Received 23/05/17

OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF FUNCTIONS WITH BOUNDED VARIATION

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we establish some Ostrowski and generalized trapezoid type inequalities for the Riemann-Liouville fractional integrals of functions of bounded variation and of Lipschitzian functions. Applications for mid-point and trapezoid inequalities are provided as well. They generalize the know results holding for the classical Riemann integral.

1. Introduction

Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt$$

for $a<x\le b$ and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt$$

for $a\le x<b$, where Γ is the *Gamma function*. For $\alpha=0$, they are defined as

$$J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)\ \text{ for } x\in(a,b).$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[6], [16]-[26] and the references therein.

The following Ostrowski type inequalities for functions of bounded variation generalize the corresponding results for the Riemann integral obtained in [9], [11], [10] and have been established recently by the author in [15] :

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be a complex valued function of bounded variation on the real interval $[a,b]$.*

1991 *Mathematics Subject Classification.* 26D15, 26D10, 26D07, 26A33.

Key words and phrases. Riemann-Liouville fractional integrals, Functions of bounded variation, Lipshitzian functions, Ostrowski type inequalities, Trapezoid inequalities.

(i) *For any* $x \in (a,b)$ *we have*

$$\begin{aligned}
&\left| J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) - \frac{f(x)}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x (x-t)^{\alpha-1} \bigvee_t^x (f)\, dt + \int_x^b (t-x)^{\alpha-1} \bigvee_x^t (f)\, dt \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} \bigvee_a^x (f) + (b-x)^{\alpha} \bigvee_x^b (f) \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \\
&\times \begin{cases} \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\ \\ \left((x-a)^{\alpha p} + (b-x)^{\alpha p} \right)^{1/p} \left(\left(\bigvee_a^x (f) \right)^q + \left(\bigvee_x^b (f) \right)^q \right)^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \left((x-a)^{\alpha} + (b-x)^{\alpha} \right), \end{cases}
\end{aligned} \tag{1.1}$$

and

$$\begin{aligned}
&\left| J_{x+}^{\alpha} f(b) + J_{x-}^{\alpha} f(a) - \frac{f(x)}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_x^b (b-t)^{\alpha-1} \bigvee_x^t (f)\, dt + \int_a^x (t-a)^{\alpha-1} \bigvee_t^x (f)\, dt \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} \bigvee_a^x (f) + (b-x)^{\alpha} \bigvee_x^b (f) \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \\
&\times \begin{cases} \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\ \\ \left((x-a)^{\alpha p} + (b-x)^{\alpha p} \right)^{1/p} \left(\left(\bigvee_a^x (f) \right)^q + \left(\bigvee_x^b (f) \right)^q \right)^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \left((x-a)^{\alpha} + (b-x)^{\alpha} \right). \end{cases}
\end{aligned} \tag{1.2}$$

(ii) *For any $x \in [a,b]$ we have*

$$\left|\frac{J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)}{2} - \frac{1}{\Gamma(\alpha+1)} f(x)(b-a)^{\alpha}\right| \tag{1.3}$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_a^x \frac{(b-t)^{\alpha-1} + (t-a)^{\alpha-1}}{2} \bigvee_t^x (f)\, dt$$
$$+ \frac{1}{\Gamma(\alpha)} \int_x^b \frac{(b-t)^{\alpha-1} + (t-a)^{\alpha-1}}{2} \bigvee_x^t (f)\, dt$$
$$\leq \frac{1}{2\Gamma(\alpha+1)} \left[(b-a)^{\alpha} + (x-a)^{\alpha} - (b-x)^{\alpha}\right] \bigvee_a^x (f)$$
$$+ \frac{1}{2\Gamma(\alpha+1)} \left[(b-a)^{\alpha} + (b-x)^{\alpha} - (x-a)^{\alpha}\right] \bigvee_x^b (f)$$
$$\leq \frac{1}{2\Gamma(\alpha+1)} \begin{cases} \left[(b-a)^{\alpha} + \left|(x-a)^{\alpha} - (b-x)^{\alpha}\right|\right] \bigvee_a^b (f), \\ (b-a)^{\alpha} \left[\bigvee_a^b (f) + \left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right]. \end{cases}$$

The following mid-point inequalities that can be derived from Theorem 1 are of interest as well:

$$\left|J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)\right| \tag{1.4}$$
$$\leq \frac{1}{\Gamma(\alpha)}$$
$$\times \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right)^{\alpha-1} \bigvee_t^{\frac{a+b}{2}} (f)\, dt + \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right)^{\alpha-1} \bigvee_{\frac{a+b}{2}}^t (f)\, dt\right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} (b-a)^{\alpha} \bigvee_a^b (f),$$

$$\left|J_{\frac{a+b}{2}+}^{\alpha} f(b) + J_{\frac{a+b}{2}-}^{\alpha} f(a) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)\right| \tag{1.5}$$
$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_{\frac{a+b}{2}}^b (b-t)^{\alpha-1} \bigvee_{\frac{a+b}{2}}^t (f)\, dt + \int_a^{\frac{a+b}{2}} (t-a)^{\alpha-1} \bigvee_t^{\frac{a+b}{2}} (f)\, dt\right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} (b-a)^{\alpha} \bigvee_a^b (f)$$

and

$$\left|\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}-\frac{1}{\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha}\right| \tag{1.6}$$
$$\leq\frac{1}{\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\bigvee_t^{\frac{a+b}{2}}(f)\,dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\bigvee_{\frac{a+b}{2}}^{t}(f)\,dt$$
$$\leq\frac{1}{2\Gamma(\alpha+1)}(b-a)^{\alpha}\bigvee_a^b(f).$$

