha, 1+1/q),
$$

and

$$
\frac{B(\alpha, 1+1/q)}{\Gamma(\alpha)} = \frac{\Gamma(1+1/q)}{\Gamma(\alpha+1/q+1)}
$$

which proves (3.2). □

We have the following mid-point inequalities:

Corollary 3. *With the assumptions of Theorem 5 we have*

$$
\begin{aligned}
(3.4)\qquad & \left| J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right)^{\alpha+1/q-1} \|f'\|_{\left[t,\frac{a+b}{2}\right],p} \, dt \\
& + \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right)^{\alpha+1/q-1} \|f'\|_{\left[\frac{a+b}{2},t\right],p} \, dt \\
& \leq \frac{1}{2^{\alpha+1/q}(\alpha+1/q)\Gamma(\alpha)} (b-a)^{\alpha+1/q} \left[\|f'\|_{\left[a,\frac{a+b}{2}\right],p} + \|f'\|_{\left[\frac{a+b}{2},b\right],p} \right]
\end{aligned}
$$

and

$$
\begin{aligned}
(3.5)\qquad & \left| J_{\frac{a+b}{2}+}^{\alpha} f(b) + J_{\frac{a+b}{2}-}^{\alpha} f(a) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} (t-a)^{\alpha-1} \left(\frac{a+b}{2} - t\right)^{1/q} \|f'\|_{\left[t,\frac{a+b}{2}\right],p} \, dt \\
& + \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^{b} (b-t)^{\alpha-1} \left(t - \frac{a+b}{2}\right)^{1/q} \|f'\|_{\left[\frac{a+b}{2},t\right],p} \, dt \\
& \leq \frac{\Gamma(1+1/q)}{2^{\alpha+1/q}\Gamma(\alpha+1+1/q)} (b-a)^{\alpha+1/q} \left[\|f'\|_{\left[a,\frac{a+b}{2}\right],p} + \|f'\|_{\left[\frac{a+b}{2},b\right],p} \right]
\end{aligned}
$$

Remark 1. *If we take $m = q$ and $n = p$, $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ in (3.1) and (3.2), then we get*

$$
\begin{aligned}
(3.6)\qquad & \left| J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) - \frac{f(x)}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha}\right] \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x (x-t)^{\alpha+1/q-1} \|f'\|_{[t,x],p} \, dt + \int_x^b (t-x)^{\alpha+1/q-1} \|f'\|_{[x,t],p} \, dt \right] \\
& \leq \frac{1}{(\alpha+1/q)\Gamma(\alpha)} \left[\|f'\|_{[a,x],p} (x-a)^{\alpha+1/q} + \|f'\|_{[x,b],p} (b-x)^{\alpha+1/q} \right] \\
& \leq \frac{1}{(\alpha+1/q)\Gamma(\alpha)} \left[(x-a)^{q\alpha+1} + (b-x)^{q\alpha+1} \right]^{1/q} \|f'\|_{[a,b],p}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.7)\qquad & \left| J_{x+}^{\alpha} f(b) + J_{x-}^{\alpha} f(a) - \frac{f(x)}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha}\right] \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} (x-t)^{1/q} \|f'\|_{[t,x],p} \, dt \\
& + \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} (t-x)^{1/q} \|f'\|_{[x,t],p} \, dt \\
& \leq \frac{\Gamma(1+1/q)}{\Gamma(\alpha+1+1/q)} \left[(x-a)^{\alpha+1/q} \|f'\|_{[a,x],p} + (b-x)^{\alpha+1/q} \|f'\|_{[x,b],p} \right] \\
& \leq \frac{\Gamma(1+1/q)}{\Gamma(\alpha+1+1/q)} \left[(x-a)^{q\alpha+1} + (b-x)^{q\alpha+1} \right]^{1/q} \|f'\|_{[a,b],p}
\end{aligned}
$$

for any $x \in (a,b)$.

If we take $\alpha = 1$ in (3.5), then we get

$$\left|\int_a^b f(t)\,dt - (b-a) f(x)\right| \tag{3.8}$$
$$\le \int_a^x (x-t)^{1/q} \|f'\|_{[t,x],p}\,dt + \int_x^b (t-x)^{1/q} \|f'\|_{[x,t],p}\,dt$$
$$\le \frac{1}{(1+1/q)} \left[\|f'\|_{[a,x],p} (x-a)^{1+1/q} + \|f'\|_{[x,b],p} (b-x)^{1+1/q}\right]$$
$$\le \frac{1}{(1+1/q)}$$
$$\times \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{1+1/q} \left[\|f'\|_{[a,x],p} + \|f'\|_{[x,b],p}\right]; \\ \\ \left[(x-a)^{m(1+1/q)} + (b-x)^{m(1+1/q)}\right]^{1/m} \left[\|f'\|^n_{[a,x],p} + \|f'\|^n_{[x,b],p}\right]^{1/n} \\ \text{with } m,\ n > 1,\ \frac{1}{m} + \frac{1}{n} = 1 \\ \\ \left[\max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,b],p}\right\}\right] \left[(x-a)^{1+1/q} + (b-x)^{1+1/q}\right]. \end{cases}$$

for any $x \in (a,b)$.

From (3.6) we have

$$\left|\int_a^b f(t)\,dt - (b-a) f(x)\right| \tag{3.9}$$
$$\le \int_a^x (x-t)^{1/q} \|f'\|_{[t,x],p}\,dt + \int_x^b (t-x)^{1/q} \|f'\|_{[x,t],p}\,dt$$
$$\le \frac{1}{(1+1/q)} \left[\|f'\|_{[a,x],p} (x-a)^{1+1/q} + \|f'\|_{[x,b],p} (b-x)^{1+1/q}\right]$$
$$\le \frac{1}{(1+1/q)} \left[(x-a)^{q+1} + (b-x)^{q+1}\right]^{1/q} \|f'\|_{[a,b],p}$$

for any $x \in (a,b)$.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa