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COMPOSITE OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF FUNCTIONS WITH BOUNDED VARIATION

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Abstract. In this paper we establish some composite Ostrowski and generalized trapezoid type inequalities for the Riemann-Liouville fractional integrals of functions of bounded variation. Applications for composite mid-point and trapezoid inequalities are provided as well. They generalize the know results holding for the classical Riemann integral.

1. Introduction

Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)\,dt$$

for $a<x\le b$ and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)\,dt$$

for $a\le x<b$, where Γ is the *Gamma function*. For $\alpha=0$, they are defined as

$$J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)\text{ for }x\in(a,b).$$

The following Ostrowski type inequalities for functions of bounded variation generalize the corresponding results for the Riemann integral obtained in [9], [11], [10] and have been established recently by the author in [15] :

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be a complex valued function of bounded variation on the real interval $[a,b]$.*

(i) *For any $x\in(a,b)$ we have*

$$\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \le\frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}(x-t)^{\alpha-1}\bigvee_{t}^{x}(f)\,dt+\int_{x}^{b}(t-x)^{\alpha-1}\bigvee_{x}^{t}(f)\,dt\right] \tag{1.1}$$

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$$\leq \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}\bigvee_{a}^{x}(f)+(b-x)^{\alpha}\bigvee_{x}^{b}(f)\right]$$

$$\leq \frac{1}{\Gamma(\alpha+1)}$$

$$\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}\bigvee_{a}^{b}(f); \\ \\ ((x-a)^{\alpha p}+(b-x)^{\alpha p})^{1/p}\left(\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right]((x-a)^{\alpha}+(b-x)^{\alpha}),\end{cases}$$

and

$$\left|J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \tag{1.2}$$

$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_{x}^{b}(b-t)^{\alpha-1}\bigvee_{x}^{t}(f)\,dt+\int_{a}^{x}(t-a)^{\alpha-1}\bigvee_{t}^{x}(f)\,dt\right]$$

$$\leq \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}\bigvee_{a}^{x}(f)+(b-x)^{\alpha}\bigvee_{x}^{b}(f)\right]$$

$$\leq \frac{1}{\Gamma(\alpha+1)}$$

$$\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}\bigvee_{a}^{b}(f); \\ \\ ((x-a)^{\alpha p}+(b-x)^{\alpha p})^{1/p}\left(\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right]((x-a)^{\alpha}+(b-x)^{\alpha}).\end{cases}$$

(ii) *For any* $x\in[a,b]$ *we have*

$$\left|\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}-\frac{1}{\Gamma(\alpha+1)}f(x)(b-a)^{\alpha}\right| \tag{1.3}$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\bigvee_{t}^{x}(f)\,dt$$

$$+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\bigvee_{x}^{t}(f)\,dt$$

$$\leq \frac{1}{2\Gamma(\alpha+1)}\left[(b-a)^{\alpha}+(x-a)^{\alpha}-(b-x)^{\alpha}\right]\bigvee_{a}^{x}(f)$$

$$+\frac{1}{2\Gamma(\alpha+1)}\left[(b-a)^{\alpha}+(b-x)^{\alpha}-(x-a)^{\alpha}\right]\bigvee_{x}^{b}(f)$$

$$\leq \frac{1}{2\Gamma(\alpha+1)}\begin{cases}[(b-a)^{\alpha}+|(x-a)^{\alpha}-(b-x)^{\alpha}|]\bigvee_{a}^{b}(f), \\ (b-a)^{\alpha}\left[\bigvee_{a}^{b}(f)+\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right].\end{cases}$$

The following mid-point inequalities that can be derived from Theorem 1 are of interest as well:

$$\left|J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)\right| \tag{1.4}$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^{\alpha-1}\bigvee_{t}^{\frac{a+b}{2}}(f)\,dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)^{\alpha-1}\bigvee_{\frac{a+b}{2}}^{t}(f)\,dt$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)}(b-a)^{\alpha}\bigvee_{a}^{b}(f),$$

$$\left|J_{\frac{a+b}{2}+}^{\alpha}f(b)+J_{\frac{a+b}{2}-}^{\alpha}f(a)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)\right| \tag{1.5}$$
$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha-1}\bigvee_{\frac{a+b}{2}}^{t}(f)\,dt+\int_{a}^{\frac{a+b}{2}}(t-a)^{\alpha-1}\bigvee_{t}^{\frac{a+b}{2}}(f)\,dt\right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)}(b-a)^{\alpha}\bigvee_{a}^{b}(f)$$

and

$$\left|\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}-\frac{1}{\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha}\right| \tag{1.6}$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_{a}^{\frac{a+b}{2}}\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\bigvee_{t}^{\frac{a+b}{2}}(f)\,dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\bigvee_{\frac{a+b}{2}}^{t}(f)\,dt$$
$$\leq \frac{1}{2\Gamma(\alpha+1)}(b-a)^{\alpha}\bigvee_{a}^{b}(f).$$

Motivated by the above results, in this paper we establish some composite Ostrowski and generalized trapezoid type inequalities for the Riemann-Liouville fractional integrals of functions of bounded variation. Applications for composite midpoint and trapezoid inequalities are provided as well. They generalize the know results holding for the classical Riemann integral.

