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OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF ABSOLUTELY CONTINUOUS FUNCTIONS WITH BOUNDED DERIVATIVES

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Abstract. In this paper we establish some Ostrowski and trapezoid type inequalities for the Riemann-Liouville fractional integrals of absolutely continuous functions with bounded derivatives. Applications for mid-point and trapezoid inequalities are provided as well. They generalize the know results holding for the classical Riemann integral.

1. Introduction

In 2002 [12], we proved the following *Ostrowski type inequality* for convex functions $f:[a,b]\to\mathbb{R}$ and $x\in(a,b)$

$$\frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right]\le\int_a^b f(t)\,dt-(b-a)f(x)\le\frac{1}{2}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right]. \tag{1.1}$$

In particular, we have the *mid-point inequalities*

$$\frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right](b-a)^2\le\int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right)\le\frac{1}{8}\left[f'_-(b)-f'_+(a)\right](b-a)^2, \tag{1.2}$$

with the constant $\frac{1}{8}$ as best possible in both inequalities.

In the same year [13], we also obtained the following *generalized trapezoid type inequality* for convex functions $f:[a,b]\to\mathbb{R}$ and $x\in(a,b)$

$$\frac{1}{2}\left[(b-x)^2 f'_+(x)-(x-a)^2 f'_-(x)\right]\le(x-a)f(a)+(b-x)f(b)-\int_a^b f(t)\,dt\le\frac{1}{2}\left[(b-x)^2 f'_-(b)-(x-a)^2 f'_+(a)\right]. \tag{1.3}$$

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In particular, we have the *trapezoid inequality*

$$\frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a)^{2}\leq\frac{f(a)+f(b)}{2}(b-a)-\int_{a}^{b}f(t)\,dt \tag{1.4}$$
$$\leq\frac{1}{8}\left[f'_{-}(b)-f'_{+}(a)\right](b-a)^{2},$$

with the constant $\frac{1}{8}$ as best possible in both inequalities.

These results were generalized in the following manner:

Theorem 1 (Dragomir, 2003 [14]). *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $x\in[a,b]$. Suppose that there exist the functions m_i, $M_i:[a,b]\to\mathbb{R}$ $\left(i=\overline{1,2}\right)$ with the properties:*

$$m_1(x)\leq f'(t)\leq M_1(x)\quad \text{for a.e. } t\in[a,x] \tag{1.5}$$

and

$$m_2(x)\leq f'(t)\leq M_2(x)\quad \text{for a.e. } t\in(x,b]. \tag{1.6}$$

Then we have the inequalities:

$$\frac{1}{2(b-a)}\left[m_1(x)(x-a)^2-M_2(x)(b-x)^2\right]\leq f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{1.7}$$
$$\leq\frac{1}{2(b-a)}\left[M_1(x)(x-a)^2-m_2(x)(b-x)^2\right].$$

The constant $\frac{1}{2}$ is sharp on both sides.

If we assume global bounds for the derivative, then we have:

Corollary 1 (Dragomir, 2003 [14]). *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and the derivative $f':[a,b]\to\mathbb{R}$ is bounded above and below, that is,*

$$-\infty<m\leq f'(t)\leq M<\infty \text{ for a.e. } t\in[a,b], \tag{1.8}$$

then we have the inequality

$$\frac{1}{2(b-a)}\left[m(x-a)^2-M(b-x)^2\right]\leq f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{1.9}$$
$$\leq\frac{1}{2(b-a)}\left[M(x-a)^2-m(b-x)^2\right]$$

for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best in both inequalities.

In particular, we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{8}(M-m)(b-a), \tag{1.10}$$

with $\frac{1}{8}$ as the best possible constant.

In order to extend these results for fractional integrals we need the following definitions.

Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt$$

for $a < x \le b$ and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt$$

for $a \le x < b$, where Γ is the *Gamma function*. For $\alpha = 0$, they are defined as

$$J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x) \text{ for } x \in (a,b).$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[6], [19]-[29] and the references therein.

Motivated by the above results, we obtain in this paper some inequalities for the Riemann-Liouville fractional integrals of absolutely continuous functions with bounded derivatives and of convex functions. Applications for mid-point and trapezoid inequalities are provided as well.

