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A VERSION OF THE HADAMARD INEQUALITY FOR CAPUTO FRACTIONAL DERIVATIVES AND RELATED RESULTS

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Abstract. In this paper we are interested to give the Hadamard inequality for n-times differentiable convex functions via Caputo fractional derivatives. We also found bounds of a difference of this inequality.

1. introduction

Fractional calculus was mainly a study kept for the finest minds in mathematics. The history of fractional calculus is as old as the history of differential calculus. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables. Fourier, Euler, Laplace are among those mathematicians who showed a casual interest by fractional calculus and mathematical consequences. A lot of them established definitions by means of their own notion and style. Most renowned of these definitions are the Grunwald-Letnikov and Riemann-Liouville definitions [5].

In the following we give the definition of Caputo fractional derivatives [4].

Definition 1. *Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$. The Caputo fractional derivatives of order α are defined as follows:*

$$(1.1) \qquad {}^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, x > a$$

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and

$$(1.2)\qquad {}^{C}D^{\alpha}_{b-}f(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, x<b.$$

If $\alpha = n \in \{1,2,3,...\}$ *and usual derivative of* f *of order* n *exists, then Caputo fractional derivative* $({}^{C}D^{\alpha}_{a+}f)(x)$ *coincides with* $f^{(n)}(x)$*. In particular we have*

$$({}^{C}D^{0}_{a+}f)(x)=({}^{C}D^{0}_{b-}f)(x)=f(x)$$

where $n=1$ *and* $\alpha=0$*.*

In [9], Sarikaya et al. proved following the Hadamard-type inequalities for Riemann-Liouville fractional integrals:

Theorem 1. *Let* $f:[a,b]\to\mathbb{R}$ *be a positive function with* $0\le a<b$ *and* $f\in L_1[a,b]$*. If* f *is a convex function on [a,b], then the following inequalities for fractional integrals hold*

$$(1.3)\qquad f\left(\frac{a+b}{2}\right)\le\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{(\frac{a+b}{2})+}f(b)+I^{\alpha}_{(\frac{a+b}{2})-}f(a)\right]\le\frac{f(a)+f(b)}{2}$$

with $\alpha>0$*.*

Theorem 2. *Let* $f:[a,b]\to\mathbb{R}$ *be a differential mapping on (a,b) with* $a<b$*. If* $|f'|^q$ *is convex on* $[a,b]$ *for* $q\ge1$*, then the following inequality for fractional integrals holds*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}[I^{\alpha}_{(\frac{a+b}{2})+}f(b)+I^{\alpha}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|\\ (1.4)\qquad &\le\frac{b-a}{4(\alpha+1)}\left(\frac{1}{2(\alpha+2)}\right)^{\frac{1}{q}}\left[((\alpha+1)|f'(a)|^q+(\alpha+3)|f'(b)|^q)^{\frac{1}{q}}\right.\\ &\left.+((\alpha+3)|f'(a)|^q+(\alpha+1)|f'(b)|^q)^{\frac{1}{q}}\right].\end{aligned}$$

with $\alpha>0$*.*

Theorem 3. *Let* $f:[a,b]\to\mathbb{R}$ *be a differential mapping on (a,b) with* $a<b$*. If* $|f'|^q$ *is convex on* $[a,b]$ *for* $q>1$*, then the following inequality* *for fractional integral holds*

(1.5)

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}[I^{\alpha}_{(\frac{a+b}{2})+}f(b)+I^{\alpha}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q+3|f'(b)|q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}[|f'(a)|+|f'(b)|],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In this paper we are interested to give versions of inequalities (1.3), (1.4) and (1.5) for n-times differentiable convex functions via Caputo fractional derivatives.
In the whole paper $C^n[a,b]$ denotes the space of n-times differentiable functions such that $f^{(n)}$ are continuous on $[a,b]$.

2. Hadamard-type inequalities for Caputo fractional derivatives

In this section we give a version of the Hadamard inequality via Caputo fractional derivatives. First we prove the follwing lemma.