We say that the function $f:[a,b]\to\mathbb{C}$ is *r-H-Hölder continuous* on $[a,b]$ with $r\in(0,1]$ and $H>0$ if

$$|f(t)-f(s)|\leq H|t-s|^{r} \tag{1.7}$$

for any $t,\ s\in[a,b]$. If $r=1$ and $H=L$ we call the function *L-Lipschitzian* on $[a,b]$.

Theorem 2. *Assume that $f:[a,b]\to\mathbb{C}$ is r-H-Hölder continuous on $[a,b]$ with $r\in(0,1]$ and $H>0$. Then we have*

$$\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \tag{1.8}$$
$$\leq\frac{H}{(r+\alpha)\Gamma(\alpha)}\left[(x-a)^{r+\alpha}+(b-x)^{r+\alpha}\right]$$

and

$$\left|J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \tag{1.9}$$
$$\leq\frac{H}{\Gamma(\alpha)}B(\alpha,r+1)\left[(x-a)^{\alpha+r}+(b-x)^{\alpha+r}\right]$$

for any $x\in(a,b)$, where

$$B(\alpha,\beta)=\int_0^1 s^{\alpha-1}(1-s)^{\beta-1}\,ds,\ \ \alpha,\ \ \beta>0$$

is the Beta function.

If $f:[a,b]\to\mathbb{C}$ is L-Lipschitzian on $[a,b]$, then we have

$$\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \tag{1.10}$$
$$\leq\frac{L}{(1+\alpha)\Gamma(\alpha)}\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]$$

and

$$\left|J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \tag{1.11}$$
$$\leq\frac{L}{\Gamma(\alpha+2)}\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]$$

for any $x \in (a,b)$.

These results generalize the corresponding inequalities for Hölder and Lipschitzian functions that hold for Riemann integrals, see for instance [8] and the survey paper [14].

The following midpoint inequalities for r-H-Hölder continuous functions are of interest

$$\left|J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)\right| \leq \frac{H}{(r+\alpha)\, 2^{r+\alpha-1}\Gamma(\alpha)}(b-a)^{r+\alpha} \tag{1.12}$$

and

$$\left|J_{\frac{a+b}{2}+}^{\alpha} f(b)+J_{\frac{a+b}{2}-}^{\alpha} f(a)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)\right| \leq \frac{H}{2^{r+\alpha-1}\Gamma(\alpha)} B(\alpha, r+1)(b-a)^{r+\alpha}. \tag{1.13}$$

In particular, if f is L-Lipschitzian we have

$$\left|J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)\right| \leq \frac{L}{(1+\alpha)\, 2^{\alpha}\Gamma(\alpha)}(b-a)^{\alpha+1} \tag{1.14}$$

and

$$\left|J_{\frac{a+b}{2}+}^{\alpha} f(b)+J_{\frac{a+b}{2}-}^{\alpha} f(a)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)\right| \leq \frac{L}{2^{\alpha}\Gamma(\alpha+2)}(b-a)^{\alpha+1}. \tag{1.15}$$

Motivated by the above results, in this paper we establish some Ostrowski and generalized trapezoid type inequalities for the Riemann-Liouville fractional integrals of functions of bounded variation and of Lipschitzian functions. Applications for mid-point and trapezoid inequalities are provided as well. They generalize the know results holding for the classical Riemann integral.

2. Some Identities

We have the following representation:

Lemma 1. *Let* $f:[a,b]\to\mathbb{C}$ *be a function of bounded variation on* $[a,b]$.

(i) *For any* $x\in(a,b)$ *we have*

$$J_{a+}^{\alpha} f(x)+J_{b-}^{\alpha} f(x)=\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha} f(a)+(b-x)^{\alpha} f(b)\right] + \frac{1}{\Gamma(\alpha+1)}\left[\int_a^x (x-t)^{\alpha}\, df(t)-\int_x^b (t-x)^{\alpha}\, df(t)\right]. \tag{2.1}$$

(ii) *For any* $x \in (a,b)$ *we have*

$$J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] f(x) + \frac{1}{\Gamma(\alpha+1)} \left[\int_x^b (b-t)^{\alpha} df(t) - \int_a^x (t-a)^{\alpha} df(t) \right]. \tag{2.2}$$

Proof. (i) Since $f : [a,b] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$, then the Riemann-Stieltjes integrals

$$\int_a^x (x-t)^{\alpha} df(t) \text{ and } \int_x^b (t-x)^{\alpha} df(t)$$

exist and integrating by parts, we have

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)} \int_a^x (x-t)^{\alpha} df(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt - \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha} f(a) \\ &= J_{a+}^{\alpha} f(x) - \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha} f(a) \end{aligned} \tag{2.3}$$

for $a < x \le b$ and

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)} \int_x^b (t-x)^{\alpha} df(t) \\ &= \frac{1}{\Gamma(\alpha+1)} (b-x)^{\alpha} f(b) - \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt \\ &= \frac{1}{\Gamma(\alpha+1)} (b-x)^{\alpha} f(b) - J_{b-}^{\alpha} f(x) \end{aligned} \tag{2.4}$$

for $a \le x < b$.

