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ON THE SIMPSON TYPE INEQUALITIES FOR $s-$CONVEX AND CONVEX FUNCTIONS

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ABSTRACT. In this paper, some new inequalities of Simpson type are obtained whose fourth derivatives absolute value are $s-$convex and convex.

1. INTRODUCTION

We will start with the definitions of $s-$convex and convex functions.

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$. (One could equivalenly take to t to be the closed interval $[0,1]$.) It is called strictly convex provied that the inequalities (1) is strict for $x \neq y$.

A function $f : \mathbb{R}^+ \to \mathbb{R}$ where $\mathbb{R}^+ = [0,\infty)$, is said to be $s-$convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y) \tag{1.2}$$

for all $x, y \in \mathbb{R}^+$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed s$\in (0,1]$. We denote by K_s^2 the class of $s-$convex function. If we choose $s = 1$ in this definition $s-$convexity reduces to the convexity in $\mathbb{R}^+$.

The following inequality is well-known in the literature as Simpson inequality:

Suppose $f : [a,b] \to \mathbb{R}$ is four times continously diferrentiable mapping on (a,b) and $\left|\left|f^{(4)}\right|\right|_\infty = \sup\left|f^{(4)}\right| < \infty$. The following the inequality

$$\left|\frac{1}{3}\left[2f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{2880}\left|\left|f^{(4)}\right|\right|_\infty (b-a)^4 \tag{1.3}$$

holds.

For some results, generalizations and improvements about convexity, $s-$convexity and Simpson inequality see the papers [2]-[10].

The main aim of this paper is to prove some new integral inequalities which are Simpson type for $s-$convex and convex functions.

We will use an integral identity from [1] which is emboided in the following Lemma to obtain our results.

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Lemma 1. *Let $f''' : I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I°, where $a, b \in I$ with $a < b$. If $f^{(4)} \in L[a,b]$ then the following equality holds:*

$$\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] = (b-a)^5 \int_0^1 p(t) f^{(4)}(tb + (1-t)a)\,dt \tag{1.4}$$

where

$$p(t) = \begin{cases} \frac{1}{24} t^3 \left(t - \frac{2}{3}\right), & 0 \le t \le \frac{1}{2} \\ \frac{1}{24}(t-1)^3 \left(t - \frac{1}{3}\right), & \frac{1}{2} < t \le 1. \end{cases} \tag{1.5}$$

2. Main Results

We will start with the following theorem.

Theorem 1. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a mapping and let $f''' : I \subseteq [0,\infty) \to \mathbb{R}$ be an absolutely continuous mapping on I° such that $f^{(4)} \in L[a,b]$, where $a, b \in I$ with $a < b$. If $\left|f^{(4)}\right|^q$ is $s-$convex on $[a,b]$ for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ \le\ & \frac{(b-a)^4}{24}\left(\left(\frac{3}{2}\right)^{-1-4p} \beta_{\frac{3}{4}}(1+3p, 1+p)\right)^{\frac{1}{p}} \\ & \left[\left(\frac{\left|f^{(4)}(b)\right|^q}{2^{s+1}(s+1)} + \frac{(2^{s+1}-1)\left|f^{(4)}(a)\right|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}} + \left(\frac{(2^{s+1}-1)\left|f^{(4)}(b)\right|^q}{2^{s+1}(s+1)} + \frac{\left|f^{(4)}(a)\right|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right] \end{aligned} \tag{2.1}$$

for all $x \in [a,b]$ and $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1 and using the properties of modulus, we have

$$\begin{aligned} A &= \left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ &\le \frac{(b-a)^4}{24}\left[\int_0^{\frac{1}{2}} t^3\left(\frac{2}{3} - t\right)\left|f^{(4)}(tb + (1-t)a)\right| dt \right. \\ &\quad \left. + \int_{\frac{1}{2}}^1 (1-t)^3\left(t - \frac{1}{3}\right)\left|f^{(4)}(tb + (1-t)a)\right| dt\right]. \end{aligned}$$

By the Hölder inequality we can write

$$\begin{aligned} A &\leq \frac{(b-a)^4}{24}\left[\left(\int_0^{\frac{1}{2}}\left[t^3\left(\frac{2}{3}-t\right)\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f^{(4)}(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\quad \left.+\left(\int_{\frac{1}{2}}^1\left[\left(t-\frac{1}{3}\right)(1-t)^3\right]^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f^{(4)}(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right]. \end{aligned}$$