2. Some Identities

We have the following two parameter identity:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$ and $\lambda,\ \mu\in\mathbb{C}$.*

(i) If $x\in(a,b)$, then we have the representations

$$J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha+1)}\left[\lambda(x-a)^{\alpha}+\mu(b-x)^{\alpha}\right]+\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\left[f(t)-\lambda\right]dt+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}\left[f(t)-\mu\right]dt. \tag{2.1}$$

and

$$J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)=\frac{1}{\Gamma(\alpha+1)}\left[\lambda(x-a)^{\alpha}+\mu(b-x)^{\alpha}\right]+\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(t-a)^{\alpha-1}\left[f(t)-\lambda\right]dt+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(b-t)^{\alpha-1}\left[f(t)-\mu\right]dt. \tag{2.2}$$

(ii) We have the representation

$$\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}=\frac{\lambda+\mu}{2}\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}+\frac{1}{2\Gamma(\alpha)}\left[\int_{a}^{b}(b-t)^{\alpha-1}\left[f(t)-\lambda\right]dt+\int_{a}^{b}(t-a)^{\alpha-1}\left[f(t)-\mu\right]dt\right]. \tag{2.3}$$

Proof. (i) We have that

$$\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\left[f(t)-\lambda\right]dt=J_{a+}^{\alpha}f(x)-\frac{\lambda}{\Gamma(\alpha+1)}(x-a)^{\alpha} \tag{2.4}$$

for $a<x\le b$ and

$$\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}\left[f(t)-\mu\right]dt=J_{b-}^{\alpha}f(x)-\frac{\mu}{\Gamma(\alpha+1)}(b-x)^{\alpha} \tag{2.5}$$

for $a\le x<b$.

By adding these equalities for $x\in(a,b)$ we get the representation (2.1).

By the definition of fractional integrals we have

$$J_{x+}^{\alpha}f(b)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(b-t)^{\alpha-1}f(t)\,dt$$

for $a\le x<b$ and

$$J_{x-}^{\alpha}f(a)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(t-a)^{\alpha-1}f(t)\,dt$$

for $a<x\le b$.

Therefore

$$\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(t-a)^{\alpha-1}\left[f(t)-\lambda\right]dt=J_{x-}^{\alpha}f(a)-\frac{\lambda}{\Gamma(\alpha+1)}(x-a)^{\alpha}$$

and

$$\frac{1}{\Gamma(\alpha)}\int_x^b (b-t)^{\alpha-1}[f(t)-\mu]\,dt = J_{x+}^{\alpha} f(b) - \frac{\mu}{\Gamma(\alpha+1)}(b-x)^{\alpha}.$$

By adding these equalities for $x \in (a,b)$ we get the representation (2.2).

From (2.4) for $x = b$ we have

$$\frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}[f(t)-\lambda]\,dt = J_{a+}^{\alpha} f(b) - \frac{\lambda}{\Gamma(\alpha+1)}(b-a)^{\alpha}. \tag{2.6}$$

From (2.4) for $x = a$ we have

$$\frac{1}{\Gamma(\alpha)}\int_a^b (t-a)^{\alpha-1}[f(t)-\mu]\,dt = J_{b-}^{\alpha} f(a) - \frac{\mu}{\Gamma(\alpha+1)}(b-a)^{\alpha}. \tag{2.7}$$

If we add the equalities (2.6) and (2.11) and divide by 2 we get the equality (2.3). □

Corollary 1. *With the assumptions of Lemma 1 we have*

$$\begin{aligned} J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) &= \frac{1}{2^{\alpha}\Gamma(\alpha+1)}(\lambda+\mu)(b-a)^{\alpha} \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^{\alpha-1}[f(t)-\lambda]\,dt \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)^{\alpha-1}[f(t)-\mu]\,dt \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} J_{\frac{a+b}{2}-}^{\alpha} f(a) + J_{\frac{a+b}{2}+}^{\alpha} f(b) &= \frac{1}{2^{\alpha}\Gamma(\alpha+1)}(\lambda+\mu)(b-a)^{\alpha} \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}(t-a)^{\alpha-1}[f(t)-\lambda]\,dt \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^b (b-t)^{\alpha-1}[f(t)-\mu]\,dt. \end{aligned} \tag{2.9}$$

Corollary 2. *With the assumptions of Lemma 1 we have*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \frac{J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x)}{2}dx \\ &= \frac{1}{\Gamma(\alpha+2)}\left(\frac{\lambda+\mu}{2}\right)(b-a)^{\alpha} \\ &+ \frac{1}{2\Gamma(\alpha)}\frac{1}{b-a}\int_a^b\left(\int_a^x (x-t)^{\alpha-1}[f(t)-\lambda]\,dt\right)dx \\ &+ \frac{1}{2\Gamma(\alpha)}\frac{1}{b-a}\int_a^b\left(\int_x^b (t-x)^{\alpha-1}[f(t)-\mu]\,dt\right)dx \end{aligned} \tag{2.10}$$

and

$$\frac{1}{b-a}\int_a^b \frac{J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)}{2}dx \tag{2.11}$$
$$=\frac{1}{\Gamma(\alpha+2)}\left(\frac{\lambda+\mu}{2}\right)(b-a)^{\alpha}$$
$$+\frac{1}{\Gamma(\alpha)}\frac{1}{b-a}\int_a^b\left(\int_a^x (t-a)^{\alpha-1}[f(t)-\lambda]\,dt\right)dx$$
$$+\frac{1}{\Gamma(\alpha)}\frac{1}{b-a}\int_a^b\left(\int_x^b (b-t)^{\alpha-1}[f(t)-\mu]\,dt\right)dx.$$

The above inequalities (2.1), (2.2) and (2.3) have some particular cases of interest out of which we list the following.

