Received 23/05/17

On Some Hermite-Hadamard Type Inequalities for Twice Differentiable $(\alpha,m)-$convex functions and Applications

UĞUR S. KIRMACI
Atatürk University, K.K.Education Faculty
Department of Mathematics, 25240
Erzurum-TURKEY
kirmaci@atauni.edu.tr

Abstract: Some Hermite-Hadamard-type inequalities for twice differentiable m-convex functions and $(\alpha,m)-$convex functions are presented. Some applications to special means of real numbers are also given.

Keywords: Convex functions, m-convex functions, $(\alpha,m)-$convex functions, power-mean integral inequality, Hölder's integral inequality.

1. Introduction

Let $f: \mathrm{I}\subset \mathrm{R}\rightarrow\mathrm{R}$ be a convex function on the interval I of real numbers and a,b$\in$I with a<b. The inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}$$

is known as Hermite-Hadamard's inequality for convex functions [9].

In [13] and [3, p. 295], G.H. Toader defines the m-convexity:

The function $f:[0,b]\rightarrow R$ is said to be m-convex , where $m\in[0,1]$, if for every $x,y\in[0,b]$ and $t\in[0,1]$, we have

$$f\left(tx+m(1-t)y\right)\leq tf(x)+m(1-t)f(y).$$

In [10] and [3, p. 301], V.G.Miheşan introduced the following class of functions:

The function $f:[0,b]\rightarrow R$ is said to be (α,m)-convex , where $(\alpha,m)\in[0,1]^2$, if for every $x,y\in[0,b]$ and $t\in[0,1]$, we have

$$f\left(tx+m(1-t)y\right)\leq t^\alpha f(x)+m(1-t^\alpha)f(y).$$

In [4] and [3, pp.38-49], S.S. Dragomir et al. gave some trapezoid type inequalities for twice differentiable mappings. In [8], M.K. Bakula et al. gave some general companion inequalities related to Jensen's inequality for the classes of m-convex and $(\alpha,m)-$convex functions. In [2], Rui-Fang Bai et al. deduced some Hermite-Hadamard type inequalities for the m- and $(\alpha,m)-$ logarithmically convex functions.

For several recent results concerning the Hermite-Hadamard type inequalities and convex functions, we refer the reader to [1,5,6,7,11,12].

In this paper, we give some Hermite-Hadamard-type inequalities for twice differentiable m-convex functions and $(\alpha,m)-$convex functions. Also, we write some applications to special means of real numbers.

2. Main Results

Firstly, we start by the following lemma:

Lemma : Let $f: I \subset R \to R$ be twice differentiable function on I^0 with f'' integrable on $[a,b] \subset I^0$. Then we have

$$\frac{(b-a)^2}{2}(I_1+I_2)=\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right] \tag{1}$$

where,

$$I_1=\int_0^{1/2} t\left(t-\frac{1}{2}\right)f''(ta+(1-t)b)dt,\quad I_2=\int_{1/2}^{1}\left(t-\frac{1}{2}\right)(t-1)f''(ta+(1-t)b)dt.$$

and I^0 denotes the interior of I.

Proof: By integration by parts twice, we have,

$$I_1=\int_0^{1/2} t\left(t-\frac{1}{2}\right)f''(ta+(1-t)b)dt=\frac{-1}{2(a-b)^2}\left[f\left(\frac{a+b}{2}\right)+f(b)\right]+\frac{2}{(a-b)^3}\int_b^{\frac{a+b}{2}} f(s)ds$$

and

$$I_2=\int_{1/2}^{1}\left(t-\frac{1}{2}\right)(t-1)f''(ta+(1-t)b)dt=\frac{-1}{2(a-b)^2}\left[f(a)+f\left(\frac{a+b}{2}\right)\right]+\frac{2}{(a-b)^3}\int_{\frac{a+b}{2}}^{a} f(s)ds$$

By adding these equalities, we get

$$I_1+I_2=\frac{-1}{2(a-b)^2}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right]+\frac{2}{(b-a)^3}\int_a^b f(s)ds$$

Hence, we obtain

$$\frac{(b-a)^2}{2}(I_1+I_2)=\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]$$

which completes the proof.