If we use the $s-$convexity of $\left|f^{(4)}\right|^q$, then we have

$$\begin{aligned} A &\leq \frac{(b-a)^4}{24}\left[\left(\int_0^{\frac{1}{2}}\left[t^3\left(\frac{2}{3}-t\right)\right]^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} t^s\left|f^{(4)}(b)\right|^q+(1-t)^s\left|f^{(4)}(a)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\quad \left.+\left(\int_{\frac{1}{2}}^1\left[\left(t-\frac{1}{3}\right)(1-t)^3\right]^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 t^s\left|f^{(4)}(b)\right|^q+(1-t)^s\left|f^{(4)}(a)\right|^q dt\right)^{\frac{1}{q}}\right] \\ &\leq \frac{(b-a)^4}{24}\left[\left(\left(\frac{3}{2}\right)^{-1-4p}\beta_{\frac{3}{4}}(1+3p,1+p)\right)^{\frac{1}{p}}\left(\frac{\left|f^{(4)}(b)\right|^q}{2^{s+1}(s+1)}+\frac{(2^{s+1}-1)\left|f^{(4)}(a)\right|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right. \\ &\quad \left.+\left(\left(\frac{3}{2}\right)^{-1-4p}\beta_{\frac{3}{4}}(1+3p,1+p)\right)^{\frac{1}{p}}\left(\frac{(2^{s+1}-1)\left|f^{(4)}(b)\right|^q}{2^{s+1}(s+1)}+\frac{\left|f^{(4)}(a)\right|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right] \end{aligned}$$

where

$$\beta_z(a,b)=\int_0^z u^{a-1}(1-u)^{b-1}du$$

is the incomplete Beta function which is a generalization of the complete Beta function.

The proof is completed. □

Corollary 1. *If we choose $s=1$ in Theorem 1, we have the following inequality for convex functions:*

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\left[\frac{1}{6}f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\right| \\ &\leq \frac{(b-a)^4}{24}\left(\left(\frac{3}{2}\right)^{-1-4p}\beta_{\frac{3}{4}}(1+3p,1+p)\right)^{\frac{1}{p}} \\ &\left[\left(\frac{\left|f^{(4)}(b)\right|^q}{8}+\frac{3\left|f^{(4)}(a)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|f^{(4)}(b)\right|^q}{8}+\frac{\left|f^{(4)}(a)\right|^q}{8}\right)^{\frac{1}{q}}\right] \end{aligned} \tag{2.2}$$

Theorem 2. *Under the assumptions of Theorem 1, we have the following inequality*

(2.3)

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\left[\frac{1}{6}f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\right| \\
\leq\ & \frac{(b-a)^4}{24}\left(\frac{3p+5}{2^{3p+2}3(3p+1)(3p+2)}\right)^{\frac{1}{p}} \\
&\left[\left(\frac{(s+5)}{2^{s+2}3(s+1)(s+2)}\left|f^{(4)}(b)\right|^q+\left[\frac{2s+1}{3(s+1)(s+2)}+\frac{1-s}{2^{s+2}3(s+1)(s+2)}\right]\left|f^{(4)}(a)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left[\frac{2s+1}{3(s+1)(s+2)}+\frac{1-s}{2^{s+2}3(s+1)(s+2)}\right]\left|f^{(4)}(b)\right|^q+\frac{(s+5)}{2^{s+2}3(s+1)(s+2)}\left|f^{(4)}(a)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. From Lemma 1 and using the properties of modulus, we have

$$\begin{aligned}
A &= \left|\frac{1}{b-a}\int_a^b f(x)\,dx-\left[\frac{1}{6}f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq (b-a)^4\int_0^1 |p(t)|\left|f^{(4)}(tb+(1-t)a)\right|dt \\
&\leq \frac{(b-a)^4}{24}\left[\int_0^{\frac{1}{2}} t^3\left(\frac{2}{3}-t\right)\left|f^{(4)}(tb+(1-t)a)\right|dt\right. \\
&\quad\left.+\int_{\frac{1}{2}}^1 (1-t)^3\left(t-\frac{1}{3}\right)\left|f^{(4)}(tb+(1-t)a)\right|dt\right]
\end{aligned}$$