1. If we take $\mu=\lambda$ in Lemma 1, then for $f\in L[a,b]$ and $x\in(a,b)$ we get the equalities

$$J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)=\frac{\lambda}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right] \tag{2.12}$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}[f(t)-\lambda]\,dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}[f(t)-\lambda]\,dt,$$

$$J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)=\frac{\lambda}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right] \tag{2.13}$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1}[f(t)-\lambda]\,dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_x^b (b-t)^{\alpha-1}[f(t)-\lambda]\,dt$$

and

$$\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}=\frac{\lambda}{\Gamma(\alpha+1)}(b-a)^{\alpha} \tag{2.14}$$
$$+\frac{1}{2\Gamma(\alpha)}\int_a^b\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right][f(t)-\lambda]\,dt$$

for $\lambda\in\mathbb{C}$.

If we take in (2.12)-(2.14) $\lambda=f(x)$, $x\in(a,b)$ then we get, see also [15]

$$J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)=\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right] \tag{2.15}$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}[f(t)-f(x)]\,dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}[f(t)-f(x)]\,dt,$$

$$J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)=\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]+\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(t-a)^{\alpha-1}\left[f(t)-f(x)\right]dt+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(b-t)^{\alpha-1}\left[f(t)-f(x)\right]dt \tag{2.16}$$

and

$$\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}=\frac{f(x)}{\Gamma(\alpha+1)}(b-a)^{\alpha}+\frac{1}{2\Gamma(\alpha)}\int_{a}^{b}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\left[f(t)-f(x)\right]dt. \tag{2.17}$$

In particular, we have the mid-point equalities, see also [15]

$$J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(b-a)^{\alpha}f\left(\frac{a+b}{2}\right)+\frac{1}{\Gamma(\alpha)}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^{\alpha-1}\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)^{\alpha-1}\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt, \tag{2.18}$$

$$J_{\frac{a+b}{2}-}^{\alpha}f(a)+J_{\frac{a+b}{2}+}^{\alpha}f(b)=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(b-a)^{\alpha}f\left(\frac{a+b}{2}\right)+\frac{1}{\Gamma(\alpha)}\int_{a}^{\frac{a+b}{2}}(t-a)^{\alpha-1}\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha-1}\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt \tag{2.19}$$

and

$$\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}=\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}f\left(\frac{a+b}{2}\right)+\frac{1}{2\Gamma(\alpha)}\int_{a}^{b}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt. \tag{2.20}$$

If we take in (2.12)-(2.14) $\lambda = \frac{1}{b-a}\int_a^b f(s)\,ds$, $x \in (a,b)$ then we get

$$J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha} + (b-x)^{\alpha}\right]\frac{1}{b-a}\int_a^b f(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left[f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right]dt + \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\left[f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right]dt, \tag{2.21}$$

$$J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha} + (b-x)^{\alpha}\right]\frac{1}{b-a}\int_a^b f(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1}\left[f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right]dt + \frac{1}{\Gamma(\alpha)}\int_x^b (b-t)^{\alpha-1}\left[f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right]dt \tag{2.22}$$

and

$$\frac{J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)}{2} = \frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}\frac{1}{b-a}\int_a^b f(s)\,ds + \frac{1}{2\Gamma(\alpha)}\int_a^b \left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right]\left[f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right]dt. \tag{2.23}$$

If we take in (2.12)-(2.14) $\lambda = \frac{1}{2}\left[f(a) + f(b)\right]$, $x \in (a,b)$ then we get

$$J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha} + (b-x)^{\alpha}\right]\frac{1}{2}\left[f(a) + f(b)\right] + \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\left[f(t) - \frac{1}{2}\left[f(a) + f(b)\right]\right]dt + \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\left[f(t) - \frac{1}{2}\left[f(a) + f(b)\right]\right]dt, \tag{2.24}$$

$$J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha} + (b-x)^{\alpha}\right]\frac{1}{2}\left[f(a) + f(b)\right] + \frac{1}{\Gamma(\alpha)}\int_a^x (t-a)^{\alpha-1}\left[f(t) - \frac{1}{2}\left[f(a) + f(b)\right]\right]dt + \frac{1}{\Gamma(\alpha)}\int_x^b (b-t)^{\alpha-1}\left[f(t) - \frac{1}{2}\left[f(a) + f(b)\right]\right]dt \tag{2.25}$$

and

$$\begin{aligned}
&\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2} \\
&=\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}\frac{1}{2}[f(a)+f(b)] \\
&+\frac{1}{2\Gamma(\alpha)}\int_{a}^{b}\left[(b-t)^{\alpha-1}+(t-a)^{\alpha-1}\right]\left[f(t)-\frac{1}{2}[f(a)+f(b)]\right]dt.
\end{aligned} \tag{2.26}$$

2. For $\mu\neq\lambda$ we can get other identities. For instance, if we take $\lambda=f(a)$ and $\mu=f(b)$ in (2.12)-(2.14) then we get for $x\in(a,b)$

$$\begin{aligned}
J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)&=\frac{1}{\Gamma(\alpha+1)}\left[f(a)(x-a)^{\alpha}+f(b)(b-x)^{\alpha}\right] \\
&+\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}[f(t)-f(a)]dt \\
&+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}[f(t)-f(b)]dt,
\end{aligned} \tag{2.27}$$

$$\begin{aligned}
J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)&=\frac{1}{\Gamma(\alpha+1)}\left[f(a)(x-a)^{\alpha}+f(b)(b-x)^{\alpha}\right] \\
&+\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(t-a)^{\alpha-1}[f(t)-f(a)]dt \\
&+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(b-t)^{\alpha-1}[f(t)-f(b)]dt.
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}
\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}&=\frac{f(a)+f(b)}{2}\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha} \\
&+\frac{1}{2\Gamma(\alpha)}\int_{a}^{b}(b-t)^{\alpha-1}[f(t)-f(a)]dt \\
&+\frac{1}{2\Gamma(\alpha)}+\int_{a}^{b}(t-a)^{\alpha-1}[f(t)-f(b)]dt.
\end{aligned} \tag{2.29}$$