Now, we write the following results:

Theorem 1: Let $f: I \to R, I \subset [0,\infty)$ be twice differentiable function on I^0 such that $f'' \in L[a,b]$, $0 \le a < b < \infty$. If $|f''|$ is (α,m)-convex function with $(\alpha,m) \in (0,1]^2$, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|\leq$$

$$\leq\frac{(b-a)^2}{2}\left[(\eta+\mu)|f''(a)|+m\left(\frac{1}{24}-(\eta+\mu)\right)\left|f''(\frac{b}{m})\right|\right] \quad (2)$$

where, $\eta+\mu=\dfrac{\alpha-1}{2(\alpha+1)(\alpha+2)(\alpha+3)}+\dfrac{1}{2^{\alpha+2}}\dfrac{1}{(\alpha+1)(\alpha+2)}$.

Proof: Since $|f''|$ is (α,m)-convex, we have, for every $x,y\in[0,b]$ and $t\in[0,1]$

$$|f''(tx+m(1-t)y)|\leq t^\alpha|f''(x)|+m(1-t^\alpha)|f''(y)|,$$

which gives

$$|f''(tx+(1-t)y)|\leq t^\alpha|f''(x)|+m(1-t^\alpha)\left|f''(\frac{y}{m})\right|, \text{ for all } t\in[0,1]. \quad (3)$$

From (1) and by inequality (3), it follows that

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|\leq$$

$$\leq\frac{(b-a)^2}{2}\left[\int_0^{1/2}|t|\left|t-\frac{1}{2}\right||f''(ta+(1-t)b)|dt+\int_{1/2}^1\left|t-\frac{1}{2}\right||t-1||f''(ta+(1-t)b)|dt\right]$$

$$\leq\frac{(b-a)^2}{2}\left[\int_0^{1/2}\left(t^{\alpha+1}\left(\frac{1}{2}-t\right)|f''(a)|+t(1-t^\alpha)\left(\frac{1}{2}-t\right)m\left|f''(\frac{b}{m})\right|\right)dt\right]+$$

$$+\frac{(b-a)^2}{2}\left[\int_{1/2}^1\left(t^\alpha\left(t-\frac{1}{2}\right)(1-t)|f''(a)|+m\left(t-\frac{1}{2}\right)(1-t)(1-t^\alpha)\left|f''(\frac{b}{m})\right|\right)dt\right]$$

$$\leq\frac{(b-a)^2}{2}\left[(\eta+\mu)|f''(a)|+m\left(\frac{1}{24}-(\eta+\mu)\right)\left|f''(\frac{b}{m})\right|\right].$$

Where we have used the facts that,

$$\int_0^{1/2}t^{\alpha+1}\left(\frac{1}{2}-t\right)dt=\frac{1}{2^{\alpha+3}(\alpha+2)(\alpha+3)}=\eta,\qquad \int_0^{1/2}t\left(\frac{1}{2}-t\right)(1-t^\alpha)dt=\frac{1}{48}-\eta \quad (4)$$

$$\int_{1/2}^1\left(t-\frac{1}{2}\right)(1-t)t^\alpha dt=\frac{3}{2}\frac{1}{\alpha+2}\left(1-\frac{1}{2^{\alpha+2}}\right)+\frac{1}{\alpha+1}\left(\frac{1}{2^{\alpha+2}}-\frac{1}{2}\right)+\frac{1}{\alpha+3}\left(\frac{1}{2^{\alpha+3}}-1\right)=\mu, \quad (5)$$

$$\int_{1/2}^1\left(t-\frac{1}{2}\right)(1-t)(1-t^\alpha)dt=\frac{1}{48}-\mu \text{ and } \eta+\mu=\frac{\alpha-1}{2(\alpha+1)(\alpha+2)(\alpha+3)}+\frac{1}{2^{\alpha+2}}\frac{1}{(\alpha+1)(\alpha+2)}.$$

This concludes the proof.

Corollary 1: Let $f:I\to R, I\subset[0,\infty)$ be twice differentiable function on I^0 such that $f''\in L[a,b]$, $0\leq a<b<\infty$. If $|f''|$ is m-convex function with $m\in(0,1]$, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|\le\frac{(b-a)^2}{96}\left[\left|f''(a)\right|+m\left|f''\left(\frac{b}{m}\right)\right|\right] \tag{6}$$

Proof: Choosing $\alpha=1$ in the inequality (2), we obtain (6).