2. Some Identities

We have the following representation:

Lemma 1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$.*

(i) *For any $x \in (a,b)$ we have*

$$\begin{aligned} &J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) \\ &= \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b) \right] \\ &+ \frac{1}{\Gamma(\alpha+1)} \left[\int_a^x (x-t)^{\alpha} f'(t)\, dt - \int_x^b (t-x)^{\alpha} f'(t)\, dt \right]. \end{aligned} \tag{2.1}$$

(ii) *For any $x \in (a,b)$ we have*

$$\begin{aligned} &J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b) \\ &= \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] f(x) \\ &+ \frac{1}{\Gamma(\alpha+1)} \left[\int_x^b (b-t)^{\alpha} f'(t)\, dt - \int_a^x (t-a)^{\alpha} f'(t)\, dt \right]. \end{aligned} \tag{2.2}$$

Proof. (i) Since $f : [a,b] \to \mathbb{C}$ is an absolutely continuous function on $[a,b]$, then the Lebesgue integrals

$$\int_a^x (x-t)^{\alpha} f'(t)\, dt \text{ and } \int_x^b (t-x)^{\alpha} f'(t)\, dt$$

exist and integrating by parts, we have

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)} \int_a^x (x-t)^{\alpha} f'(t)\, dt \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt - \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha} f(a) \\ &= J_{a+}^{\alpha} f(x) - \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha} f(a) \end{aligned} \tag{2.3}$$

for $a < x \leq b$ and

$$\frac{1}{\Gamma(\alpha+1)}\int_x^b (t-x)^{\alpha} f'(t)\,dt = \frac{1}{\Gamma(\alpha+1)}(b-x)^{\alpha} f(b) - \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt = \frac{1}{\Gamma(\alpha+1)}(b-x)^{\alpha} f(b) - J_{b-}^{\alpha} f(x) \tag{2.4}$$

for $a \leq x < b$.

From (2.3) we have

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha} f(a) + \frac{1}{\Gamma(\alpha+1)}\int_a^x (x-t)^{\alpha} f'(t)\,dt$$

for $a < x \leq b$ and from (2.4) we have

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)}(b-x)^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)}\int_x^b (t-x)^{\alpha} f'(t)\,dt,$$

for $a \leq x < b$, which by addition give (2.1).

(ii) We have

$$J_{x+}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha)}\int_x^b (b-t)^{\alpha-1} f(t)\,dt$$

for $a \leq x < b$ and

$$J_{x-}^{\alpha} f(a) = \frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1} f(t)\,dt$$

for $a < x \leq b$.

Since $f : [a,b] \to \mathbb{C}$ is an absolutely continuous function $[a,b]$, then the Lebesgue integrals

$$\int_a^x (t-a)^{\alpha} f'(t)\,dt \text{ and } \int_x^b (b-t)^{\alpha} f'(t)\,dt$$

exist and integrating by parts, we have

$$\frac{1}{\Gamma(\alpha+1)}\int_a^x (t-a)^{\alpha} f'(t)\,dt = \frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha} f(x) - \frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1} f(t)\,dt = \frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha} f(x) - J_{x-}^{\alpha} f(a) \tag{2.5}$$

for $a < x \leq b$ and

$$\frac{1}{\Gamma(\alpha+1)}\int_x^b (b-t)^{\alpha} f'(t)\,dt = \frac{1}{\Gamma(\alpha)}\int_x^b (b-t)^{\alpha-1} f(t)\,dt - \frac{1}{\Gamma(\alpha+1)}(b-x)^{\alpha} f(x) = J_{x+}^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)}(b-x)^{\alpha} f(x) \tag{2.6}$$

for $a \leq x < b$.