If we take $\lambda=\frac{f(a)+f(x)}{2}$ and $\mu=\frac{f(x)+f(b)}{2}$ in (2.12)-(2.14) then we get for $x\in(a,b)$ that

$$\begin{aligned}
&J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x) \\
&=\frac{1}{2\Gamma(\alpha+1)}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right] \\
&+\frac{1}{2\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x) \\
&+\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\left[f(t)-\frac{f(a)+f(x)}{2}\right]dt \\
&+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}\left[f(t)-\frac{f(x)+f(b)}{2}\right]dt,
\end{aligned} \tag{2.30}$$

$$
\begin{aligned}
& J_{x-}^{\alpha} f(a)+J_{x+}^{\alpha} f(b) \\
& =\frac{1}{2\Gamma(\alpha+1)}\left[(x-a)^{\alpha} f(a)+(b-x)^{\alpha} f(b)\right] \\
& +\frac{1}{2\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right] f(x) \\
& +\frac{1}{\Gamma(\alpha)} \int_{a}^{x}(t-a)^{\alpha-1}\left[f(t)-\frac{f(a)+f(x)}{2}\right] dt \\
& +\frac{1}{\Gamma(\alpha)} \int_{x}^{b}(b-t)^{\alpha-1}\left[f(t)-\frac{f(x)+f(b)}{2}\right] dt.
\end{aligned} \tag{2.31}
$$

and

$$
\begin{aligned}
& \frac{J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)}{2} \\
& =\frac{1}{2} \frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}\left[\frac{f(a)+f(b)}{2}+f(x)\right] \\
& +\frac{1}{2\Gamma(\alpha)} \int_{a}^{b}(b-t)^{\alpha-1}\left[f(t)-\frac{f(a)+f(x)}{2}\right] dt \\
& +\frac{1}{2\Gamma(\alpha)} \int_{a}^{b}(t-a)^{\alpha-1}\left[f(t)-\frac{f(x)+f(b)}{2}\right] dt.
\end{aligned} \tag{2.32}
$$

We can also take $\lambda=f\left(\frac{a+x}{2}\right)$ and $\mu=f\left(\frac{x+b}{2}\right)$ or $\lambda=\frac{1}{x-a}\int_a^x f(s)\,ds$ and $\mu=\frac{1}{b-x}\int_x^b f(s)\,ds$ in (2.12)-(2.14) to get other similar equalities. The details are not presented here.

3. Inequalities for Bounded Function

Now, for $\phi,\ \Phi\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$
\begin{aligned}
& \bar{U}_{[a,b]}(\phi,\Phi) \\
& :=\left\{f:[a,b]\to\mathbb{C} \mid \operatorname{Re}\left[(\Phi-f(t))\left(\overline{f(t)}-\bar{\phi}\right)\right]\geq 0 \text{ for almost every } t\in[a,b]\right\}
\end{aligned}
$$

and

$$
\bar{\Delta}_{[a,b]}(\phi,\Phi):=\left\{f:[a,b]\to\mathbb{C} \mid \left|f(t)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi| \text{ for a.e. } t\in[a,b]\right\}.
$$

The following representation result may be stated.

Proposition 1. *For any $\phi,\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that $\bar{U}_{[a,b]}(\phi,\Phi)$ and $\bar{\Delta}_{[a,b]}(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$
\bar{U}_{[a,b]}(\phi,\Phi)=\bar{\Delta}_{[a,b]}(\phi,\Phi). \tag{3.1}
$$

Proof. We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$
\left|z-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|
$$

if and only if

$$
\operatorname{Re}\left[(\Phi-z)(\bar{z}-\phi)\right]\geq 0.
$$

This follows by the equality

$$\frac{1}{4}|\Phi-\phi|^2-\left|z-\frac{\phi+\Phi}{2}\right|^2=\operatorname{Re}\left[(\Phi-z)(\bar{z}-\phi)\right]$$

that holds for any $z\in\mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 3. *For any $\phi,\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that*

$$\bar{U}_{[a,b]}(\phi,\Phi)=\{f:[a,b]\to\mathbb{C}\mid (\operatorname{Re}\Phi-\operatorname{Re}f(t))(\operatorname{Re}f(t)-\operatorname{Re}\phi) \\ +(\operatorname{Im}\Phi-\operatorname{Im}f(t))(\operatorname{Im}f(t)-\operatorname{Im}\phi)\geq 0 \text{ for a.e. } t\in[a,b]\}. \tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Phi)\geq\operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi)\geq\operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\phi,\Phi):=\{f:[a,b]\to\mathbb{C}\mid \operatorname{Re}(\Phi)\geq\operatorname{Re}f(t)\geq\operatorname{Re}(\phi) \\ \text{and } \operatorname{Im}(\Phi)\geq\operatorname{Im}f(t)\geq\operatorname{Im}(\phi) \text{ for a.e. } t\in[a,b]\}. \tag{3.3}$$