Remark 1: With the assumptions in Corollary 1, with the condition that $\|f''\|_\infty=\sup_{x\in[a,b]}|f''(x)|<\infty$ on $[a,b]\subset I^o$ and $m\in(0,1]$, we have the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|\le\frac{(b-a)^2}{96}(1+m)\|f''\|_\infty$$

Also, putting $m=1$, we get the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|\le\frac{(b-a)^2}{48}\|f''\|_\infty .$$

Theorem 2: Let $f:I\to R, I\subset[0,\infty)$ be twice differentiable function on I^0 such that $f''\in L[a,b]$, $0\le a<b<\infty$. If $|f''|^{p/(p-1)}$ is (α,m)-convex function with $(\alpha,m)\in(0,1]^2$, then we have, for p>1 and $\frac{1}{p}+\frac{1}{q}=1$

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|\le$$

$$\le\frac{(b-a)^2}{2}\left[\sigma^{1/p}\left(\gamma|f''(a)|^q+m\left(\frac{1}{2}-\gamma\right)\left|f''(\frac{b}{m})\right|^q\right)^{1/q}+\beta^{1/p}\left(\rho|f''(a)|^q+m\left(\frac{1}{2}-\rho\right)\left|f''(\frac{b}{m})\right|^q\right)^{1/q}\right] \tag{7}$$

where,

$$\sigma=\frac{1}{2^{p+2}}\left(\frac{1}{2p+1}+\frac{1}{p+1}\right),\beta=\frac{1}{2p+1}(2^{2p-2}-\frac{1}{8})+(1+\frac{1}{2^p})\frac{1}{p+1}(2^{2p-2}-\frac{2^p}{8})+\frac{2^p}{8}, \tag{8}$$

$$\gamma=\frac{1}{(\alpha+1)2^{\alpha+1}},\rho=\frac{1}{\alpha+1}-\frac{1}{(\alpha+1)2^{\alpha+1}} .$$

Proof: Since $|f''|^q$ is (α,m)-convex, we have, for every $x,y\in[0,b]$ and $t\in[0,1]$,

$$|f''(tx+(1-t)y)|^q\le t^\alpha|f''(x)|^q+m(1-t^\alpha)\left|f''(\frac{y}{m})\right|^q$$

By (1) and from Hölder's integral inequality, we have inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|\le$$

$$\le\frac{(b-a)^2}{2}\left[\int_0^{1/2}|t|\left|t-\frac{1}{2}\right||f''(ta+(1-t)b)|dt+\int_{1/2}^1\left|t-\frac{1}{2}\right||t-1||f''(ta+(1-t)b)|dt\right]$$

$$\leq \frac{(b-a)^2}{2}\left[\left(\int_0^{1/2} t^p\left|t-\frac{1}{2}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1/2}\left|f''(ta+(1-t)b\right|^q dt\right)^{\frac{1}{q}}+\left(\int_{1/2}^{1}\left|t-\frac{1}{2}\right|^p\left|t-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_{1/2}^{1}\left|f''(ta+(1-t)b\right|^q dt\right)^{\frac{1}{q}}\right]$$

where 1/p+1/q=1. From (3), we obtain,

$$\int_0^{1/2}\left|f''(ta+(1-t)b\right|^q dt \leq \int_0^{1/2}\left[t^\alpha\left|f''(a)\right|^q+m(1-t^\alpha)\left|f''(\frac{b}{m})\right|^q\right]dt$$

$$\leq \frac{1}{(\alpha+1)2^{\alpha+1}}\left|f''(a)\right|^q+m\left[\frac{1}{2}-\frac{1}{(\alpha+1)2^{\alpha+1}}\right]\left|f''(\frac{b}{m})\right|^q$$

and

$$\int_{1/2}^{1}\left|f''(ta+(1-t)b\right|^q dt \leq \int_{1/2}^{1}\left[t^\alpha\left|f''(a)\right|^q+m(1-t^\alpha)\left|f''(\frac{b}{m})\right|^q\right]dt$$

$$\leq\left[\frac{1}{\alpha+1}-\frac{1}{(\alpha+1)2^{\alpha+1}}\right]\left|f''(a)\right|^q+m\left[\frac{1}{2}-\left[\frac{1}{\alpha+1}-\frac{1}{(\alpha+1)2^{\alpha+1}}\right]\right]\left|f''(\frac{b}{m})\right|^q$$