By the Hölder inequality

$$\begin{aligned}
A \leq\ & \frac{(b-a)^4}{24}\left[\left(\int_0^{\frac{1}{2}} t^{3p}\left(\frac{2}{3}-t\right)dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\frac{2}{3}-t\right)\left|f^{(4)}(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left(t-\frac{1}{3}\right)(1-t)^{3p}\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left(t-\frac{1}{3}\right)\left|f^{(4)}(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right]
\end{aligned}$$

If we use the $s-$convexity of $\left|f^{(4)}\right|^{q}$, then we have

$$\begin{aligned}
A &\leq \frac{(b-a)^{4}}{24}\left[\left(\int_{0}^{\frac{1}{2}} t^{3p}\left(\frac{2}{3}-t\right)dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\left(\frac{2}{3}-t\right)\left[t^{s}\left|f^{(4)}(b)\right|^{q}+(1-t)^{s}\left|f^{(4)}(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_{\frac{1}{2}}^{1}\left(t-\frac{1}{3}\right)(1-t)^{3p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left(t-\frac{1}{3}\right)\left[t^{s}\left|f^{(4)}(b)\right|^{q}+(1-t)^{s}\left|f^{(4)}(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(b-a)^{4}}{24}\left[\left(\frac{3p+5}{2^{3p+2}3(3p+1)(3p+2)}\right)^{\frac{1}{p}}\right. \\
&\quad\times\left(\frac{\left|f^{(4)}(b)\right|^{q}(s+5)}{2^{s+2}3(s+1)(s+2)}+\left[\frac{2s+1}{3(s+1)(s+2)}+\frac{1-s}{2^{s+2}3(s+1)(s+2)}\right]\left|f^{(4)}(a)\right|^{q}\right)^{\frac{1}{q}} \\
&\quad+\left(\frac{3p+5}{2^{3p+2}3(3p+1)(3p+2)}\right)^{\frac{1}{p}} \\
&\quad\left.\times\left(\left[\frac{2s+1}{3(s+1)(s+2)}+\frac{1-s}{2^{s+2}3(s+1)(s+2)}\right]\left|f^{(4)}(b)\right|^{q}+\frac{(s+5)}{2^{s+2}3(s+1)(s+2)}\left|f^{(4)}(a)\right|^{q}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

The proof is completed. □

Corollary 2. *If we choose $s=1$ in Theorem 2, we have the following inequality for convex functions:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_{a}^{b}f(x)\,dx-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq \frac{(b-a)^{4}}{4608}\left[\frac{6p+10}{(3p+1)(3p+2)}\right]^{\frac{1}{p}} \\
&\quad\times\left[\left(4\left|f^{(4)}(b)\right|^{q}+\left|f^{(4)}(a)\right|^{q}\right)^{\frac{1}{q}}+\left(\left|f^{(4)}(b)\right|^{q}+4\left|f^{(4)}(a)\right|^{q}\right)^{\frac{1}{q}}\right]
\end{aligned} \tag{2.4}$$

Following inequalities are obtained for convex functions:

Theorem 3. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a mapping and let $f''': I\subseteq[0,\infty)\to\mathbb{R}$ be an absolutely continuous mapping on I° such that $f^{(4)}\in L[a,b]$, where $a,b\in I$ with $a<b$. If $\left|f^{(4)}\right|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_{a}^{b}f(x)\,dx-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq \frac{(b-a)^{4}}{5760}\left[\left|f^{(4)}(a)\right|+\left|f^{(4)}(b)\right|\right].
\end{aligned} \tag{2.5}$$