From (2.5) we have

$$J_{x-}^{\alpha}f(a)=\frac{1}{\Gamma(\alpha+1)}(x-a)^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}\int_a^x(t-a)^{\alpha}f'(t)\,dt \tag{2.7}$$

for $a<x\le b$ and from (2.6)

$$J_{x+}^{\alpha}f(b)=\frac{1}{\Gamma(\alpha+1)}(b-x)^{\alpha}f(x)+\frac{1}{\Gamma(\alpha+1)}\int_x^b(b-t)^{\alpha}f'(t)\,dt, \tag{2.8}$$

for $a\le x<b$, which by addition produce (2.2). □

Corollary 2. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. We have the midpoint equalities*

$$\begin{aligned}&J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\\&=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}\\&+\frac{1}{\Gamma(\alpha+1)}\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^{\alpha}f'(t)\,d-\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)^{\alpha}f'(t)\,dt\right]\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}&J_{\frac{a+b}{2}-}^{\alpha}f(a)+J_{\frac{a+b}{2}+}^{\alpha}f(b)\\&=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha}\\&+\frac{1}{\Gamma(\alpha+1)}\left[\int_{\frac{a+b}{2}}^b(b-t)^{\alpha}f'(t)\,dt-\int_a^{\frac{a+b}{2}}(t-a)^{\alpha}f'(t)\,dt\right]\end{aligned} \tag{2.10}$$

and the trapezoid equality

$$\begin{aligned}\frac{J_{b-}^{\alpha}f(a)+J_{a+}^{\alpha}f(b)}{2}&=\frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^{\alpha}\\&+\frac{1}{\Gamma(\alpha+1)}\int_a^b\frac{(b-t)^{\alpha}-(t-a)^{\alpha}}{2}f'(t)\,dt.\end{aligned} \tag{2.11}$$

Proof. Equality (2.9) follows by (2.1) for $x=\frac{a+b}{2}$ while the equality (2.10) follows by (2.2).

For $x=b$ in (2.7) we have

$$J_{b-}^{\alpha}f(a)=\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\int_a^b(t-a)^{\alpha}f'(t)\,dt$$

while from (2.8) we have for $x=a$ that

$$J_{a+}^{\alpha}f(b)=\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}f(a)+\frac{1}{\Gamma(\alpha+1)}\int_a^b(b-t)^{\alpha}f'(t)\,dt.$$

If we add these two equalities and divide by 2, we get (2.11). □

3. Inequalities for Functions with Bounded Derivatives

We have:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $x\in(a,b)$ and there exists the real numbers $m_1(x)$, $M_1(x)$, $m_2(x)$, $M_2(x)$ such that*

$$m_1(x)\le f'(t)\le M_1(x)\text{ for a.e. }t\in(a,x)\tag{3.1}$$

and

$$m_2(x)\le f'(t)\le M_2(x)\text{ for a.e. }t\in(x,b)\tag{3.2}$$

then

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+2)}\left[m_2(x)(b-x)^{\alpha+1}-M_1(x)(x-a)^{\alpha+1}\right]\\&\le\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right]-J_{a+}^{\alpha}f(x)-J_{b-}^{\alpha}f(x)\\&\le\frac{1}{\Gamma(\alpha+2)}\left[M_2(x)(b-x)^{\alpha+1}-m_1(x)(x-a)^{\alpha+1}\right]\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+2)}\left[m_2(x)(b-x)^{\alpha+1}-M_1(x)(x-a)^{\alpha+1}\right]\\&\le J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)\\&\le\frac{1}{\Gamma(\alpha+2)}\left[M_2(x)(b-x)^{\alpha+1}-m_1(x)(x-a)^{\alpha+1}\right].\end{aligned}\tag{3.4}$$

Proof. We have

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right]-J_{a+}^{\alpha}f(x)-J_{b-}^{\alpha}f(x)\\&=\frac{1}{\Gamma(\alpha+1)}\left[\int_x^b(t-x)^{\alpha}f'(t)\,dt-\int_a^x(x-t)^{\alpha}f'(t)\,dt\right]\end{aligned}\tag{3.5}$$

for any $x\in(a,b)$.

Using the conditions (3.1) and (3.2) we have

$$m_2(x)\int_x^b(t-x)^{\alpha}dt\le\int_x^b(t-x)^{\alpha}f'(t)\,dt\le M_2(x)\int_x^b(t-x)^{\alpha}dt$$

and

$$m_1(x)\int_a^x(x-t)^{\alpha}dt\le\int_a^x(x-t)^{\alpha}f'(t)\,dt\le M_1(x)\int_a^x(x-t)^{\alpha}dt$$

namely

$$\frac{1}{\alpha+1}m_2(x)(b-x)^{\alpha+1}\le\int_x^b(t-x)^{\alpha}f'(t)\,dt\le\frac{1}{\alpha+1}M_2(x)(b-x)^{\alpha+1}$$

and

$$\frac{1}{\alpha+1}m_1(x)(x-a)^{\alpha+1}\le\int_a^x(x-t)^{\alpha}f'(t)\,dt\le\frac{1}{\alpha+1}M_1(x)(x-a)^{\alpha+1}.$$