One can easily observe that $\bar{S}_{[a,b]}(\phi,\Phi)$ is closed, convex and

$$\emptyset\neq\bar{S}_{[a,b]}(\phi,\Phi)\subseteq\bar{U}_{[a,b]}(\phi,\Phi). \tag{3.4}$$

We have

Theorem 2. *Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$ and $\phi,\ \Phi\in\mathbb{C}$, $\phi\neq\Phi$ such that $f\in\bar{\Delta}_{[a,b]}(\phi,\Phi)$.*

(i) For any $x\in(a,b)$ we have

$$\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\frac{\phi+\Phi}{2}\right| \\ \leq\frac{1}{\Gamma(\alpha+1)}|\Phi-\phi|\left[\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{2}\right] \tag{3.5}$$

and, in particular

$$\left|J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(b-a)^{\alpha}\frac{\phi+\Phi}{2}\right| \\ \leq\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}|\Phi-\phi|(b-a)^{\alpha}. \tag{3.6}$$

(ii) For any $x\in(a,b)$ we have

$$\left|J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\frac{\phi+\Phi}{2}\right| \\ \leq\frac{1}{\Gamma(\alpha+1)}|\Phi-\phi|\left[\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{2}\right], \tag{3.7}$$

and, in particular

$$\left|J_{\frac{a+b}{2}-}^{\alpha}f(a)+J_{\frac{a+b}{2}+}^{\alpha}f(b)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(b-a)^{\alpha}\frac{\phi+\Phi}{2}\right| \\ \leq\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}|\Phi-\phi|(b-a)^{\alpha}. \tag{3.8}$$

(iii) We have

$$\left| \frac{J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)}{2} - \frac{1}{\Gamma(\alpha+1)} (b-a)^{\alpha} \frac{\phi+\Phi}{2} \right| \leq \frac{1}{2\Gamma(\alpha+1)} |\Phi - \phi| (b-a)^{\alpha}. \tag{3.9}$$

Proof. (i) From the identity (2.12) we have for $x \in (a,b)$ that

$$\begin{aligned} & J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) - \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \frac{\phi+\Phi}{2} \\ & = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left[f(t) - \frac{\phi+\Phi}{2} \right] dt \\ & + \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left[f(t) - \frac{\phi+\Phi}{2} \right] dt. \end{aligned} \tag{3.10}$$

If $f \in \bar{\Delta}_{[a,b]}(\phi, \Phi)$, then by taking the modulus in (3.10)

$$\begin{aligned} & \left| J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) - \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \frac{\phi+\Phi}{2} \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \left| \int_a^x (x-t)^{\alpha-1} \left[f(t) - \frac{\phi+\Phi}{2} \right] dt \right| \\ & + \frac{1}{\Gamma(\alpha)} \left| \int_x^b (t-x)^{\alpha-1} \left[f(t) - \frac{\phi+\Phi}{2} \right] dt \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left| f(t) - \frac{\phi+\Phi}{2} \right| dt \\ & + \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left| f(t) - \frac{\phi+\Phi}{2} \right| dt \\ & \leq \frac{1}{2\Gamma(\alpha)} |\Phi - \phi| \left[\int_a^x (x-t)^{\alpha-1} dt + \int_x^b (t-x)^{\alpha-1} dt \right], \\ & = \frac{1}{2\alpha\Gamma(\alpha)} |\Phi - \phi| \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \\ & = \frac{1}{\Gamma(\alpha+1)} |\Phi - \phi| \left[\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{2} \right], \end{aligned}$$

which proves (3.5).

(ii) From the identity (2.13) we have for $x \in (a,b)$ that

$$\begin{aligned} & J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \frac{\phi+\Phi}{2} \\ & = \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} \left[f(t) - \frac{\phi+\Phi}{2} \right] dt \\ & + \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} \left[f(t) - \frac{\phi+\Phi}{2} \right] dt. \end{aligned}$$

Now, by employing a similar argument as in (i) we deduce the desired result (3.7).

(iii) From the identity (2.14) we have

$$\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}-\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}\frac{\phi+\Phi}{2} \tag{3.11}$$
$$=\frac{1}{\Gamma(\alpha)}\int_a^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\left[f(t)-\frac{\phi+\Phi}{2}\right]dt.$$

If $f\in\bar{\Delta}_{[a,b]}(\phi,\Phi)$, then by taking the modulus in (3.11) we get

$$\left|\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}-\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}\frac{\phi+\Phi}{2}\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\int_a^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}\left|f(t)-\frac{\phi+\Phi}{2}\right|dt$$
$$\leq\frac{1}{2\Gamma(\alpha)}|\Phi-\phi|\int_a^b\frac{(b-t)^{\alpha-1}+(t-a)^{\alpha-1}}{2}dt$$
$$=\frac{1}{2\Gamma(\alpha+1)}|\Phi-\phi|(b-a)^{\alpha},$$

which proves the inequality (3.9). □

Corollary 4. *With the assumptions of Theorem 2 we have*

$$\left|\frac{1}{b-a}\int_a^b\frac{J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)}{2}dx-\frac{1}{\Gamma(\alpha+2)}\frac{\phi+\Phi}{2}(b-a)^{\alpha}\right| \tag{3.12}$$
$$\leq\frac{1}{2\Gamma(\alpha+2)}|\Phi-\phi|(b-a)^{\alpha}$$

and

$$\left|\frac{1}{b-a}\int_a^b\frac{J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)}{2}dx-\frac{1}{\Gamma(\alpha+2)}\frac{\phi+\Phi}{2}(b-a)^{\alpha}\right| \tag{3.13}$$
$$\leq\frac{1}{2\Gamma(\alpha+2)}|\Phi-\phi|(b-a)^{\alpha}.$$