Proof. From Lemma 1 and using the properties of modulus, we have

$$\begin{aligned}
A &= \left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq \frac{(b-a)^4}{24}\left[\int_0^{\frac{1}{2}} t^3\left(\frac{2}{3}-t\right)\left|f^{(4)}(tb+(1-t)a)\right|dt + \int_{\frac{1}{2}}^1 (1-t)^3\left(t-\frac{1}{3}\right)\left|f^{(4)}(tb+(1-t)a)\right|dt\right] \\
&\leq \frac{(b-a)^4}{24}\left[\int_0^{\frac{1}{2}} t^3\left(\frac{2}{3}-t\right)t\left|f^{(4)}(b)\right|dt\right. \\
&\quad + \int_0^{\frac{1}{2}} t^3\left(\frac{2}{3}-t\right)(1-t)\left|f^{(4)}(a)\right|dt \\
&\quad + \int_{\frac{1}{2}}^1 (1-t)^3\left(t-\frac{1}{3}\right)t\left|f^{(4)}(b)\right|dt \\
&\quad \left.+ \int_{\frac{1}{2}}^1 (1-t)^3\left(t-\frac{1}{3}\right)(1-t)\left|f^{(4)}(a)\right|dt\right].
\end{aligned}$$

If we calculate the integrals above we get the desired result. □

Theorem 4. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a mapping and let $f''': I \subseteq [0,\infty) \to \mathbb{R}$ be an*

absolutely continuous mapping on I° such that $f^{(4)} \in L[a,b]$, where $a,b \in I$ with $a<b$. If $\left|f^{(4)}\right|^q$ is convex on $[a,b]$, then the following inequality holds:

$$\begin{aligned}
(2.6)\qquad & \left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq \frac{(b-a)^4}{24}\left(\left(\frac{3}{2}\right)^{-1-4p}\beta_{\frac{3}{4}}(1+3p,1+p)\right)^{\frac{1}{p}} \\
&\quad \times\left[\left(\frac{2\left|f^{(4)}(b)\right|^q+3\left|f^{(4)}(a)\right|^q}{320}\right)^{\frac{1}{q}}+\left(\frac{3\left|f^{(4)}(b)\right|^q+2\left|f^{(4)}(a)\right|^q}{320}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

for all $x \in [a,b]$ and $q>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and using the properties of modulus, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq (b-a)^4\int_0^1 |p(t)|\left|f^{(4)}(tb+(1-t)a)\right|dt \\
&\leq \frac{(b-a)^4}{24}\left[\int_0^{\frac{1}{2}} t^3\left(\frac{2}{3}-t\right)\left|f^{(4)}(tb+(1-t)a)\right|dt + \int_{\frac{1}{2}}^1 (1-t)^3\left(t-\frac{1}{3}\right)\left|f^{(4)}(tb+(1-t)a)\right|dt\right]
\end{aligned}$$

By the Hölder inequality and convexity of $\left|f^{(4)}\right|^{q}$, we can write

$$\begin{aligned} A &\leq \frac{(b-a)^4}{24}\left[\left(\int_0^{\frac{1}{2}} t^3\left(\frac{2}{3}-t\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} t^3\left|f^{(4)}(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_{\frac{1}{2}}^1 (1-t)^3\left(t-\frac{1}{3}\right)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 (1-t)^3\left|f^{(4)}(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right] \\ &\leq \frac{(b-a)^4}{24}\left[\left(\int_0^{\frac{1}{2}} t^{3p}\left(\frac{2}{3}-t\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} t^4\left|f^{(4)}(b)\right|^q+(t^3-t^4)\left|f^{(4)}(a)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_{\frac{1}{2}}^1 (1-t)^{3p}\left(t-\frac{1}{3}\right)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 (1-t)^3 t\left|f^{(4)}(b)\right|^q+(1-t)^4\left|f^{(4)}(a)\right|^q dt\right)^{\frac{1}{q}}\right]. \end{aligned}$$

If we calculate the integrals above we get the desired result. □

Theorem 5. *Under the assumptions of Teorem 4, following inequality holds:*

$$\begin{aligned} &(2.7)\quad \left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\ &\leq \frac{(b-a)^4}{24}\left(\frac{1}{(3p+1)\,2^{3p+1}}\right)^{\frac{1}{p}} \\ &\times\left[\left(\left|f^{(4)}(b)\right|^q\frac{4^{q+2}-(3q+7)}{(q+1)(q+2)\,6^{q+2}}+\left|f^{(4)}(a)\right|^q\frac{2^{2q+1}(3q+4)-(3q+5)}{(q+1)(q+2)\,6^{q+2}}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\left|f^{(4)}(b)\right|^q\frac{2^{2q+1}(3q+4)-(3q+5)}{(q+1)(q+2)\,6^{q+2}}+\left|f^{(4)}(a)\right|^q\frac{4^{q+2}-(3q+7)}{(q+1)(q+2)\,6^{q+2}}\right)^{\frac{1}{q}}\right] \end{aligned}$$