Lemma 1. *Let $g:[a,b]\to\mathbb{R}$ be a function such that $g\in C^n[a,b]$, also let g^n is integrable and symmetric to $\frac{a+b}{2}$, then we have*

$${}^{C}D^{\alpha}_{a+}g(b)=(-1)^{n C}D^{\alpha}_{b-}g(a)=\frac{1}{2}[{}^{C}D^{\alpha}_{a+}g(b)+(-1)^{n C}D^{\alpha}_{b-}g(a)].$$

Proof. By symmetricity of $g^{(n)}$ we have $g^{(n)}(a+b-x)=g^{(n)}(x)$, where $x\in[a,b]$. Setting $x=a+b-x$ in the following integral we have

$$\begin{aligned}
{}^{C}D^{\alpha}_{a+}g(b) &= \frac{1}{\Gamma(n-\alpha)}\int_a^b \frac{g^{(n)}(x)}{(b-x)^{\alpha-n+1}}dx\\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^b \frac{g^{(n)}(a+b-x)}{(x-a)^{\alpha-n+1}}dx\\
&= \frac{1}{\Gamma(n-\alpha)}\int_a^b \frac{g^{(n)}(x)}{(x-a)^{\alpha-n+1}}dx\\
&= (-1)^{n C}D^{\alpha}_{b-}g(a).
\end{aligned}$$

□

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in C^n[a, b]$. If $f^{(n)}$ is a convex function on $[a, b]$, then the following inequalities for Caputo fractional derivatives hold*

$$
\begin{aligned}
& f^{(n)}\left(\frac{a+b}{2}\right) \\
& \le \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[{}^C D^{\alpha}_{(\frac{a+b}{2})+} f(b) + (-1)^{n C} D^{\alpha}_{(\frac{a+b}{2})-} f(a)\right] \\
& \le \frac{f^{(n)}(a) + f^{(n)}(b)}{2}. \qquad (2.1)
\end{aligned}
$$

Proof. From convexity of $f^{(n)}$ we have

$$f^{(n)}\left(\frac{x+y}{2}\right) \le \frac{f^{(n)}(x) + f^{(n)}(y)}{2}. \qquad (2.2)$$

Setting $x = \frac{t}{2}a + \frac{(2-t)}{2}b, y = \frac{(2-t)}{2}a + \frac{t}{2}b$ for $t \in [0, 1]$. Then $x, y \in [a, b]$ and above equation gives

$$2f^{(n)}\left(\frac{a+b}{2}\right) \le f^{(n)}\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f^{(n)}\left(\frac{2-t}{2}a + \frac{t}{2}b\right), \qquad (2.3)$$

multiplying both sides of above inequality with $t^{n-\alpha-1}$ and integrating over $[0, 1]$ we have

$$
\begin{aligned}
& 2f^{(n)}\left(\frac{a+b}{2}\right)\int_0^1 t^{n-\alpha-1}dt \\
& \le \int_0^1 t^{n-\alpha-1} f^{(n)}\left(\frac{t}{2}a + \frac{2-t}{2}b\right)dt + \int_0^1 t^{n-\alpha-1} f^{(n)}\left(\frac{2-t}{2}a + \frac{t}{2}b\right)dt \\
& = \frac{2^{n-\alpha}\Gamma(\alpha)}{(b-a)^{n-\alpha}}\left[{}^C D^{\alpha}_{(\frac{a+b}{2})+} f(b) + (-1)^{n C} D^{\alpha}_{(\frac{a+b}{2})-} f(a)\right],
\end{aligned}
$$

from which one can have

$$
\begin{aligned}
& f^{(n)}\left(\frac{a+b}{2}\right) \\
& \le \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[{}^C D^{\alpha}_{(\frac{a+b}{2})+} f(b) + (-1)^{n C} D^{\alpha}_{(\frac{a+b}{2})-} f(a)\right]. \qquad (2.4)
\end{aligned}
$$

On the other hand convexity of $f^{(n)}$ gives

$$f^{(n)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+f^{(n)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)$$
$$\leq \frac{t}{2}f^{(n)}(a)+\frac{2-t}{2}f^{(n)}(b)+\frac{2-t}{2}f^{(n)}(a)+\frac{t}{2}f^{(n)}(b),$$

multiplying both sides of above inequality with $t^{n-\alpha-1}$ and integrating over $[0,1]$ we have

$$\int_0^1 t^{n-\alpha-1}f^{(n)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt+\int_0^1 t^{n-\alpha-1}f^{(n)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt$$
$$\leq \left[f^{(n)}(a)+f^{(n)}(b)\right]\int_0^1 t^{n-\alpha-1}dt,$$

from which one can have

$$\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f(b)+(-1)^{n}{}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f(a)\right]$$
$$\leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}. \tag{2.5}$$

Combining inequality (2.4) and inequality (2.5) we get inequality (2.1). □

3. Caputo fractional inequalities related to the Hadamard inequality

In this Section we give the bounds of a difference of the Hadamard inequality proved in previous Section. For our results we use the following lemma.