These imply that

$$\begin{aligned}
&\frac{1}{\alpha+1}\left[m_2(x)(b-x)^{\alpha+1}-M_1(x)(x-a)^{\alpha+1}\right]\\
&\leq \int_x^b (t-x)^{\alpha} f'(t)\,dt-\int_a^x (x-t)^{\alpha} f'(t)\,dt\\
&\leq \frac{1}{\alpha+1}\left[M_2(x)(b-x)^{\alpha+1}-m_1(x)(x-a)^{\alpha+1}\right]
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+2)}\left[m_2(x)(b-x)^{\alpha+1}-M_1(x)(x-a)^{\alpha+1}\right]\\
&\leq \frac{1}{\Gamma(\alpha+1)}\left[\int_x^b (t-x)^{\alpha} f'(t)\,dt-\int_a^x (x-t)^{\alpha} f'(t)\,dt\right]\\
&\leq \frac{1}{\Gamma(\alpha+2)}\left[M_2(x)(b-x)^{\alpha+1}-m_1(x)(x-a)^{\alpha+1}\right].
\end{aligned}$$

By using the equality (3.5) we get (3.3).

From (2.2) we have

$$\begin{aligned}
(3.6)\qquad &J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)\\
&=\frac{1}{\Gamma(\alpha+1)}\left[\int_x^b (b-t)^{\alpha} f'(t)\,dt-\int_a^x (t-a)^{\alpha} f'(t)\,dt\right].
\end{aligned}$$

In a similar way, we have

$$\frac{1}{\alpha+1}m_2(x)(b-x)^{\alpha+1}\leq \int_x^b (b-t)^{\alpha} f'(t)\,dt\leq \frac{1}{\alpha+1}M_2(x)(b-x)^{\alpha+1}$$

and

$$\frac{1}{\alpha+1}m_1(x)(x-a)^{\alpha+1}\leq \int_a^x (t-a)^{\alpha} f'(t)\,dt\leq \frac{1}{\alpha+1}M_1(x)(x-a)^{\alpha+1},$$

which implies that

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+2)}\left[m_2(x)(b-x)^{\alpha+1}-M_1(x)(x-a)^{\alpha+1}\right]\\
&\leq \frac{1}{\Gamma(\alpha+1)}\left[\int_x^b (b-t)^{\alpha} f'(t)\,dt-\int_a^x (t-a)^{\alpha} f'(t)\,dt\right]\\
&\leq \frac{1}{\Gamma(\alpha+2)}\left[M_2(x)(b-x)^{\alpha+1}-m_1(x)(x-a)^{\alpha+1}\right]
\end{aligned}$$

and by (3.6) we get (3.4). □

Remark 1. *If we take $\alpha = 1$ in (3.3), then we get*

$$\frac{1}{2}\left[m_2(x)(b-x)^2 - M_1(x)(x-a)^2\right] \le (x-a)f(a) + (b-x)f(b) - \int_a^b f(t)\,dt \le \frac{1}{2}\left[M_2(x)(b-x)^2 - m_1(x)(x-a)^2\right] \tag{3.7}$$

for any $x \in (a,b)$. If we take $\alpha = 1$ in (3.4), then we get (1.7).

Corollary 3. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If there exists the real numbers m_1, M_1, m_2, M_2 such that*

$$m_1 \le f'(t) \le M_1 \text{ for a.e. } t \in \left(a, \frac{a+b}{2}\right) \tag{3.8}$$

and

$$m_2 \le f'(t) \le M_2 \text{ for a.e. } t \in \left(\frac{a+b}{2}, b\right) \tag{3.9}$$

then

$$\begin{aligned} &\frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(m_2 - M_1) \\ &\le \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}(b-a)^{\alpha} - J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) - J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) \\ &\le \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(M_2 - m_1) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &\frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(m_2 - M_1) \\ &\le J_{\frac{a+b}{2}-}^{\alpha} f(a) + J_{\frac{a+b}{2}+}^{\alpha} f(b) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)(b-a)^{\alpha} \\ &\le \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(M_2 - m_1). \end{aligned} \tag{3.11}$$