From (2.3) we have

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha} f(a) + \frac{1}{\Gamma(\alpha+1)} \int_a^x (x-t)^{\alpha} df(t)$$

for $a < x \le b$ and from (2.4) we have

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)} (b-x)^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)} \int_x^b (t-x)^{\alpha} df(t),$$

for $a \le x < b$, which by addition give (2.1).

(ii) We have

$$J_{x+}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} f(t)\, dt$$

for $a \le x < b$ and

$$J_{x-}^{\alpha} f(a) = \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} f(t)\, dt$$

for $a < x \le b$.

Since $f:[a,b]\to\mathbb{C}$ is a function of bounded variation on $[a,b]$, then the Riemann-Stieltjes integrals

$$\int_a^x (t-a)^\alpha df(t) \text{ and } \int_x^b (b-t)^\alpha df(t)$$

exist and integrating by parts, we have

$$\begin{aligned}\frac{1}{\Gamma(\alpha+1)}\int_a^x (t-a)^\alpha df(t) &\\ = \frac{1}{\Gamma(\alpha+1)}(x-a)^\alpha f(x) - \frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1} f(t)\,dt &\\ = \frac{1}{\Gamma(\alpha+1)}(x-a)^\alpha f(x) - J_{x-}^\alpha f(a)&\end{aligned} \tag{2.5}$$

for $a<x\le b$ and

$$\begin{aligned}\frac{1}{\Gamma(\alpha+1)}\int_x^b (b-t)^\alpha df(t) &\\ = \frac{1}{\Gamma(\alpha)}\int_x^b (b-t)^{\alpha-1} f(t)\,dt - \frac{1}{\Gamma(\alpha+1)}(b-x)^\alpha f(x) &\\ = J_{x+}^\alpha f(b) - \frac{1}{\Gamma(\alpha+1)}(b-x)^\alpha f(x)&\end{aligned} \tag{2.6}$$

for $a\le x<b$.

From (2.5) we have

$$J_{x-}^\alpha f(a) = \frac{1}{\Gamma(\alpha+1)}(x-a)^\alpha f(x) - \frac{1}{\Gamma(\alpha+1)}\int_a^x (t-a)^\alpha df(t) \tag{2.7}$$

for $a<x\le b$ and from (2.6)

$$J_{x+}^\alpha f(b) = \frac{1}{\Gamma(\alpha+1)}(b-x)^\alpha f(x) + \frac{1}{\Gamma(\alpha+1)}\int_x^b (b-t)^\alpha df(t), \tag{2.8}$$

for $a\le x<b$, which by addition produce (2.2). $\square$

Corollary 1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. We have the midpoint equalities*

$$\begin{aligned}J_{a+}^\alpha f\left(\frac{a+b}{2}\right) + J_{b-}^\alpha f\left(\frac{a+b}{2}\right) &\\ = \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2} &\\ + \frac{1}{\Gamma(\alpha+1)}\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^\alpha df(t) - \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)^\alpha df(t)\right]&\end{aligned} \tag{2.9}$$

and

$$J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha}+\frac{1}{\Gamma(\alpha+1)}\left[\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha}df(t)-\int_{a}^{\frac{a+b}{2}}(t-a)^{\alpha}df(t)\right]\tag{2.10}$$

and the trapezoid equality

$$\frac{J^{\alpha}_{b-}f(a)+J^{\alpha}_{a+}f(b)}{2}=\frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^{\alpha}+\frac{1}{\Gamma(\alpha+1)}\int_{a}^{b}\frac{(b-t)^{\alpha}-(t-a)^{\alpha}}{2}df(t).\tag{2.11}$$

Proof. Equality (2.9) follows by (2.1) for $x=\frac{a+b}{2}$ while the equality (2.10) follows by (2.2).

For $x=b$ in (2.7) we have

$$J^{\alpha}_{b-}f(a)=\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\int_{a}^{b}(t-a)^{\alpha}df(t)$$

while from (2.8) we have for $x=a$ that

$$J^{\alpha}_{a+}f(b)=\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}f(a)+\frac{1}{\Gamma(\alpha+1)}\int_{a}^{b}(b-t)^{\alpha}df(t).$$

If we add these two equalities and divide by 2 we get (2.11). □

3. Inequalities for Functions of Bounded Variation

The following lemma is of interest in itself as well [2, p. 177], see also [12] for a generalization.

Lemma 2. *Let $f,\ u:[a,b]\to\mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and*

$$\left|\int_{a}^{b}f(t)\,du(t)\right|\le\int_{a}^{b}|f(t)|\,d\left(\bigvee_{a}^{t}(u)\right)\le\max_{t\in[a,b]}|f(t)|\bigvee_{a}^{b}(u),\tag{3.1}$$

where $\bigvee_a^t(u)$ denotes the total variation of u on $[a,t]$, $t\in[a,b]$.