Also, using the fact that,

$$\left|a+b\right|^p \leq 2^{p-1}\left(\left|a\right|^p+\left|b\right|^p\right)$$

for $p \geq 0, a, b \in R$, we obtain

$$\int_0^{1/2} t^p\left|t-\frac{1}{2}\right|^p dt \leq 2^{p-1}\int_0^{1/2} t^p\left(\left|t\right|^p+\left|\frac{1}{2}\right|^p\right)dt=\frac{1}{2^{p+2}}\left(\frac{1}{2p+1}+\frac{1}{p+1}\right)=\sigma$$

$$\int_{1/2}^{1}\left|t-\frac{1}{2}\right|^p\left|t-1\right|^p dt \leq 2^{2p-2}\int_{1/2}^{1}\left(t^p+\frac{1}{2^p}\right)\left(t^p+1\right)dt \leq$$

$$\leq 2^{2p-2}\int_{1/2}^{1}\left(t^{2p}+(1+\frac{1}{2^p})t^p+\frac{1}{2^p}\right)dt=\frac{1}{2p+1}(2^{2p-2}-\frac{1}{8})+(1+\frac{1}{2^p})\frac{1}{p+1}(2^{2p-2}-\frac{2^p}{8})+\frac{2^p}{8}=\beta$$

Combining all these inequalities, we deduce

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right| \leq$$

$$\leq \frac{(b-a)^2}{2}\left[\sigma^{1/p}\left(\gamma\left|f''(a)\right|^q+m\left(\frac{1}{2}-\gamma\right)\left|f''(\frac{b}{m})\right|^q\right)^{1/q}+\beta^{1/p}\left(\rho\left|f''(a)\right|^q+m\left(\frac{1}{2}-\rho\right)\left|f''(\frac{b}{m})\right|^q\right)^{1/q}\right]$$

Where,

$$\gamma=\int_0^{1/2}t^{\alpha}dt=\frac{1}{(\alpha+1)2^{\alpha+1}},\ \int_0^{1/2}(1-t^{\alpha})dt=\frac{1}{2}-\gamma,\ \rho=\int_{1/2}^{1}t^{\alpha}dt=\frac{1}{\alpha+1}-\frac{1}{(\alpha+1)2^{\alpha+1}},$$

$$\int_{1/2}^{1}(1-t^{\alpha})dt=\frac{1}{2}-\rho$$

This concludes the proof.

Corollary 2: Let $f:I\to R, I\subset[0,\infty)$ be twice differentiable function on I^0 such that $f''\in L[a,b]$, $0\le a<b<\infty$. If $|f''|^{p/(p-1)}$ is m-convex function with $m\in(0,1]$, then we have, for p>1 and $\frac{1}{p}+\frac{1}{q}=1$

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|\le$$

$$\le\frac{(b-a)^2}{2}\left[\sigma^{1/p}\left(\frac{|f''(a)|^q+3m\left|f''(\frac{b}{m})\right|^q}{8}\right)^{1/q}+\beta^{1/p}\left(\frac{3|f''(a)|^q+m\left|f''(\frac{b}{m})\right|^q}{8}\right)^{1/q}\right]\tag{9}$$

where, σ,β are given by (8).

Proof: Choosing $\alpha=1$ in the inequality (7), we obtain (9).

Remark 2: With the assumptions in Corollary 2, with the condition that $\|f''\|_{\infty}=\sup_{x\in[a,b]}|f''(x)|<\infty$ on $[a,b]\subset I^o$ and $m\in(0,1]$, we have the inequality, for $\sigma^{1/p},\beta^{1/p}<1$

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|\le\frac{(b-a)^2}{2}\left[\left(\frac{1+3m}{8}\right)^{1/q}+\left(\frac{3+m}{8}\right)^{1/q}\right]\|f''\|_{\infty}$$

Also, putting $m=1$, for $(1/2)^{1/q}<1$, we get the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|\le(b-a)^2\|f''\|_{\infty}$$

Theorem 3: Let $f:I\to R, I\subset[0,\infty)$ be twice differentiable function on I^0 such that $f''\in L[a,b]$, $0\le a<b<\infty$. If $|f''|^p$ is (α,m)-convex function with $(\alpha,m)\in(0,1]^2$, then we have, for p>1

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right|\le$$

$$\le\frac{(b-a)^2}{96}\left(\frac{1}{48}\right)^{-1/p}\left[\eta\left[|f''(a)|^p+m\left(\frac{1}{48}-\eta\right)\left|f''(\frac{b}{m})\right|^p\right]^{1/p}+\left[\mu|f''(a)|^p+m\left(\frac{1}{48}-\mu\right)\left|f''(\frac{b}{m})\right|^p\right]^{1/p}\right]\tag{10}$$

where η, μ are given by (4) and (5) respectively.