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f\in C^{n+1}[a,b]$, then the following equality for Caputo fractional derivatives holds*

$$\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}[{}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f(b)+(-1)^{n}{}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f(a)]$$
$$-f^{(n)}\left(\frac{a+b}{2}\right)$$
$$=\frac{b-a}{4}\left[\int_0^1 t^{n-\alpha}f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\right.$$
$$\left.-\int_0^1 t^{n-\alpha}f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right]. \tag{3.1}$$

Proof. One can note that

$$\frac{b-a}{4}\left[\int_0^1 t^{n-\alpha} f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt\right]$$
$$=\frac{b-a}{4}\left[t^{n-\alpha}\frac{2}{a-b}f^{(n)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)|_0^1\right.$$
$$\left.-\int_0^1 \alpha t^{n-\alpha-1}\frac{2}{a-b}f^{(n)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]$$
$$=\frac{b-a}{4}\left[-\frac{2}{b-a}f^{(n)}\left(\frac{a+b}{2}\right)\right.$$
$$\left.-\frac{2\alpha}{(a-b)}\int_b^{\frac{a+b}{2}}\left(\frac{2}{b-a}(b-x)\right)^{n-\alpha-1}\frac{2}{a-b}f^{(n)}(x)dx\right]$$
(3.2)
$$=\frac{b-a}{4}\left[-\frac{2}{b-a}f^{(n)}\left(\frac{a+b}{2}\right)+\frac{2^{n-\alpha+1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}}(-1)^{n\,C}D^{\alpha}_{(\frac{a+b}{2})-}f(b)\right].$$

Similarly

$$-\frac{b-a}{4}\left[\int_0^1 t^{n-\alpha} f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right]$$
(3.3)
$$=-\frac{b-a}{4}\left[\frac{2}{b-a}f^{(n)}\left(\frac{a+b}{2}\right)-\frac{2^{n-\alpha+1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha+1}}{}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f^{(n)}(a)\right].$$

Combining (3.2) and (3.3) one can have (3.1). □

Using the above lemma we give following Caputo fractional Hadamard-type inequality.

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be a differential mapping on (a,b) with $a<b$ and $f\in C^{n+1}[a,b]$. If $|f^{(n+1)}|^q$ is convex on $[a,b]$ for $q\geq 1$, then*

the following inequality for Caputo fractional derivatives holds

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^{n}({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.$$
$$\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4(n-\alpha+1)}\left(\frac{1}{2(n-\alpha+2)}\right)^{\frac{1}{q}}\left[\left[(n-\alpha+1)\,|f^{(n+1)}(a)|^{q}\right.\right.$$
$$\left.+(n-\alpha+3)\,|f^{(n+1)}(b)|^{q}\right]^{\frac{1}{q}}+\left[(n-\alpha+3)\,|f^{(n+1)}(a)|^{q}\right.$$
$$\left.\left.+(n-\alpha+1)\,|f^{(n+1)}(b)|^{q}\right]^{\frac{1}{q}}\right].$$

Proof. From Lemma 2 and convexity of $|f^{(n+1)}|$ and for $q=1$ we have

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}[{}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f(b)+{}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f(a)]-f^{(n)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4}\int_{0}^{1}t^{n-\alpha}\left(\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt+\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\right)dt.$$
$$=\frac{b-a}{4\,(n-\alpha+1)}[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|].$$

For $q>1$ using Lemma (2) we have

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}[{}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f(b)+(-1)^{nC}D^{\alpha}_{(\frac{a+b}{2})-}f(a)]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4}\left[\int_{0}^{1}t^{n-\alpha}\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right.$$
$$\left.+\int_{0}^{1}t^{n-\alpha}\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right].$$

Using power mean inequality we get

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{c}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^{n}({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.$$
$$\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4}\left(\frac{1}{n-\alpha+1}\right)^{\frac{1}{p}}\left[\left[\int_{0}^{1}t^{n-\alpha}\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^{q}dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_{0}^{1}t^{n-\alpha}\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^{q}dt\right]^{\frac{1}{q}}\right].$$