In particular, we have the simpler inequalities:

Corollary 4. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If there exists the real numbers m, M, such that $m \le f'(t) \le M$ for a.e. $t \in (a,b)$, then*

$$\begin{aligned} &\left|\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}(b-a)^{\alpha} - J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) - J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)\right| \\ &\le \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(M - m) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &\left|J_{\frac{a+b}{2}-}^{\alpha} f(a) + J_{\frac{a+b}{2}+}^{\alpha} f(b) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right)(b-a)^{\alpha}\right| \\ &\le \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(M - m). \end{aligned} \tag{3.13}$$

Remark 2. *If we take $\alpha = 1$ in (3.12), then we get*

$$\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)^2(M-m) \tag{3.14}$$

while from (3.13) we get (1.10).

We also have the following trapezoid type result:

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If there exists the real numbers m, M, such that $m \le f'(t) \le M$ for a.e. $t\in(a,b)$, then*

$$\begin{aligned} &\left|\frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^\alpha - \frac{J_{b-}^\alpha f(a)+J_{a+}^\alpha f(b)}{2}\right| \\ &\le \frac{2^\alpha-1}{2^{\alpha+1}\Gamma(\alpha+2)}(M-m)(b-a)^{\alpha+1}. \end{aligned} \tag{3.15}$$

Proof. We have by (2.11) that

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^\alpha - \frac{J_{b-}^\alpha f(a)+J_{a+}^\alpha f(b)}{2} \\ &= \frac{1}{\Gamma(\alpha+1)}\int_a^b \frac{(t-a)^\alpha-(b-t)^\alpha}{2} f'(t)\,dt. \end{aligned}$$

Observe also that

$$\begin{aligned} &\int_a^b \frac{(t-a)^\alpha-(b-t)^\alpha}{2}\left(f'(t)-\frac{m+M}{2}\right)dt \\ &= \int_a^b \frac{(t-a)^\alpha-(b-t)^\alpha}{2} f'(t)\,dt - \frac{m+M}{2}\int_a^b \frac{(t-a)^\alpha-(b-t)^\alpha}{2}dt \end{aligned}$$

and since

$$\int_a^b \left[(t-a)^\alpha-(b-t)^\alpha\right]dt = \frac{(b-a)^{\alpha+1}}{\alpha+1}-\frac{(b-a)^{\alpha+1}}{\alpha+1}=0,$$

then we have the following identity of interest

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^\alpha - \frac{J_{b-}^\alpha f(a)+J_{a+}^\alpha f(b)}{2} \\ &= \frac{1}{\Gamma(\alpha+1)}\int_a^b \frac{(t-a)^\alpha-(b-t)^\alpha}{2}\left(f'(t)-\frac{m+M}{2}\right)dt. \end{aligned} \tag{3.16}$$

By taking the modulus in (3.16), we get

$$\begin{aligned} &\left|\frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^\alpha - \frac{J_{b-}^\alpha f(a)+J_{a+}^\alpha f(b)}{2}\right| \\ &\le \frac{1}{\Gamma(\alpha+1)}\int_a^b \left|\frac{(t-a)^\alpha-(b-t)^\alpha}{2}\right|\left|f'(t)-\frac{m+M}{2}\right|dt \\ &\le \frac{1}{4}(M-m)\frac{1}{\Gamma(\alpha+1)}\int_a^b \left|(b-t)^\alpha-(t-a)^\alpha\right|dt. \end{aligned} \tag{3.17}$$

The function $h:[a,b]\to[0,\infty)$, $h(t)=|(b-t)^{\alpha}-(t-a)^{\alpha}|$ is symmetric on $[a,b]$, then