Proof: Since $|f''|^p$ is (α, m)-convex function, we have, for every $x, y \in [0, b]$, $t \in [0,1]$ and p>1

$$|f''(tx+(1-t)y)|^p \leq t^\alpha |f''(x)|^p + m(1-t^\alpha)\left|f''(\frac{y}{m})\right|^p \tag{11}$$

From (1), we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq$$

$$\leq \frac{(b-a)^2}{2}\left[\int_0^{1/2} |t|\left|t-\frac{1}{2}\right||f''(ta+(1-t)b)|dt + \int_{1/2}^1 \left|t-\frac{1}{2}\right||t-1||f''(ta+(1-t)b)|dt\right]$$

By the power-mean integral inequality, we have inequalities

$$\int_0^{1/2} t\left|t-\frac{1}{2}\right||f''(ta+(1-t)b)|dt \leq \left(\int_0^{1/2} t\left|t-\frac{1}{2}\right|dt\right)^{1-\frac{1}{p}}\left(\int_0^{1/2} t\left|t-\frac{1}{2}\right||f''(ta+(1-t))b|^p\right)^{1/p}$$

and

$$\int_{1/2}^1 |t-1|\left|t-\frac{1}{2}\right||f''(ta+(1-t)b)|dt \leq \left(\int_{1/2}^1 \left|t-\frac{1}{2}\right||t-1|dt\right)^{1-\frac{1}{p}}\left(\int_{1/2}^1 \left|t-\frac{1}{2}\right||t-1||f''(ta+(1-t)b)|^p\right)^{1/p}$$

From inequality (11), we obtain,

$$\int_0^{1/2} t\left|t-\frac{1}{2}\right||f''(ta+(1-t)b)|^p dt \leq \int_0^{1/2} t\left(\frac{1}{2}-t\right)\left[t^\alpha |f''(a)|^p + m(1-t^\alpha)\left|f''(\frac{b}{m})\right|^p\right]dt$$

$$\leq \eta|f''(a)|^p + m\left(\frac{1}{48}-\eta\right)\left|f''(\frac{b}{m})\right|^p$$

and

$$\int_{1/2}^1 \left|t-\frac{1}{2}\right||t-1||f''(ta+(1-t)b)|^p dt \leq \int_{1/2}^1 \left(t-\frac{1}{2}\right)(1-t)\left[t^\alpha |f''(a)|^p + m(1-t^\alpha)\left|f''(\frac{b}{m})\right|^p\right]dt$$

$$\leq \mu|f''(a)|^p + m\left(\frac{1}{48}-\mu\right)\left|f''(\frac{b}{m})\right|^p$$

where η, μ are given by (4) and (5) respectively and

$$\int_0^{1/2} t\left|t-\frac{1}{2}\right|dt = \int_{1/2}^1 \left|t-\frac{1}{2}\right||t-1|dt = \frac{1}{48}$$

Combining all these inequalities, we deduce

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq$$

$$\leq \frac{(b-a)^2}{96}\left(\frac{1}{48}\right)^{-1/p}\left[\eta\left[\left|f''(a)\right|^p + m\left(\frac{1}{48}-\eta\right)\left|f''(\frac{b}{m})\right|^p\right]^{1/p} + \left[\mu\left|f''(a)\right|^p + m\left(\frac{1}{48}-\mu\right)\left|f''(\frac{b}{m})\right|^p\right]^{1/p}\right]$$

Hence, we have the conclusion.

Corollary 3: Let $f: I \to R, I \subset [0,\infty)$ be twice differentiable function on I^0 such that $f'' \in L[a,b]$, $0 \leq a < b < \infty$. If $|f''|^p$ is m-convex function with $m \in (0,1]$, then we have, for p>1

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq$$
$$\leq \frac{(b-a)^2}{96} 4^{-\frac{1}{p}}\left[\left[\left|f''(a)\right|^p + 3m\left|f''(\frac{b}{m})\right|^p\right]^{1/p} + \left[3\left|f''(a)\right|^p + m\left|f''(\frac{b}{m})\right|^p\right]^{1/p}\right] \tag{12}$$

Proof: Choosing $\alpha = 1$ in the inequality (10), we obtain (12).