Remark 1. *If the function $f:[a,b]\to\mathbb{R}$ is measurable and there exists the constants m, M such that $m\leq f(t)\leq M$ for a.e. $t\in[a,b]$, then for any $x\in(a,b)$ we have by (3.5) and (3.6) that*

$$\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}[(x-a)^{\alpha}+(b-x)^{\alpha}]\frac{m+M}{2}\right| \tag{3.14}$$
$$\leq\frac{1}{\Gamma(\alpha+1)}(M-m)\left[\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{2}\right]$$

and, in particular

$$\left|J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(b-a)^{\alpha}\frac{m+M}{2}\right| \tag{3.15}$$
$$\leq\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(M-m)(b-a)^{\alpha}.$$

By (3.7) and (3.8) we have that

$$\left| J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} + (b-x)^{\alpha} \right] \frac{m+M}{2} \right| \leq \frac{1}{\Gamma(\alpha+1)} (M-m) \left[\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{2} \right], \tag{3.16}$$

and, in particular

$$\left| J_{\frac{a+b}{2}-}^{\alpha} f(a) + J_{\frac{a+b}{2}+}^{\alpha} f(b) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} (b-a)^{\alpha} \frac{m+M}{2} \right| \leq \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} (M-m)(b-a)^{\alpha}. \tag{3.17}$$

From (3.9) we have

$$\left| \frac{J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)}{2} - \frac{1}{\Gamma(\alpha+1)} (b-a)^{\alpha} \frac{m+M}{2} \right| \leq \frac{1}{2\Gamma(\alpha+1)} (M-m)(b-a)^{\alpha}. \tag{3.18}$$

4. Composite Inequalities for Functions of Bounded Variation

Let $f : [a,b] \to \mathbb{C}$ be a Lebesgue integrable function on $[a,b]$. For $\gamma \in [0,1]$ and $x \in (a,b)$ we have from (2.1)-(2.3) for $\lambda = (1-\gamma) f(a) + \gamma f(x)$ and $\mu = (1-\gamma) f(x) + \gamma f(b)$ the following 3-point representations

$$\begin{aligned} J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha+1)} \left[(1-\gamma)(x-a)^{\alpha} f(a) + \gamma (b-x)^{\alpha} f(b) \right] \\ &+ \frac{1}{\Gamma(\alpha+1)} \left[\gamma (x-a)^{\alpha} + (1-\gamma)(b-x)^{\alpha} \right] f(x) \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left[f(t) - (1-\gamma) f(a) - \gamma f(x) \right] dt \\ &+ \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left[f(t) - (1-\gamma) f(x) - \gamma f(b) \right] dt, \end{aligned} \tag{4.1}$$

$$\begin{aligned} J_{x-}^{\alpha} f(a) + J_{x+}^{\alpha} f(b) &= \frac{1}{\Gamma(\alpha+1)} \left[(1-\gamma)(x-a)^{\alpha} f(a) + \gamma (b-x)^{\alpha} f(b) \right] \\ &+ \frac{1}{\Gamma(\alpha+1)} \left[\gamma (x-a)^{\alpha} + (1-\gamma)(b-x)^{\alpha} \right] f(x) \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^{\alpha-1} \left[f(t) - (1-\gamma) f(a) - \gamma f(x) \right] dt \\ &+ \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} \left[f(t) - (1-\gamma) f(x) - \gamma f(b) \right] dt. \end{aligned} \tag{4.2}$$

and

$$\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2} \tag{4.3}$$
$$=\frac{1}{2}\left[(1-\gamma)f(a)+\gamma f(b)+f(x)\right]\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}$$
$$+\frac{1}{2\Gamma(\alpha)}\int_{a}^{b}(b-t)^{\alpha-1}\left[f(t)-(1-\gamma)f(a)-\gamma f(x)\right]dt$$
$$+\frac{1}{2\Gamma(\alpha)}\int_{a}^{b}(t-a)^{\alpha-1}\left[f(t)-(1-\gamma)f(x)-\gamma f(b)\right]dt.$$

Theorem 3. *Assume that $f:[a,b]\to\mathbb{C}$ is a function of bounded variation on $[a,b]$ and $\gamma\in[0,1]$*

(i) If $x\in(a,b)$, then

$$\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}\left[(1-\gamma)(x-a)^{\alpha}f(a)+\gamma(b-x)^{\alpha}f(b)\right]\right. \tag{4.4}$$
$$\left.-\frac{1}{\Gamma(\alpha+1)}\left[\gamma(x-a)^{\alpha}+(1-\gamma)(b-x)^{\alpha}\right]f(x)\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\left[(1-\gamma)\bigvee_{a}^{t}(f)+\gamma\bigvee_{t}^{x}(f)\right]dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}\left[(1-\gamma)\bigvee_{x}^{t}(f)+\gamma\bigvee_{t}^{b}(f)\right]dt$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\max\{1-\gamma,\gamma\}\left[(x-a)^{\alpha}\bigvee_{a}^{x}(f)+(b-x)^{\alpha}\bigvee_{x}^{b}(f)\right]$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\max\{1-\gamma,\gamma\}$$
$$\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}\bigvee_{a}^{b}(f);\\ \\ \left((x-a)^{\alpha p}+(b-x)^{\alpha p}\right)^{1/p}\left(\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right]\left((x-a)^{\alpha}+(b-x)^{\alpha}\right),\end{cases}$$