$$\begin{aligned}&\int_a^b|(b-t)^{\alpha}-(t-a)^{\alpha}|\,dt\\&=2\int_a^{\frac{a+b}{2}}[(b-t)^{\alpha}-(t-a)^{\alpha}]\,dt\\&=2\left[-\left.\frac{(b-t)^{\alpha+1}}{\alpha+1}\right|_a^{\frac{a+b}{2}}-\left.\frac{(t-a)^{\alpha+1}}{\alpha+1}\right|_a^{\frac{a+b}{2}}\right]\\&=2\left[-\frac{\left(b-\frac{a+b}{2}\right)^{\alpha+1}}{\alpha+1}+\frac{(b-a)^{\alpha+1}}{\alpha+1}-\frac{\left(\frac{a+b}{2}-a\right)^{\alpha+1}}{\alpha+1}\right]\\&=2\left[\frac{(b-a)^{\alpha+1}}{\alpha+1}-\frac{(b-a)^{\alpha+1}}{2^{\alpha}(\alpha+1)}\right]=\frac{2^{\alpha}-1}{2^{\alpha-1}(\alpha+1)}(b-a)^{\alpha+1}.\end{aligned}$$

□

4. Inequalities for Convex Functions

We have the following result for convex functions:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $x\in(a,b)$, then we have the inequalities*

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+2)}\left[f'_+(x)(b-x)^{\alpha+1}-f'_-(x)(x-a)^{\alpha+1}\right]\\&\le\frac{1}{\Gamma(\alpha+1)}[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)]-J^{\alpha}_{a+}f(x)-J^{\alpha}_{b-}f(x)\\&\le\frac{1}{\Gamma(\alpha+2)}\left[f'_-(b)(b-x)^{\alpha+1}-f'_+(a)(x-a)^{\alpha+1}\right]\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+2)}\left[f'_+(x)(b-x)^{\alpha+1}-f'_-(x)(x-a)^{\alpha+1}\right]\\&\le J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(b)-\frac{1}{\Gamma(\alpha+1)}[(x-a)^{\alpha}+(b-x)^{\alpha}]f(x)\\&\le\frac{1}{\Gamma(\alpha+2)}\left[f'_-(b)(b-x)^{\alpha+1}-f'_+(a)(x-a)^{\alpha+1}\right],\end{aligned}\tag{4.2}$$

where $f'_{\pm}(\cdot)$ are the lateral derivatives of f.

Proof. Since f is convex, then the derivative f' exists almost everywhere on $[a,b]$ and

$$f'_+(a)\le f'(t)\le f'_-(x)\text{ for a.e. }t\in(a,x)$$

and

$$f'_+(x)\le f'(t)\le f'_-(b)\text{ for a.e. }t\in(x,b).$$

Now, writing the inequalities (3.3) and (3.4) for $m_1(x)=f'_+(a)$, $M_1(x)=f'_-(x)$, $m_2(x)=f'_+(x)$ and $M_2(x)=f'_-(b)$ we get the desired results (4.1) and (4.2). □

Remark 3. *If we take $\alpha = 1$ in (4.1) and (4.2), then we recapture (1.3) and (1.1) that hold for convex functions.*

Corollary 5. *Let $f : [a, b] \to \mathbb{R}$ be a convex function, then we have the inequalities*

$$
\begin{aligned}
(4.3)\quad 0 &\le \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b-a)^{\alpha+1} \\
&\le \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}(b-a)^{\alpha} - J^{\alpha}_{a+}f\left(\frac{a+b}{2}\right) - J^{\alpha}_{b-}f\left(\frac{a+b}{2}\right) \\
&\le \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_-(b) - f'_+(a)\right](b-a)^{\alpha+1},
\end{aligned}
$$

$$
\begin{aligned}
(4.4)\quad 0 &\le \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b-a)^{\alpha+1} \\
&\le J^{\alpha}_{\frac{a+b}{2}-}f(a) + J^{\alpha}_{\frac{a+b}{2}+}f(b) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha} \\
&\le \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_-(b) - f'_+(a)\right](b-a)^{\alpha+1},
\end{aligned}
$$

and

$$
\begin{aligned}
(4.5)\quad 0 &\le \frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^{\alpha} - \frac{J^{\alpha}_{b-}f(a) + J^{\alpha}_{a+}f(b)}{2} \\
&\le \frac{2^{\alpha}-1}{2^{\alpha+1}\Gamma(\alpha+2)}\left(f'_-(b) - f'_+(a)\right)(b-a)^{\alpha+1}.
\end{aligned}
$$

If we take $\alpha = 1$ in (4.3) and (4.4), then we recapture the midpoint and trapezoid inequalities for convex functions mentioned in the introduction.

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