We have:

Theorem 3. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then for any $x\in(a,b)$ we have*

$$\begin{aligned}
&\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right]\right| \\
&\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^x(x-t)^{\alpha-1}\bigvee_a^t(f)\,dt+\int_x^b(t-x)^{\alpha-1}\bigvee_t^b(f)\,dt\right] \\
&\leq\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}\bigvee_a^x(f)+(b-x)^{\alpha}\bigvee_x^b(f)\right] \\
&\leq\frac{1}{\Gamma(\alpha+1)} \\
&\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}\bigvee_a^b(f); \\ \\ \left[(x-a)^{\alpha p}+(b-x)^{\alpha p}\right]^{1/p}\left[\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right]^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}
\end{aligned}\tag{3.2}$$

and, see also (1.2),

$$\begin{aligned}
&\left|J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)\right| \\
&\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^x(t-a)^{\alpha-1}\bigvee_t^x(f)\,dt+\int_x^b(b-t)^{\alpha}\bigvee_x^t(f)\,d(t)\right] \\
&\leq\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}\bigvee_a^x(f)+(b-x)^{\alpha}\bigvee_x^b(f)\right] \\
&\leq\frac{1}{\Gamma(\alpha+1)} \\
&\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}\bigvee_a^b(f); \\ \\ \left[(x-a)^{\alpha p}+(b-x)^{\alpha p}\right]^{1/p}\left[\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right]^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right].\end{cases}
\end{aligned}\tag{3.3}$$

Proof. Using the representation (2.1) we have for $x\in(a,b)$ that

$$\begin{aligned}
&\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right]\right| \\
&\leq\frac{1}{\Gamma(\alpha+1)}\left[\left|\int_a^x(x-t)^{\alpha}\,df(t)\right|+\left|\int_x^b(t-x)^{\alpha}\,df(t)\right|\right]=:K(x,\alpha).
\end{aligned}\tag{3.4}$$

By using Lemma 2 we have

$$\left|\int_a^x (x-t)^\alpha \, df(t)\right| \le \int_a^x (x-t)^\alpha \, d\left(\bigvee_a^t (f)\right)$$

and

$$\left|\int_x^b (t-x)^\alpha \, df(t)\right| \le \int_x^b (t-x)^\alpha \, d\left(\bigvee_x^t (f)\right).$$

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}\int_a^x (x-t)^\alpha \, d\left(\bigvee_a^t (f)\right) &= (x-t)^\alpha \bigvee_a^t (f)\Big|_a^x + \alpha \int_a^x (x-t)^{\alpha-1} \bigvee_a^t (f)\, dt \\ &= \alpha \int_a^x (x-t)^{\alpha-1} \bigvee_a^t (f)\, dt\end{aligned}$$

and

$$\begin{aligned}\int_x^b (t-x)^\alpha \, d\left(\bigvee_x^t (f)\right) &= (t-x)^\alpha \bigvee_x^t (f)\Big|_x^b - \alpha \int_x^b (t-x)^{\alpha-1} \bigvee_x^t (f)\, dt \\ &= (b-x)^\alpha \bigvee_x^b (f) - \alpha \int_x^b (t-x)^{\alpha-1} \bigvee_x^t (f)\, dt \\ &= \alpha \bigvee_x^b (f) \int_x^b (t-x)^{\alpha-1}\, dt - \alpha \int_x^b (t-x)^{\alpha-1} \bigvee_x^t (f)\, dt \\ &= \alpha \int_x^b \left[\bigvee_x^b (f) - \bigvee_x^t (f)\right] (t-x)^{\alpha-1}\, dt \\ &= \alpha \int_x^b (t-x)^{\alpha-1} \bigvee_t^b (f)\, dt\end{aligned}$$

for any $x \in (a,b)$.

Therefore

$$\begin{aligned}K(x,\alpha) &\le \frac{1}{\Gamma(\alpha+1)}\left[\alpha \int_a^x (x-t)^{\alpha-1} \bigvee_a^t (f)\, dt + \alpha \int_x^b (t-x)^{\alpha-1} \bigvee_t^b (f)\, dt\right] \\ &\le \frac{1}{\Gamma(\alpha)}\left[\int_a^x (x-t)^{\alpha-1} \bigvee_a^t (f)\, dt + \int_x^b (t-x)^{\alpha-1} \bigvee_t^b (f)\, dt\right]\end{aligned}$$

for any $x \in (a,b)$, which proves the first part of (3.2).

Moreover, since $\bigvee_a^t(f) \le \bigvee_a^x(f)$ for $a \le t \le x$ and $\bigvee_t^b(f) \le \bigvee_x^b(f)$ for $x \le t \le b$, then

$$\begin{aligned}
&\int_a^x (x-t)^{\alpha-1} \bigvee_a^t (f)\, dt + \int_x^b (t-x)^{\alpha-1} \bigvee_t^b (f)\, dt \\
&\le \bigvee_a^x (f) \int_a^x (x-t)^{\alpha-1}\, dt + \bigvee_x^b (f) \int_x^b (t-x)^{\alpha-1}\, dt \\
&= \frac{1}{\alpha}\left[(x-a)^\alpha \bigvee_a^x (f) + (b-x)^\alpha \bigvee_x^b (f)\right]
\end{aligned}$$

for any $x \in (a,b)$, which proves the second part of (3.2).