Remark 3: With the assumptions in Corollary 3, with the condition that $\|f''\|_\infty = \sup_{x\in[a,b]}|f''(x)| < \infty$ on $[a,b] \subset I^o$ and $m \in (0,1]$, we have the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{(b-a)^2}{96} 4^{-1/p}\left[(1+3m)^{1/p} + (3+m)^{1/p}\right]\|f''\|_\infty$$

Also, taking $m = 1$, we get the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{(b-a)^2}{48}\|f''\|_\infty .$$

3. Applications to Special Means

We shall consider the means for arbitrary real numbers $\alpha,\beta,\alpha\neq\beta$. We take

$A(\alpha,\beta) = \dfrac{\alpha+\beta}{2}$, $\alpha,\beta\in R$, (arithmetic mean)

$I(\alpha,\beta) = \dfrac{1}{e}\left(\dfrac{\beta^\beta}{\alpha^\alpha}\right)^{1/(\beta-\alpha)}$, (identric mean)

$G(\alpha,\beta) = \sqrt{\alpha\beta}$, (geometric mean)

$$L(\alpha,\beta)=\frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|}\ ,\ |\alpha|\neq|\beta|,\ \ \alpha\beta\neq 0, \qquad \text{(logarithmic mean)}$$

$$\mathrm{L_n}(\alpha,\beta)=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{1/n},\ n\in Z\backslash\{-1,0\},\ \alpha,\beta\in R,\ \alpha\neq\beta,\ \text{(generalized log-mean)}$$

Now, using the results of Section 2, we give some applications to special means of real numbers.

Proposition 1: Let $a,b\in[0,\infty), a<b$ and $n\in Z^{+}, n\geq 2$. Then we have

$$\left|L_n^n(a,b)-\frac{1}{2}\left[A(a^n,b^n)+A^n(a,b)\right]\right|\leq\frac{(b-a)^2}{48}n(n-1)A\left(|a|^{n-2},|b|^{n-2}\right).$$

Proof: The assertion follows from Corollary 1 applied to the 1-convex function $f(x)=x^n, f:[0,\infty)\to R$.

Proposition 2: Let $a,b\in(0,\infty), a<b$ Then we have

$$\left|\ln\left[\frac{I^{(b-a)}(a+1,b+1)}{\sqrt{G(a+1,b+1)A(a+1,b+1)}}\right]\right|\leq\frac{(b-a)^2}{96}\left[\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}\right].$$

Proof: The assertion follows from Corollary 1 applied to the 1-convex function $f(x)=-\ln(x+1), f:(0,\infty)\to R$.

Proposition 3: Let $a,b\in[0,\infty), a<b$ and $n\in Z^{+}, n\geq 2$. Then we have, for all q>1

$$\left|L_n^n(a,b)-\frac{1}{2}\left[A(a^n,b^n)+A^n(a,b)\right]\right|\leq\frac{(b-a)^2}{2}n(n-1)\Big[\sigma^{1/p}\left(A(|a|^{(n-2)q},3|b|^{(n-2)q})\right)^{1/q}+$$
$$+\beta^{1/p}\left(A(3|a|^{(n-2)q},|b|^{(n-2)q})\right)^{1/q}\Big].$$

Proof: The assertion follows from Corollary 2 applied to the 1-convex function $f(x)=x^n, f:[0,\infty)\to R$.

Proposition 4: Let $a,b\in[0,\infty), a<b$. Then we have, for all q>1

$$\left|\frac{b_1-a_1}{b-a}-\frac{1}{2}(A(a_2,b_2)+\cosh A(a,b))\right|\leq\frac{(b-a)^2}{2}\left[\sigma^{1/p}\left(\frac{a_2^q+3b_2^q}{8}\right)^{1/q}+\beta^{1/p}\left(\frac{3a_2^q+b_2^q}{8}\right)^{1/q}\right]$$

where $a_1=\sinh a, b_1=\sinh b, a_2=\cosh a, b_2=\cosh b$.

Proof: The assertion follows from Corollary 2 applied to the 1-convex function $f(x)=\cosh x, f:[0,\infty)\to R^{+}$.

Proposition 5: Let $a,b\in[0,\infty), a<b$ and $n\in Z^{+}, n\geq 2$. Then we have, for all p>1

$$\left|L_n^n(a,b)-\frac{1}{2}\left[A(a^n,b^n)+A^n(a,b)\right]\right|\le\frac{(b-a)^2}{48}n(n-1)4^{-1/p}\left[\left(A(|a|^{(n-2)p},3|b|^{(n-2)p})\right)^{1/p}+\right.$$
$$\left.+\left(A(3|a|^{(n-2)p},|b|^{(n-2)p})\right)^{1/p}\right]$$

Proof: The assertion follows from Corollary 3 applied to the 1-convex function $f(x)=x^n, f:[0,\infty)\to R$.