and

$$\left|J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\left[(1-\gamma)(x-a)^{\alpha}f(a)+\gamma(b-x)^{\alpha}f(b)\right]\right. \tag{4.5}$$
$$\left.-\frac{1}{\Gamma(\alpha+1)}\left[\gamma(x-a)^{\alpha}+(1-\gamma)(b-x)^{\alpha}\right]f(x)\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(t-a)^{\alpha-1}\left[(1-\gamma)\bigvee_{a}^{t}(f)+\gamma\bigvee_{t}^{x}(f)\right]dt$$
$$+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(b-t)^{\alpha-1}\left[(1-\gamma)\bigvee_{x}^{t}(f)+\gamma\bigvee_{t}^{b}(f)\right]dt$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \max\{1-\gamma,\gamma\} \left[(x-a)^{\alpha} \bigvee_a^x (f) + (b-x)^{\alpha} \bigvee_x^b (f) \right]$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \max\{1-\gamma,\gamma\}$$

$$\times \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{\alpha} \bigvee_a^b (f); \\ \\ ((x-a)^{\alpha p} + (b-x)^{\alpha p})^{1/p} \left(\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q \right)^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] ((x-a)^{\alpha} + (b-x)^{\alpha}). \end{cases}$$

(ii) If $x \in [a,b]$, then

$$\left| \frac{J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)}{2} - \frac{1}{2}[(1-\gamma) f(a) + \gamma f(b) + f(x)] \frac{1}{\Gamma(\alpha+1)} (b-a)^{\alpha} \right| \tag{4.6}$$

$$\leq \frac{1}{2\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} \left[(1-\gamma) \bigvee_a^t (f) + \gamma \bigvee_t^x (f) \right] dt$$

$$+ \frac{1}{2\Gamma(\alpha)} \int_a^b (t-a)^{\alpha-1} \left[(1-\gamma) \bigvee_x^t (f) + \gamma \bigvee_t^b (f) \right] dt$$

$$\leq \frac{1}{2\Gamma(\alpha+1)} (b-a)^{\alpha} \max\{1-\gamma,\gamma\} \bigvee_a^b (f).$$

Proof. (i) Since f is of bounded variation, then we have for $a \leq t \leq x$ that

$$\begin{aligned} &|f(t) - (1-\gamma) f(a) - \gamma f(x)| \\ &= |(1-\gamma) f(t) + \gamma f(t) - (1-\gamma) f(a) - \gamma f(x)| \\ &= |(1-\gamma)[f(t) - f(a)] + \gamma [f(t) - f(x)]| \\ &\leq (1-\gamma)|f(t) - f(a)| + \gamma |f(x) - f(t)| \\ &\leq (1-\gamma) \bigvee_a^t (f) + \gamma \bigvee_t^x (f) \leq \max\{1-\gamma,\gamma\} \bigvee_a^x (f) \end{aligned} \tag{4.7}$$

and for $x \leq t \leq b$ that

$$\begin{aligned} &|f(t) - (1-\gamma) f(x) - \gamma f(b)| \\ &= |(1-\gamma) f(t) + \gamma f(t) - (1-\gamma) f(x) - \gamma f(b)| \\ &= |(1-\gamma)[f(t) - f(x)] + \gamma [f(t) - f(b)]| \\ &\leq (1-\gamma)|f(t) - f(x)| + \gamma |f(b) - f(t)| \\ &\leq (1-\gamma) \bigvee_x^t (f) + \gamma \bigvee_t^b (f) \leq \max\{1-\gamma,\gamma\} \bigvee_x^b (f). \end{aligned} \tag{4.8}$$

Using (4.1) for $x \in (a,b)$ we have

$$\begin{aligned}
&\left| J_{a+}^{\alpha} f(x) + J_{b-}^{\alpha} f(x) - \frac{1}{\Gamma(\alpha+1)} \left[(1-\gamma)(x-a)^{\alpha} f(a) + \gamma (b-x)^{\alpha} f(b) \right] \right. \\
&\left. - \frac{1}{\Gamma(\alpha+1)} \left[\gamma (x-a)^{\alpha} + (1-\gamma)(b-x)^{\alpha} \right] f(x) \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left| f(t) - (1-\gamma) f(a) - \gamma f(x) \right| dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left| f(t) - (1-\gamma) f(x) - \gamma f(b) \right| dt \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left[(1-\gamma) \bigvee_a^t (f) + \gamma \bigvee_t^x (f) \right] dt \\
&+ \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left[(1-\gamma) \bigvee_x^t (f) + \gamma \bigvee_t^b (f) \right] dt \\
&\leq \max\{1-\gamma, \gamma\} \frac{1}{\Gamma(\alpha)} \left[\bigvee_a^x (f) \int_a^x (x-t)^{\alpha-1} dt + \bigvee_x^b (f) \int_x^b (t-x)^{\alpha-1} dt \right] \\
&= \max\{1-\gamma, \gamma\} \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} \bigvee_a^x (f) + (b-x)^{\alpha} \bigvee_x^b (f) \right],
\end{aligned}$$

which prove the first two inequalities in (4.4).

The last part is obvious by making use of the elementary Hölder type inequalities for positive real numbers c, d, m, $n \geq 0$

$$mc + nd \leq \begin{cases} \max\{m,n\}(c+d); \\ \\ (m^p + n^p)^{1/p} (c^q + d^q)^{1/q} \text{ with } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1. \end{cases}$$

The inequality (4.5) follows in a similar way and we omit the details.