The last part is obvious by making use of the elementary Hölder type inequalities for positive real numbers c, d, m, $n \ge 0$

$$mc + nd \le \begin{cases} \max\{m,n\}(c+d); \\ \\ (m^p+n^p)^{1/p}(c^q+d^q)^{1/q} \text{ with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1. \end{cases}$$

We give another proof for (3.3) than the one from [15].

By the use of the identity (2.2), we have for $x \in (a,b)$

$$\begin{aligned}
(3.5)\qquad &\left| J_{x-}^\alpha f(a) + J_{x+}^\alpha f(b) - \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^\alpha + (b-x)^\alpha\right] f(x) \right| \\
&\le \frac{1}{\Gamma(\alpha+1)}\left[\left|\int_a^x (t-a)^\alpha\, df(t)\right| + \left|\int_x^b (b-t)^\alpha\, df(t)\right|\right] =: M(x,\alpha)
\end{aligned}$$

By using Lemma 2 we have

$$\left|\int_a^x (t-a)^\alpha\, df(t)\right| \le \int_a^x (t-a)^\alpha\, d\left(\bigvee_a^t (f)\right)$$

and

$$\left|\int_x^b (b-t)^\alpha\, df(t)\right| \le \int_x^b (b-t)^\alpha\, d\left(\bigvee_x^t (f)\right).$$

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}
\int_a^x (t-a)^\alpha\, d\left(\bigvee_a^t (f)\right) &= (t-a)^\alpha \bigvee_a^t (f)\Big|_a^x - \alpha \int_a^x (t-a)^{\alpha-1} \bigvee_a^t (f)\, dt \\
&= (x-a)^\alpha \bigvee_a^x (f) - \alpha \int_a^x (t-a)^{\alpha-1} \bigvee_a^t (f)\, dt \\
&= \alpha \bigvee_a^x (f) \int_a^x (t-a)^{\alpha-1} - \alpha \int_a^x (t-a)^{\alpha-1} \bigvee_a^t (f)\, dt \\
&= \alpha \int_a^x \left[\bigvee_a^x (f) - \bigvee_a^t (f)\right](t-a)^{\alpha-1}\, dt \\
&= \alpha \int_a^x (t-a)^{\alpha-1} \bigvee_t^x (f)\, dt
\end{aligned}$$

and

$$\begin{aligned}\int_x^b (b-t)^\alpha d\left(\bigvee_x^t (f)\right) &= (b-t)^\alpha \bigvee_x^t (f)\bigg|_x^b + \alpha\int_x^b (b-t)^\alpha \bigvee_x^t (f)\, d(t)\\ &= \alpha\int_x^b (b-t)^\alpha \bigvee_x^t (f)\, d(t)\end{aligned}$$

for any $x \in (a,b)$.

Therefore

$$\begin{aligned}M(x,\alpha) &\le \frac{1}{\Gamma(\alpha+1)}\left[\alpha\int_a^x (t-a)^{\alpha-1}\bigvee_t^x (f)\, dt + \alpha\int_x^b (b-t)^\alpha \bigvee_x^t (f)\, d(t)\right]\\ &= \frac{1}{\Gamma(\alpha)}\left[\int_a^x (t-a)^{\alpha-1}\bigvee_t^x (f)\, dt + \int_x^b (b-t)^\alpha \bigvee_x^t (f)\, d(t)\right],\end{aligned}$$

which proves the first inequality in (3.3).

We also have

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\left[\int_a^x (t-a)^{\alpha-1}\bigvee_t^x (f)\, dt + \int_x^b (b-t)^\alpha \bigvee_x^t (f)\, d(t)\right]\\ &\le \frac{1}{\Gamma(\alpha)}\left[\bigvee_a^x (f)\int_a^x (t-a)^{\alpha-1} dt + \bigvee_x^b (f)\int_x^b (b-t)^\alpha d(t)\right]\\ &= \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^\alpha \bigvee_a^x (f) + (b-x)^\alpha \bigvee_x^b (f)\right],\end{aligned}$$

which proves the second inequality in (3.3). The rest is obvious. □

Corollary 2. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then we have the mid-point inequalities*

$$\begin{aligned}&\left|J_{a+}^\alpha f\left(\frac{a+b}{2}\right) + J_{b-}^\alpha f\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{1}{\Gamma(\alpha)}\\ &\times\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^{\alpha-1}\bigvee_a^t (f)\, dt + \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)^{\alpha-1}\bigvee_t^b (f)\, dt\right]\\ &\le \frac{1}{2^\alpha\Gamma(\alpha+1)}(b-a)^\alpha \bigvee_a^b (f)\end{aligned}\tag{3.6}$$

and

$$\left|J^{\alpha}_{\frac{a+b}{2}+}f(b)+J^{\alpha}_{\frac{a+b}{2}-}f(a)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)\right| \tag{3.7}$$
$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^{\frac{a+b}{2}}(t-a)^{\alpha-1}\bigvee_t^{\frac{a+b}{2}}(f)\,dt+\int_{\frac{a+b}{2}}^b(b-t)^{\alpha}\bigvee_{\frac{a+b}{2}}^t(f)\,d(t)\right]$$
$$\leq\frac{1}{2^{\alpha}\Gamma(\alpha+1)}(b-a)^{\alpha}\bigvee_a^b(f).$$