Convexity of $|f^{(n+1)}|^q$ gives

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^CD^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^CD^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.$$
$$\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{n-\alpha+1}\right)^{\frac{1}{p}}\left[\left[\int_0^1 t^{n-\alpha}\left(\frac{t}{2}|f^{(n+1)}(a)|^q+\frac{2-t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 t^{n-\alpha}\left(\frac{2-t}{2}|f^{(n+1)}(a)|^q+\frac{t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right]$$
$$=\frac{b-a}{4}\left(\frac{1}{n-\alpha+1}\right)^{\frac{1}{p}}\left[\left[\frac{|f^{(n+1)}(a)|^q}{2(n-\alpha+2)}+\frac{|f^{(n+1)}(b)|^q}{n-\frac{\alpha}{k}+1}-\frac{|f^{(n+1)}(b)|^q}{2(n-\alpha+2)}\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{|f^{(n+1)}(a)|^q}{n-\alpha+1}-\frac{|f^{(n+1)}(a)|^q}{2(n-\alpha+2)}+\frac{|f^{(n+1)}(b)|^q}{2(n-\alpha+2)}\right]^{\frac{1}{q}}\right],$$

which after a little computation gives the required result. □

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f\in C^n[a,b]$, $a<b$. If $|f^{(n+1)}|^q$ is convex on $[a,b]$ for $q>1$, then the following inequality for Caputo fractional derivatives holds*

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^CD^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^n({}^CD^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right. \tag{3.4}$$
$$\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4}\left(\frac{1}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f^{(n+1)}(a)|^q+3|f^{(n+1)}(b)|^q}{4}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{b-a}{4}\left(\frac{4}{3(np-\alpha p+1)}\right)^{\frac{1}{p}}[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|],$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2 we have

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^{n}({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.$$
$$\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4}\left[\int_{0}^{1}t^{n-\alpha}\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|dt\right.$$
$$\left.+\int_{0}^{1}t^{n-\alpha}\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|dt\right].$$

From $H\ddot{o}lder's$ inequality we get

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^{n}({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.$$
$$\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4}\left[\left[\int_{0}^{1}t^{np-\alpha p}dt\right]^{\frac{1}{p}}\left[\int_{0}^{1}\left|f^{(n+1)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^{q}dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_{0}^{1}t^{np-\alpha p}dt\right]^{\frac{1}{p}}\left[\int_{0}^{1}\left|f^{(n+1)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^{q}dt\right]^{\frac{1}{q}}\right].$$

Convexity of $|f^{(n+1)}|^{q}$ gives

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^{n}({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.$$
$$\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{4}\left(\frac{1}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left[\int_{0}^{1}\left(\frac{t}{2}|f^{(n+1)}(a)|^{q}+\frac{2-t}{2}|f^{(n+1)}(b)|^{q}\right)dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_{0}^{1}\left(\frac{2-t}{2}|f^{(n+1)}(a)|^{q}+\frac{t}{2}|f^{(n+1)}(b)|^{q}\right)dt\right]^{\frac{1}{q}}\right]$$
$$=\frac{b-a}{4}\left(\frac{1}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left[\frac{|f^{(n+1)}(a)|^{q}+3|f^{(n+1)}(b)|^{q}}{4}\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\frac{3|f^{(n+1)}(a)|^{q}+|f^{(n+1)}(b)|^{q}}{4}\right]^{\frac{1}{q}}\right].$$

For second inequality of (3.4) we use Minkowski's inequality as follows

$$\begin{aligned}
&\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+b}{2})+}f)(b)+(-1)^{n}({}^{C}D^{\alpha}_{(\frac{a+b}{2})-}f)(a)\right]\right.\\
&\left.-f^{(n)}\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{b-a}{16}\left(\frac{4}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left[|f^{(n+1)}(a)|^{q}+3|f^{(n+1)}(b)|^{q}\right]^{\frac{1}{q}}\right.\\
&\left.+\left[3|f^{(n+1)}(a)|^{q}+|f^{(n+1)}(b)|^{q}\right]^{\frac{1}{q}}\right]\\
&\leq \frac{b-a}{16}\left(\frac{4}{np-\alpha p+1}\right)^{\frac{1}{p}}(3^{\frac{1}{q}}+1)(|f^{(n+1)}(a)|+|f^{(n+1)}(b)|)\\
&\leq \frac{b-a}{4}\left(\frac{4}{3(np-\alpha p+1)}\right)^{\frac{1}{p}}(|f^{(n+1)}(a)|+|f^{(n+1)}(b)|).
\end{aligned}$$

□

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