(ii) By (4.3) we get

$$\begin{aligned}
&\left| \frac{J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)}{2} - \frac{1}{2} \left[(1-\gamma) f(a) + \gamma f(b) + f(x) \right] \frac{1}{\Gamma(\alpha+1)} (b-a)^{\alpha} \right| \\
&\leq \frac{1}{2\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} \left| f(t) - (1-\gamma) f(a) - \gamma f(x) \right| dt \\
&+ \frac{1}{2\Gamma(\alpha)} \int_a^b (t-a)^{\alpha-1} \left| f(t) - (1-\gamma) f(x) - \gamma f(b) \right| dt \\
&\leq \frac{1}{2\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} \left[(1-\gamma) \bigvee_a^t (f) + \gamma \bigvee_t^x (f) \right] dt \\
&+ \frac{1}{2\Gamma(\alpha)} \int_a^b (t-a)^{\alpha-1} \left[(1-\gamma) \bigvee_x^t (f) + \gamma \bigvee_t^b (f) \right] dt
\end{aligned}$$

$$\leq \max\{1-\gamma,\gamma\} \bigvee_a^x (f) \frac{1}{2\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} dt$$
$$+ \max\{1-\gamma,\gamma\} \bigvee_x^b (f) \frac{1}{2\Gamma(\alpha)} \int_a^b (t-a)^{\alpha-1} dt$$
$$= \max\{1-\gamma,\gamma\} \frac{1}{2\Gamma(\alpha+1)} (b-a)^\alpha \bigvee_a^b (f),$$

which proves (4.6). □

Corollary 5. *With the assumptions of Theorem 3, we have*

$$\begin{aligned}(4.9)\qquad & \left| J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) \right. \\ & - \frac{1}{2^\alpha \Gamma(\alpha+1)} \left[(1-\gamma) f(a) + \gamma f(b)\right] (b-a)^\alpha \\ & \left. - \frac{1}{2^\alpha \Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) (b-a)^\alpha \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right)^{\alpha-1} \left[(1-\gamma) \bigvee_a^t (f) + \gamma \bigvee_t^{\frac{a+b}{2}} (f)\right] dt \\ & + \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right)^{\alpha-1} \left[(1-\gamma) \bigvee_{\frac{a+b}{2}}^t (f) + \gamma \bigvee_t^b (f)\right] dt \\ & \leq \frac{1}{2^\alpha \Gamma(\alpha+1)} \max\{1-\gamma,\gamma\} \bigvee_a^b (f) (b-a)^\alpha\end{aligned}$$

$$\begin{aligned}(4.10)\qquad & \left| J_{\frac{a+b}{2}-}^{\alpha} f(a) + J_{\frac{a+b}{2}+}^{\alpha} f(b) - \frac{1}{2^\alpha \Gamma(\alpha+1)} \left[(1-\gamma) f(a) + \gamma f(b)\right] (b-a)^\alpha \right. \\ & \left. - \frac{1}{2^\alpha \Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) (b-a)^\alpha \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} (t-a)^{\alpha-1} \left[(1-\gamma) \bigvee_a^t (f) + \gamma \bigvee_t^{\frac{a+b}{2}} (f)\right] dt \\ & + \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b (b-t)^{\alpha-1} \left[(1-\gamma) \bigvee_{\frac{a+b}{2}}^t (f) + \gamma \bigvee_t^b (f)\right] dt \\ & \leq \frac{1}{2^\alpha \Gamma(\alpha+1)} \max\{1-\gamma,\gamma\} \bigvee_a^b (f) (b-a)^\alpha\end{aligned}$$

and

$$\begin{aligned} &\left|\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}\right. \\ &\left.-\frac{1}{2}\left[(1-\gamma)f(a)+\gamma f(b)+f\left(\frac{a+b}{2}\right)\right]\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}\right| \\ &\leq \frac{1}{2\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}\left[(1-\gamma)\bigvee_a^t(f)+\gamma\bigvee_t^{\frac{a+b}{2}}(f)\right]dt \\ &+\frac{1}{2\Gamma(\alpha)}\int_a^b (t-a)^{\alpha-1}\left[(1-\gamma)\bigvee_{\frac{a+b}{2}}^t(f)+\gamma\bigvee_t^b(f)\right]dt \\ &\leq \frac{1}{2\Gamma(\alpha+1)}(b-a)^{\alpha}\max\{1-\gamma,\gamma\}\bigvee_a^b(f). \end{aligned} \tag{4.11}$$

Remark 2. *If we take $\gamma=\frac{1}{2}$ in Corollary 5, then we get the following composite mid-point and trapezoid inequalities*

$$\begin{aligned} &\left|J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)-\frac{1}{2^{\alpha}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}(b-a)^{\alpha}\right. \\ &\left.-\frac{1}{2^{\alpha}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha}\right| \leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\bigvee_a^b(f)(b-a)^{\alpha} \end{aligned} \tag{4.12}$$

$$\begin{aligned} &\left|J_{\frac{a+b}{2}-}^{\alpha}f(a)+J_{\frac{a+b}{2}+}^{\alpha}f(b)-\frac{1}{2^{\alpha}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}(b-a)^{\alpha}\right. \\ &\left.-\frac{1}{2^{\alpha}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha}\right| \leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\bigvee_a^b(f)(b-a)^{\alpha} \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} &\left|\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\frac{1}{\Gamma(\alpha+1)}(b-a)^{\alpha}\right| \\ &\leq \frac{1}{4\Gamma(\alpha+1)}(b-a)^{\alpha}\bigvee_a^b(f). \end{aligned} \tag{4.14}$$

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