We also have the trapezoid type inequality:

Theorem 4. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then we have for $\alpha>0$ that*

$$\left|\frac{J^{\alpha}_{b-}f(a)+J^{\alpha}_{a+}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^{\alpha}\right| \tag{3.8}$$
$$\leq\frac{1}{2\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\bigvee_a^t(f)\,dt$$
$$+\frac{1}{2\Gamma(\alpha)}\int_{\frac{a+b}{2}}^b\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\bigvee_t^b(f)\,dt$$
$$\leq\frac{1}{2\Gamma(\alpha+1)}(b-a)^{\alpha}\bigvee_a^b(f)\,dt.$$

Proof. Using the identity (2.11) and Lemma 2 we have

$$\left|\frac{J^{\alpha}_{b-}f(a)+J^{\alpha}_{a+}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^{\alpha}\right| \tag{3.9}$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\left|\int_a^b\frac{(b-t)^{\alpha}-(t-a)^{\alpha}}{2}df(t)\right|$$
$$\leq\frac{1}{2}\int_a^b\left|(b-t)^{\alpha}-(t-a)^{\alpha}\right|d\left(\bigvee_a^t(f)\right)$$
$$=\frac{1}{2}\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right]d\left(\bigvee_a^t(f)\right)$$
$$+\frac{1}{2}\int_{\frac{a+b}{2}}^b\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right]d\left(\bigvee_a^t(f)\right).$$

Using integration by parts we have

$$\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right]d\left(\bigvee_a^t(f)\right)=\alpha\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\bigvee_a^t(f)\,dt$$

and

$$\begin{aligned}
&\int_{\frac{a+b}{2}}^{b} \left[(t-a)^{\alpha}-(b-t)^{\alpha}\right] d\left(\bigvee_{a}^{t}(f)\right) \\
&= \left[(t-a)^{\alpha}-(b-t)^{\alpha}\right]\left(\bigvee_{a}^{t}(f)\right)\Bigg|_{\frac{a+b}{2}}^{b} \\
&-\alpha\int_{\frac{a+b}{2}}^{b}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\bigvee_{a}^{t}(f)\,dt \\
&=(b-a)^{\alpha}\bigvee_{a}^{b}(f)-\alpha\int_{\frac{a+b}{2}}^{b}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\bigvee_{a}^{t}(f)\,dt.
\end{aligned}$$

Observe that

$$\alpha\int_{\frac{a+b}{2}}^{b}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]dt=(b-a)^{\alpha}.$$

Therefore

$$\begin{aligned}
&\int_{\frac{a+b}{2}}^{b} \left[(t-a)^{\alpha}-(b-t)^{\alpha}\right] d\left(\bigvee_{a}^{t}(f)\right) \\
&=\alpha\bigvee_{a}^{b}(f)\int_{\frac{a+b}{2}}^{b}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]dt \\
&-\alpha\int_{\frac{a+b}{2}}^{b}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\bigvee_{a}^{t}(f)\,dt \\
&=\alpha\int_{\frac{a+b}{2}}^{b}\left[\bigvee_{a}^{b}(f)-\bigvee_{a}^{t}(f)\right]\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]dt \\
&=\alpha\int_{\frac{a+b}{2}}^{b}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\bigvee_{t}^{b}(f)\,dt
\end{aligned}$$

and by (3.9) we get the first inequality in (3.8).

We have

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_{a}^{\frac{a+b}{2}}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\bigvee_{a}^{t}(f)\,dt \\
&+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\bigvee_{t}^{b}(f)\,dt \\
&\leq\frac{1}{\Gamma(\alpha)}\bigvee_{a}^{\frac{a+b}{2}}(f)\int_{a}^{\frac{a+b}{2}}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]dt \\
&+\frac{1}{\Gamma(\alpha)}\bigvee_{\frac{a+b}{2}}^{b}(f)\int_{\frac{a+b}{2}}^{b}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]dt
\end{aligned}\tag{3.10}$$

and since

$$\int_{\frac{a+b}{2}}^{b}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]dt=\int_{a}^{\frac{a+b}{2}}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]dt=\frac{(b-a)^{\alpha}}{\alpha},$$

then by (3.10) we get the last part of (3.8). □

4. Inequalities for Lipschitzian Functions

We use the following well known fact:

Lemma 3. *Let f, $u:[a,b]\to\mathbb{C}$. If f is Riemann integrable on $[a,b]$ and u is Lipschitzian on $[a,b]$ with the constant $L>0$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and*

$$\left|\int_a^b f(t)\,du(t)\right|\le L\int_a^b |f(t)|\,dt. \tag{4.1}$$

We have the following result:

Theorem 5. *Let $f:[a,b]\to\mathbb{C}$ be a Lipschitzian function on $[a,b]$ with the constant $L>0$. Then for any $x\in(a,b)$ we have*

$$\begin{aligned}&\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right]\right|\\&\le\frac{L}{\Gamma(\alpha+2)}\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\end{aligned}\tag{4.2}$$

and, see also (1.11),

$$\begin{aligned}&\left|J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)\right|\\&\le\frac{L}{\Gamma(\alpha+2)}\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right].\end{aligned}\tag{4.3}$$

Proof. Using the representation (2.1) and Lemma 3 we have for $x\in(a,b)$ that

$$\begin{aligned}&\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right]\right|\\&\le\frac{1}{\Gamma(\alpha+1)}\left[\left|\int_a^x(x-t)^{\alpha}\,df(t)\right|+\left|\int_x^b(t-x)^{\alpha}\,df(t)\right|\right]\\&\le\frac{L}{\Gamma(\alpha+1)}\left[\int_a^x(x-t)^{\alpha}\,dt+\int_x^b(t-x)^{\alpha}\,df(t)\right]\\&=\frac{L}{\Gamma(\alpha+1)}\left[\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{\alpha+1}\right]\\&=\frac{L}{\Gamma(\alpha+2)}\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right],\end{aligned}$$

which proves (4.2).

Using the representation (2.2) and Lemma 3 we have for $x \in (a,b)$ that

$$\begin{aligned}
&\left| J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] f(x) \right| \\
&\leq \frac{1}{\Gamma(\alpha+1)} \left[\left| \int_x^b (b-t)^{\alpha} \, df(t) \right| + \left| \int_a^x (t-a)^{\alpha} \, df(t) \right| \right] \\
&\leq \frac{L}{\Gamma(\alpha+1)} \left[\int_x^b (b-t)^{\alpha} \, dt + \int_a^x (t-a)^{\alpha} \, dt \right] \\
&= \frac{L}{\Gamma(\alpha+1)} \left[\frac{(b-x)^{\alpha+1} + (x-a)^{\alpha+1}}{\alpha+1} \right] \\
&= \frac{L}{\Gamma(\alpha+2)} \left[(x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right],
\end{aligned}$$

which proves (4.3). This is a different proof than the one from [15]. □

Corollary 3. *With the assumptions of Theorem 5 we have*

$$\begin{aligned}
(4.4) \quad &\left| J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} \frac{f(a)+f(b)}{2} (b-a)^{\alpha} \right| \\
&\leq \frac{1}{2^{\alpha}\Gamma(\alpha+2)} (b-a)^{\alpha+1} L
\end{aligned}$$

and

$$\begin{aligned}
(4.5) \quad &\left| J_{\frac{a+b}{2}+}^{\alpha} f(b) + J_{\frac{a+b}{2}-}^{\alpha} f(a) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \right| \\
&\leq \frac{1}{2^{\alpha}\Gamma(\alpha+2)} (b-a)^{\alpha+1} L.
\end{aligned}$$

Finally, we have

Theorem 6. *Let $f : [a,b] \to \mathbb{C}$ be a Lipschitzian function on $[a,b]$ with the constant $L > 0$. Then for $\alpha > 0$ we have the trapezoid inequality*

$$\begin{aligned}
(4.6) \quad &\left| \frac{J_{b-}^{\alpha} f(a) + J_{a+}^{\alpha} f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} \frac{f(b)+f(a)}{2} (b-a)^{\alpha} \right| \\
&\leq \frac{1}{\Gamma(\alpha+2)} \left(\frac{2^{\alpha}-1}{2^{\alpha}} \right) (b-a)^{\alpha+1} L.
\end{aligned}$$

Proof. Using the representation (2.11) and the property (4.1) we have

$$\begin{aligned}
(4.7) \quad &\left| \frac{J_{b-}^{\alpha} f(a) + J_{a+}^{\alpha} f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} \frac{f(b)+f(a)}{2} (b-a)^{\alpha} \right| \\
&\leq L \frac{1}{2\Gamma(\alpha+1)} \int_a^b \left| (b-t)^{\alpha} - (t-a)^{\alpha} \right| dt.
\end{aligned}$$

The function $h : [a, b] \to [0, \infty)$, $h(t) = |(b-t)^\alpha - (t-a)^\alpha|$ is symmetric on $[a, b]$, then

$$\begin{aligned}\int_a^b |(b-t)^\alpha - (t-a)^\alpha|\, dt &= 2\int_a^{\frac{a+b}{2}} [(b-t)^\alpha - (t-a)^\alpha]\, dt \\ &= 2\left[-\frac{\left(b-\frac{a+b}{2}\right)^{\alpha+1}}{\alpha+1} + \frac{(b-a)^{\alpha+1}}{\alpha+1} - \frac{\left(\frac{a+b}{2}-a\right)^{\alpha+1}}{\alpha+1}\right] \\ &= 2\left[\frac{(b-a)^{\alpha+1}}{\alpha+1} - \frac{(b-a)^{\alpha+1}}{2^\alpha(\alpha+1)}\right] = \frac{(b-a)^{\alpha+1}}{(\alpha+1)}\left(\frac{2^\alpha-1}{2^{\alpha-1}}\right)\end{aligned}$$

and by (4.7) we get (4.6). □

References

[1] A. Aglić Aljinović, Montgomery identity and Ostrowski type inequalities for Riemann-Liouville fractional integral. *J. Math.* **2014**, Art. ID 503195, 6 pp.