Proof. From Lemma 1,using the properties of modulus, Hölder inequality and convexity of $\left|f^{(4)}\right|^{q}$, we can write

$$\begin{aligned} A &\leq \frac{(b-a)^4}{24}\left[\left(\int_0^{\frac{1}{2}} t^{3q}dt\right)^{\frac{1}{p}}\left(\left|f^{(4)}(b)\right|^q\int_0^{\frac{1}{2}}\left(\frac{2}{3}-t\right)^q tdt+\left|f^{(4)}(a)^q\right|\int_0^{\frac{1}{2}}\left(\frac{2}{3}-t\right)^q(1-t)dt\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\int_{\frac{1}{2}}^1 (1-t)^{3q}\,dt\right)^{\frac{1}{p}}\left(\left|f^{(4)}(b)\right|^q d\int_{\frac{1}{2}}^1\left(t-\frac{1}{3}\right)^q t+\left|f^{(4)}(a)\right|^q\int_{\frac{1}{2}}^1\left(t-\frac{1}{3}\right)^q(1-t)dt\right)^{\frac{1}{q}}\right] \end{aligned}$$

If we calculate the integrals above, then we have

$$\begin{aligned} A &\leq \frac{(b-a)^4}{24}\left(\frac{1}{(3p+1)\,2^{3p+1}}\right)^{\frac{1}{p}} \\ &\left[\left(\left|f^{(4)}(b)\right|^q \frac{4^{q+2}-(3q+7)}{(q+1)(q+2)\,6^{q+2}}+\left|f^{(4)}(a)\right|^q \frac{2^{2q+1}(3q+4)-(3q+5)}{(q+1)(q+2)\,6^{q+2}}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\left|f^{(4)}(b)\right|^q \frac{2^{2q+1}(3q+4)-(3q+5)}{(q+1)(q+2)\,6^{q+2}}+\left|f^{(4)}(a)\right|^q \frac{4^{q+2}-(3q+7)}{(q+1)(q+2)\,6^{q+2}}\right)^{\frac{1}{q}}\right]. \end{aligned}$$

The proof is completed. □

Remark 1. *Some applications for special means and to Simpson's quadrature rule can be given. It is left to the interested reader.*

References

[1] Alomari, M. W., (2016). An improvement inequality of Simpson's type for $quasi-$convex mappings with applications. Arxiv: 1603.08055v1.
[2] Avcı Ardıç, M., Ozdemir, M., E., (2012). Simpson Type Inequalities for Functions Whos Third Derivatives in The Absolute Value Are $s-$Convex and $s-$Concave, Arxiv: 1112.3865v2.
[3] Ozdemir, M., E., Ekinci, A., Gürbüz, M., Akdemir, A., O., (2012). Simpson Type Inequalities for $Q-$Class Functions. Arxiv: 1207.2435v1.
[4] Ozdemir, M., E., Avcı Ardıç, M., Akdemir, A., O., (2012). Simpson Type Inequalities via $\varphi-$Convexity. Arxiv: 1205.6657v2.
[5] Pečarić, J., Varošanec, S., (2000). A Note on Simpson's Inequality for Functions of Bounded Variation, Tamkang Journal of Mathematics, Volume 31, Number:3.
[6] Dragomir, S.,S., (1998). On Simpson's Quadrature Formula for Differentiable Mappings Whose Derivatives Belong to L_p-Spaces and Aplications, RGMIA, Vol. 1.
[7] Roberts, A. W., and Varberg, D.E., (1973). Convex Functions, Academic Press, New York.
[8] Matuszewska, W., and Orlicz,W., (1981). A note on the theory of $s-$normed spaces of φ-integrable functions, Studia Math., 21, 107-115.
[9] Ozdemir, M., E., Kavurmacı, H., Yıldız, Ç., (2012). Fractional Integral Inequalities Via $s-$Convex Functions. Arxiv: 1201.4915v1.
[10] Dragomir, S. S., Agarwal, R. P., and Cerone, P., (2000). On Simpson's inequality and applications, Journal of Inequalities and Applications, vol. 5, no. 6, pp. 533–579.

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