Proposition 6: Let $a,b\in[0,\infty), a<b$. Then we have, for all p>1

$$\left|L(e^a,e^b)-\frac{1}{2}(A(e^a,e^b)+e^{A(a,b)})\right|\le\frac{(b-a)^2}{96}2^{-1/p}\left[(A(e^{ap},3e^{bp}))^{1/p}+(A(3e^{ap},e^{bp}))^{1/p}\right]$$

Proof: The assertion follows from Corollary 3 applied to the 1-convex function $f(x)=e^x, f:[0,\infty)\to R$.

Proposition 7: Let $a,b\in[0,\infty), a<b$. Then we have, for all p>1

$$\left|\frac{F(b)-F(a)}{b-a}-\frac{1}{2}\left[A(f(a),f(b))+f\left(A(a,b)\right)\right]\right|\le$$
$$\le\frac{(b-a)^2}{96}4^{-\frac{1}{p}}\left[\left[|f''(a)|^p+\frac{48}{17}\left|f''(\frac{17b}{16})\right|^p\right]^{1/p}+\left[3|f''(a)|^p+\frac{16}{17}\left|f''(\frac{17b}{16})\right|^p\right]^{1/p}\right]$$

where $F(x)=\frac{1}{12}(\frac{x^5}{5}-\frac{5}{4}x^4+3x^3-\frac{5}{2}x^2$ and $f''(x)=x^2-\frac{30}{12}x+\frac{18}{12}$.

Proof: $f:[0,\infty)\to R$ defined as $f(x)=\frac{1}{12}(x^4-5x^3+9x^2-5x)$ is $\frac{16}{17}$-convex function (see [8] and [11]). The assertion follows from Corollary 3 applied to the function $f(x)$.

References

[1] M. Alomari, M. Darus, U.S. Kirmaci, Refinements of Hadamard-type inequalities for quasi-convex functions with applictions to trapezoidal formula and to special means, Comput. Math. Appl. 59 (2010), no.1, 225-232.

[2] R.F.Bai, F. Qi, B. Xi, Hermite-Hadamard type inequalities for the m- and $(\alpha,m)-$logarithmically convex functions, Filomat, 27:1, (2013), 1-7

[3] S.S.Dragomir, Selected Topics on Hermite-Hadamard Inequalities and Applications, http://rgmia.vu.edu.au/SSDragomirWeb.html

[4] S. S. Dragomir, P. Cerone, A. Sofo, Some remarks on the trapezoid rule in numerical integration, RGMIA Res. Rep. Coll., 2 (5) (1999), Article 1.

http://rgmia.vu.edu.au/v2n5.html, Indian J. of Pure and Appl. Math., 31 (5) (2000), 475-494.

[5] U.S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comput. 147(2004) 137-146.

[6] U.S. Kirmaci, M. K. Bakula, M.E. Özdemir, J.E. Pečarić , Hadamard-type inequalities for s-convex functions, Appl. Math. Comput. 193 (2007) 26-35.

[7] U.S. Kirmaci, Improvement and further generalization of inequalities for differentiable mappings and applications, Comput. Math. Appl. 55 (2008) 485-493.

[8] M. Klaričić Bakula, J. Pečarić, M. Ribičić, Companion inequalities to Jensen's inequality for m-convex and $(\alpha, m)-$convex functions, J. Inequal. Pure and Appl. Math., **7**(5) Art. 194, 2006

[9] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, Classical and New inequalities in Analysis, Kluwer Academic Publishers, 1993

[10] V.G. Miheşan, A generalisation of the convexity, Seminar on functional equations, Approx and convex., Cluj-Napoca, 1993

[11] P.T. Mocanu, I. Serb, G. Toader, Real star-convex functions, Studia Univ. Babe¸s-Bolyai, Math., **42**(3) (1997), 65–80.

[12] C.E.M.Pearce, J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formula, Appl. Math. Lett. 13 (2000) 51-55.

[13] G.H. Toader, Some generalisations of the convexity, Proc. Colloq. Approx. Optim. Cluj-Napoca, (1985), 329-338.