[2] T. M. Apostol, *Mathematical Analysis,* Second Edition, Addison-Wesley Publishing Company, 1975.

[3] A. O. Akdemir, Inequalities of Ostrowski's type for m- and (α, m)-logarithmically convex functions via Riemann-Liouville fractional integrals. *J. Comput. Anal. Appl.* **16** (2014), no. 2, 375–383

[4] G. A. Anastassiou, Fractional representation formulae under initial conditions and fractional Ostrowski type inequalities. *Demonstr. Math.* **48** (2015), no. 3, 357–378

[5] G. A. Anastassiou, The reduction method in fractional calculus and fractional Ostrowski type inequalities. *Indian J. Math.* **56** (2014), no. 3, 333–357.

[6] H. Budak, M. Z. Sarikaya, E. Set, Generalized Ostrowski type inequalities for functions whose local fractional derivatives are generalized s-convex in the second sense. *J. Appl. Math. Comput. Mech.* **15** (2016), no. 4, 11–21.

[7] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view. *Handbook of analytic-computational methods in applied mathematics*, 135–200, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[8] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications. *Comput. Math. Appl.* **38** (1999), no. 11-12, 33–37.

[9] S. S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation. *Bull. Austral. Math. Soc.* **60** (1999), No. 3, 495–508.

[10] S. S. Dragomir, On the midpoint quadrature formula for mappings with bounded variation and applications. *Kragujevac J. Math.* **22** (2000), 13–19.

[11] S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. Appl.* **4** (2001), No. 1, 59-66. Preprint: *RGMIA Res. Rep. Coll.* **2** (1999), Art. 7, [Online: http://rgmia.org/papers/v2n1/v2n1-7.pdf]

[12] S. S. Dragomir, Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation. *Arch. Math.* (Basel) **91** (2008), no. 5, 450–460.

[13] S. S. Dragomir, Refinements of the Ostrowski inequality in terms of the cumulative variation and applications, *Analysis* (Berlin) **34** (2014), No. 2, 223–240. Preprint: *RGMIA Res. Rep. Coll.* **16** (2013), Art. 29 [Online:http://rgmia.org/papers/v16/v16a29.pdf].

[14] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results, *Australian J. Math. Anal. Appl.*, Volume **14**, Issue 1, Article 1, pp. 1-287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[15] S. S. Dragomir, Ostrowski type inequalities for Riemann-Liouville fractional integrals of bounded variation, Hölder and Lipschitzian functions, Preprint *RGMIA Res. Rep. Coll.* **20** (2017), Art. 48. [Online http://rgmia.org/papers/v20/v20a48.pdf].

[16] A. Guezane-Lakoud and F. Aissaoui, New fractional inequalities of Ostrowski type. *Transylv. J. Math. Mech.* **5** (2013), no. 2, 103–106

[17] A. Kashuri and R. Liko, Ostrowski type fractional integral inequalities for generalized (s, m, φ)-preinvex functions. *Aust. J. Math. Anal. Appl.* **13** (2016), no. 1, Art. 16, 11 pp.

[18] M. A. Noor, K. I.Noor and S. Iftikhar, Fractional Ostrowski inequalities for harmonic h-preinvex functions. *Facta Univ. Ser. Math. Inform.* **31** (2016), no. 2, 417–445

[19] M. Z. Sarikaya and H. Filiz, Note on the Ostrowski type inequalities for fractional integrals. *Vietnam J. Math.* **42** (2014), no. 2, 187–190

[20] M. Z. Sarikaya and H. Budak, Generalized Ostrowski type inequalities for local fractional integrals. *Proc. Amer. Math. Soc.* **145** (2017), no. 4, 1527–1538.

[21] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Comput. Math. Appl.* 63 (2012), no. 7, 1147–1154.

[22] M. Tunç, On new inequalities for h-convex functions via Riemann-Liouville fractional integration, *Filomat* **27**:4 (2013), 559–565.

[23] M. Tunç, Ostrowski type inequalities for m- and (α, m)-geometrically convex functions via Riemann-Louville fractional integrals. *Afr. Mat.* **27** (2016), no. 5-6, 841–850.

[24] H. Yildirim and Z. Kirtay, Ostrowski inequality for generalized fractional integral and related inequalities, *Malaya J. Mat.*, **2**(3)(2014), 322-329.

[25] C. Yildiz, E, Özdemir and Z. S. Muhamet, New generalizations of Ostrowski-like type inequalities for fractional integrals. *Kyungpook Math. J.* **56** (2016), no. 1, 161–172.

[26] H. Yue, Ostrowski inequality for fractional integrals and related fractional inequalities. *Transylv. J. Math. Mech.* **5** (2013), no. 1, 85–89.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa