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TRAPEZOID TYPE INEQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF FUNCTIONS WITH BOUNDED VARIATION

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Abstract. In this paper we establish some trapezoid type inequalities for the Riemann-Liouville fractional integrals of functions of bounded variation and of Hölder continuous functions. Applications for the *g-mean of two numbers* are provided as well. Some particular cases for Hadamard fractional integrals are also provided.

1. Introduction

Let (a,b) with $-\infty \leq a < b \leq \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\operatorname{Re}(\alpha) > 0$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Following [18, p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ by

$$I_{a+,g}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)\,f(t)\,dt}{[g(x)-g(t)]^{1-\alpha}},\ a < x \leq b \tag{1.1}$$

and

$$I_{b-,g}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)\,f(t)\,dt}{[g(t)-g(x)]^{1-\alpha}},\ a \leq x < b. \tag{1.2}$$

For $g(t)=t$ we have the classical *Riemann-Liouville fractional integrals*

$$J_{a+}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}},\ a < x \leq b \tag{1.3}$$

and

$$J_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha}},\ a \leq x < b, \tag{1.4}$$

while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [18, p. 111]

$$H_{a+}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0 \leq a < x \leq b \tag{1.5}$$

and

$$H_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0 \leq a < x < b. \tag{1.6}$$

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One can consider the function $g(t) = -t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha}f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}t^{\alpha+1}},\ 0 \le a < x \le b \tag{1.7}$$

and

$$R_{b-}^{\alpha}f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha}t^{\alpha+1}},\ 0 \le a < x < b. \tag{1.8}$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the *"β-Exponential fractional integrals"*

$$E_{a+,\beta}^{\alpha}f(x) := \frac{\beta}{\Gamma(\alpha)}\int_a^x \frac{\exp(\beta t)f(t)\,dt}{[\exp(\beta x) - \exp(\beta t)]^{1-\alpha}},\ a < x \le b \tag{1.9}$$

and

$$E_{b-,\beta}^{\alpha}f(x) := \frac{\beta}{\Gamma(\alpha)}\int_x^b \frac{\exp(\beta t)f(t)\,dt}{[\exp(\beta t) - \exp(\beta x)]^{1-\alpha}},\ a \le x < b. \tag{1.10}$$

In the recent paper [14] we obtained the following Ostrowski type inequalities for functions of bounded variation:

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For any $x \in (a,b)$ we have the inequalities*

$$\left|I_{a+,g}^{\alpha}f(x) + I_{b-,g}^{\alpha}f(x) - \frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right)f(x)\right|$$
$$\le \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\bigvee_t^x(f)\,dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)\bigvee_x^t(f)\,dt}{[g(t)-g(x)]^{1-\alpha}}\right]$$

$$\le \frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}\bigvee_a^x(f) + [g(b)-g(x)]^{\alpha}\bigvee_x^b(f)\right]$$
$$\le \frac{1}{\Gamma(\alpha+1)}\begin{cases} \left[\frac{1}{2}(g(b)-g(a)) + \left|g(x) - \frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_a^b(f); \\ \left((g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q + \left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left((g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right], \end{cases}$$

and

$$
\begin{aligned}
&\left| I_{x-,g}^{\alpha} f(a) + I_{x+,g}^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)} \left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha} \right) f(x) \right| \\
&\qquad \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) \bigvee_t^x (f)\, dt}{[g(t)-g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t) \bigvee_x^t (f)\, dt}{[g(b)-g(t)]^{1-\alpha}} \right] \\
&\qquad \leq \frac{1}{\Gamma(\alpha+1)} \left[[g(x)-g(a)]^{\alpha} \bigvee_a^x (f) + [g(b)-g(x)]^{\alpha} \bigvee_x^b (f) \right] \\
&\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}(g(b)-g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\ \left((g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p} \right)^{1/p} \left(\left(\bigvee_a^x (f) \right)^q + \left(\bigvee_x^b (f) \right)^q \right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left((g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right) \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right]. \end{cases}
\end{aligned}
$$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a, b \in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2} \right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a,b) = A(a,b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0,\infty)$ and $g(t) = \ln t$, then $M_g(a,b) = G(a,b) := \sqrt{ab}$, the *geometric mean*. If $I = (0,\infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a,b) = H(a,b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0,\infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a,b) = M_p(a,b) := \left(\frac{a^p+b^p}{2} \right)^{1/p}$, the *power mean with exponent p*. Finally, if $I = \mathbb{R}$ and $g(t) = \exp t$, then

$$M_g(a,b) = LME(a,b) := \ln\left(\frac{\exp a + \exp b}{2} \right),$$

the *LogMeanExp function*.

The following particular case for *g-mean* is of interest [14].

Corollary 1. *With the assumptions of Theorem 1 we have*

$$
\begin{aligned}
&\left| I_{a+,g}^{\alpha} f(M_g(a,b)) + I_{b-,g}^{\alpha} f(M_g(a,b)) - \frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} f(M_g(a,b)) \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) \bigvee_t^{M_g(a,b)} (f)\, dt}{[g(M_g(a,b)) - g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) \bigvee_{M_g(a,b)}^t (f)\, dt}{[g(t) - g(M_g(a,b))]^{1-\alpha}} \right] \\
&\qquad\qquad \leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} (g(b)-g(a))^{\alpha} \bigvee_a^b (f);
\end{aligned}
$$

and

$$\left| I_{M_g(a,b)-,g}^{\alpha} f(a) + I_{M_g(a,b)+,g}^{\alpha} f(b) - \frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} f(M_g(a,b)) \right|$$
$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t) \bigvee_t^{M_g(a,b)}(f)\, dt}{[g(t)-g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t) \bigvee_x^t(f)\, dt}{[g(b)-g(t)]^{1-\alpha}} \right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} (g(b)-g(a))^{\alpha} \bigvee_a^b (f).$$

Remark 1. *If we take in Theorem 1* $x = \frac{a+b}{2}$, *then we obtain similar mid-point inequalities, however the details are not presented here. Some applications for the Hadamard fractional integrals are also provided in* [14].

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[5], [16]-[27] and the references therein.

Motivated by the above results, in this paper we establish some trapezoid type inequalities for the generalized Riemann-Liouville fractional integrals of functions of bounded variation and of Hölder continuous functions. Applications for the g-*mean of two numbers* are provided as well. Some particular cases for Hadamard fractional integrals are also provided.

2. Some Identities

We have:

Lemma 1. *Let* $f : [a,b] \to \mathbb{C}$ *be Lebesgue integrable on* $[a,b]$, g *be a strictly increasing function on* (a,b), *having a continuous derivative* g' *on* (a,b) *and* λ, μ *some complex parameters:*

(i) For any $x \in (a,b)$ *we have the representation*

$$I_{a+,g}^{\alpha} f(x) + I_{b-,g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha+1)} \left(\lambda [g(x)-g(a)]^{\alpha} + \mu [g(b)-g(x)]^{\alpha} \right) + \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)[f(t)-\lambda]\, dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)[f(t)-\mu]\, dt}{[g(t)-g(x)]^{1-\alpha}} \right] \tag{2.1}$$

and

$$I_{x-,g}^{\alpha} f(a) + I_{x+,g}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha+1)} \left(\lambda [g(x)-g(a)]^{\alpha} + \mu [g(b)-g(x)]^{\alpha} \right) + \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)[f(t)-\lambda]\, dt}{[g(t)-g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t)[f(t)-\mu]\, dt}{[g(b)-g(t)]^{1-\alpha}} \right]. \tag{2.2}$$

(ii) We have

$$\frac{I_{b-,g}^{\alpha} f(a) + I_{a+,g}^{\alpha} f(b)}{2} = \frac{1}{\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} \frac{\lambda+\mu}{2} + \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{g'(t)[f(t)-\lambda]\, dt}{[g(b)-g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t)[f(t)-\mu]\, dt}{[g(t)-g(a)]^{1-\alpha}} \right]. \tag{2.3}$$

Proof. (i) We observe that

$$\begin{aligned}\tag{2.4} &\frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)[f(t)-\lambda]\,dt}{[g(x)-g(t)]^{1-\alpha}}\\ &= I_{a+,g}^{\alpha}f(x)-\lambda\frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)\,dt}{[g(x)-g(t)]^{1-\alpha}}\\ &= I_{a+,g}^{\alpha}f(x)-\frac{[g(x)-g(a)]^{\alpha}}{\alpha\Gamma(\alpha)}\lambda = I_{a+,g}^{\alpha}f(x)-\frac{[g(x)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\lambda\end{aligned}$$

for $a<x\le b$ and, similarly,

$$\frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(t)-g(x)]^{1-\alpha}} = I_{b-,g}^{\alpha}f(x)-\frac{[g(b)-g(x)]^{\alpha}}{\Gamma(\alpha+1)}\mu \tag{2.5}$$

for $a\le x<b$.

If $x\in(a,b)$, then by adding the equalities (2.4) and (2.5) we get the representation (2.1).

By the definition of fractional integrals we have

$$I_{x+,g}^{\alpha}f(b) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)f(t)\,dt}{[g(b)-g(t)]^{1-\alpha}},\ a\le x<b$$

and

$$I_{x-,g}^{\alpha}f(a) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)f(t)\,dt}{[g(t)-g(a)]^{1-\alpha}},\ a<x\le b.$$

Then

$$\frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)[f(t)-\lambda]\,dt}{[g(b)-g(t)]^{1-\alpha}} = I_{x+,g}^{\alpha}f(b)-\frac{[g(b)-g(x)]^{\alpha}}{\Gamma(\alpha+1)}\lambda \tag{2.6}$$

for $a\le x<b$ and

$$\frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)[f(t)-\mu]\,dt}{[g(t)-g(a)]^{1-\alpha}} = I_{x-,g}^{\alpha}f(a)-\frac{[g(x)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\mu \tag{2.7}$$

for $a<x\le b$.

If $x\in(a,b)$, then by adding the equalities (2.6) and (2.7) we get the representation (2.1).

If we take $x=b$ in (2.4) we get

$$\frac{1}{\Gamma(\alpha)}\int_a^b \frac{g'(t)[f(t)-\lambda]\,dt}{[g(b)-g(t)]^{1-\alpha}} = I_{a+,g}^{\alpha}f(b)-\frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\lambda \tag{2.8}$$

while from $x=a$ in (2.5) we get

$$\frac{1}{\Gamma(\alpha)}\int_a^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(t)-g(a)]^{1-\alpha}} = I_{b-,g}^{\alpha}f(a)-\frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\mu. \tag{2.9}$$

If we add (2.8) with (2.9) and divide by 2 we get (2.3). □

Remark 2. *If we take in (2.1) and (2.2)* $x = M_g(a,b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, *then we get*

$$I^{\alpha}_{a+,g}f(M_g(a,b)) + I^{\alpha}_{b-,g}f(M_g(a,b)) = \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\left(\frac{\lambda+\mu}{2}\right) + \frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)[f(t)-\lambda]\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}}\right] \tag{2.10}$$

and

$$I^{\alpha}_{M_g(a,b)-,g}f(a) + I^{\alpha}_{M_g(a,b)+,g}f(b) = \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\left(\frac{\lambda+\mu}{2}\right) + \frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)[f(t)-\lambda]\,dt}{[g(t)-g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]. \tag{2.11}$$

The above lemma provides various identities of interest by taking particular values for the parameters λ and μ, out of which we give only a few:

Corollary 2. *With the assumptions of Lemma 1 we have:*

(i) For any $x \in (a,b)$,

$$I^{\alpha}_{a+,g}f(x) + I^{\alpha}_{b-,g}f(x) = \frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right)f(x) + \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)[f(t)-f(x)]\,dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)[f(t)-f(x)]\,dt}{[g(t)-g(x)]^{1-\alpha}}\right] \tag{2.12}$$

and

$$I^{\alpha}_{x-,g}f(a) + I^{\alpha}_{x+,g}f(b) = \frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right)f(x) + \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)[f(t)-f(x)]\,dt}{[g(t)-g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t)[f(t)-f(x)]\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]. \tag{2.13}$$

(ii) For any $x \in [a,b]$,

$$\frac{I^{\alpha}_{b-,g}f(a) + I^{\alpha}_{a+,g}f(b)}{2} = \frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}f(x) + \frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)[f(t)-f(x)]\,dt}{[g(b)-g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t)[f(t)-f(x)]\,dt}{[g(t)-g(a)]^{1-\alpha}}\right]. \tag{2.14}$$

The proof is obvious by taking $\lambda = \mu = f(x)$ in Lemma 1. These identities were obtained in [14]. If we take in (2.12)-(2.14) $x = M_g(a,b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, then we get the corresponding identities were obtained in [14].

Corollary 3. *With the assumptions of Lemma 1 we have:*

$$\begin{aligned} I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x) &= \frac{1}{\Gamma(\alpha+1)}\left(\left[g(x)-g(a)\right]^{\alpha}f(a)+\left[g(b)-g(x)\right]^{\alpha}f(b)\right) \\ &+\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)\left[f(t)-f(a)\right]dt}{\left[g(x)-g(t)\right]^{1-\alpha}}+\int_x^b\frac{g'(t)\left[f(t)-f(b)\right]dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right] \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b) &= \frac{1}{\Gamma(\alpha+1)}\left(\left[g(x)-g(a)\right]^{\alpha}f(a)+\left[g(b)-g(x)\right]^{\alpha}f(b)\right) \\ &+\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)\left[f(t)-f(a)\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}}+\int_x^b\frac{g'(t)\left[f(t)-f(b)\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}}\right], \end{aligned} \tag{2.16}$$

for any $x\in(a,b)$

(ii) We also have

$$\begin{aligned} \frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2} &= \frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\frac{f(b)+f(a)}{2} \\ &+\frac{1}{2\Gamma(\alpha)}\left[\int_a^b\frac{g'(t)\left[f(t)-f(b)\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}}+\int_a^b\frac{g'(t)\left[f(t)-f(a)\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}}\right]. \end{aligned} \tag{2.17}$$

The proof of (2.15) and (2.16) are obvious by taking $\lambda=f(a)$, $\mu=f(b)$ in Lemma 1. The proof of (2.17) follows by Lemma 1 on taking $\lambda=f(b)$ and $\mu=f(a)$.

Remark 3. *If we take in (2.15) and (2.16) $x=M_g(a,b)=g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, then we get*

$$\begin{aligned} I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b)) &= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\left(\frac{f(a)+f(b)}{2}\right) \\ +\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)\left[f(t)-f(a)\right]dt}{\left[g(M_g(a,b))-g(t)\right]^{1-\alpha}}\right. &+ \left.\int_{M_g(a,b)}^b\frac{g'(t)\left[f(t)-f(b)\right]dt}{\left[g(t)-g(M_g(a,b))\right]^{1-\alpha}}\right] \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} I_{M_g(a,b)-,g}^{\alpha}f(a)+I_{M_g(a,b)+,g}^{\alpha}f(b) &= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\left(\frac{f(a)+f(b)}{2}\right) \\ +\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)\left[f(t)-f(a)\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}}\right. &+ \left.\int_{M_g(a,b)}^b\frac{g'(t)\left[f(t)-f(b)\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}}\right]. \end{aligned} \tag{2.19}$$

3. Inequalities for Bounded Functions

Now, for $\phi,\ \Phi\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions, see for instance [15]

$$\begin{aligned}&\bar{U}_{[a,b]}(\phi,\Phi)\\&:=\left\{f:[a,b]\to\mathbb{C}\,|\operatorname{Re}\left[(\Phi-f(t))\left(\overline{f(t)}-\overline{\phi}\right)\right]\geq 0\text{ for almost every }t\in[a,b]\right\}\end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\phi,\Phi):=\left\{f:[a,b]\to\mathbb{C}|\ \left|f(t)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|\ \text{ for a.e. }\ t\in[a,b]\right\}.$$

The following representation result may be stated.

Proposition 1. *For any $\phi,\ \Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that $\bar{U}_{[a,b]}(\phi,\Phi)$ and $\bar{\Delta}_{[a,b]}(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\phi,\Phi)=\bar{\Delta}_{[a,b]}(\phi,\Phi)\,. \tag{3.1}$$

Proof. We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$\left|z-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|$$

if and only if

$$\operatorname{Re}[(\Phi-z)(\bar{z}-\phi)]\geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Phi-\phi|^2-\left|z-\frac{\phi+\Phi}{2}\right|^2=\operatorname{Re}[(\Phi-z)(\bar{z}-\phi)]$$

that holds for any $z\in\mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 4. *For any $\phi,\Phi\in\mathbb{C}$, $\phi\neq\Phi$, we have that*

$$\begin{aligned}\bar{U}_{[a,b]}(\phi,\Phi)=\{f:[a,b]\to\mathbb{C}\ |\ &(\operatorname{Re}\Phi-\operatorname{Re}f(t))(\operatorname{Re}f(t)-\operatorname{Re}\phi)\\&+(\operatorname{Im}\Phi-\operatorname{Im}f(t))(\operatorname{Im}f(t)-\operatorname{Im}\phi)\geq 0\ \textit{for a.e. } t\in[a,b]\}\,.\end{aligned} \tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Phi)\geq\operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi)\geq\operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{[a,b]}(\phi,\Phi):=\{f:[a,b]\to\mathbb{C}\ |\ &\operatorname{Re}(\Phi)\geq\operatorname{Re}f(t)\geq\operatorname{Re}(\phi)\\&\text{and }\operatorname{Im}(\Phi)\geq\operatorname{Im}f(t)\geq\operatorname{Im}(\phi)\ \text{ for a.e. } t\in[a,b]\}\,.\end{aligned} \tag{3.3}$$

One can easily observe that $\bar{S}_{[a,b]}(\phi,\Phi)$ is closed, convex and

$$\emptyset\neq\bar{S}_{[a,b]}(\phi,\Phi)\subseteq\bar{U}_{[a,b]}(\phi,\Phi)\,. \tag{3.4}$$

We have:

Theorem 2. *Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$, g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b) and $\phi,\ \Phi\in\mathbb{C}$, $\phi\neq\Phi$ such that $f\in\bar{\Delta}_{[a,b]}(\phi,\Phi)$.*

(i) For any $x \in (a,b)$,

$$\left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{\phi+\Phi}{2\Gamma(\alpha+1)} \left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha} \right) \right| \leq \frac{1}{2} |\Phi-\phi| \frac{1}{\Gamma(\alpha+1)} \left[[g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha} \right] \tag{3.5}$$

and

$$\left| I^{\alpha}_{x-,g} f(a) + I^{\alpha}_{x+,g} f(b) - \frac{\phi+\Phi}{2\Gamma(\alpha+1)} \left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha} \right) \right| \leq \frac{1}{2} |\Phi-\phi| \frac{1}{\Gamma(\alpha+1)} \left[[g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha} \right]. \tag{3.6}$$

(ii) We have

$$\left| \frac{I^{\alpha}_{b-,g} f(a) + I^{\alpha}_{a+,g} f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} [g(b)-g(a)]^{\alpha} \frac{\phi+\Phi}{2} \right| \leq \frac{1}{2} |\Phi-\phi| \frac{1}{\Gamma(\alpha+1)} |\Phi-\phi| [g(b)-g(a)]^{\alpha}. \tag{3.7}$$

Proof. Using the identity (2.1) for $\lambda = \mu = \frac{\phi+\Phi}{2}$, we have

$$\begin{aligned} & I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) \\ & \quad - \frac{1}{\Gamma(\alpha+1)} \left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha} \right) \frac{\phi+\Phi}{2} \\ & = \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) \left[f(t) - \frac{\phi+\Phi}{2} \right] dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) \left[f(t) - \frac{\phi+\Phi}{2} \right] dt}{[g(t)-g(x)]^{1-\alpha}} \right] \end{aligned} \tag{3.8}$$

for any $x \in (a,b)$.

Taking the modulus in (3.8), then we get

$$\begin{aligned} & \left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{1}{\Gamma(\alpha+1)} \left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha} \right) \frac{\phi+\Phi}{2} \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t) \left| f(t) - \frac{\phi+\Phi}{2} \right| dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t) \left| f(t) - \frac{\phi+\Phi}{2} \right| dt}{[g(t)-g(x)]^{1-\alpha}} \right] \\ & \leq \frac{1}{2} |\Phi-\phi| \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)\, dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)\, dt}{[g(t)-g(x)]^{1-\alpha}} \right] \\ & = \frac{1}{2} |\Phi-\phi| \frac{1}{\Gamma(\alpha+1)} \left[[g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha} \right] \end{aligned}$$

for any $x \in (a,b)$, which proves (3.5).

The inequality (3.6) follows in a similar manner from the identity (2.2).

The inequality (3.7) follows by (2.3) for $\lambda = \mu = \frac{\phi+\Phi}{2}$. □

Corollary 5. *With the assumptions of Theorem 2 we have*

$$\left|I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))-\frac{\phi+\Phi}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\right| \leq \frac{1}{2^{\alpha}}|\Phi-\phi|\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha} \tag{3.9}$$

and

$$\left|I_{M_g(a,b)-,g}^{\alpha}f(a)+I_{M_g(a,b)+,g}^{\alpha}f(b)-\frac{\phi+\Phi}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\right| \leq \frac{1}{2^{\alpha}}|\Phi-\phi|\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}. \tag{3.10}$$

Remark 4. *If the function $f:[a,b]\to\mathbb{R}$ is measurable and there exists the constants m, M such that $m\leq f(t)\leq M$ for a.e. $t\in[a,b]$, then for any $x\in(a,b)$ we have by (3.5) and (3.6) that*

$$\left|I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)-\frac{m+M}{2\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)\right| \leq \frac{1}{2}(M-m)\frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right] \tag{3.11}$$

and

$$\left|I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)-\frac{m+M}{2\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)\right| \leq \frac{1}{2}(M-m)\frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right]. \tag{3.12}$$

In particular,

$$\left|I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))-\frac{m+M}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\right| \leq \frac{1}{2^{\alpha}}(M-m)\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha} \tag{3.13}$$

and

$$\left|I_{M_g(a,b)-,g}^{\alpha}f(a)+I_{M_g(a,b)+,g}^{\alpha}f(b)-\frac{m+M}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\right| \leq \frac{1}{2^{\alpha}}(M-m)\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}. \tag{3.14}$$

4. Trapezoid Inequalities for Functions of Bounded Variation

We have:

Theorem 3. *Let $f:[a,b]\to\mathbb{C}$ be a complex valued function of bounded variation on the real interval $[a,b]$, and g be a strictly increasing function on (a,b), having* *a continuous derivative g' on (a,b). Then we have the inequalities*

$$\begin{aligned}
&\left|I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)-\frac{[g(x)-g(a)]^{\alpha}f(a)+[g(b)-g(x)]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \\
&\le \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\bigvee_a^t(f)\,dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b \frac{g'(t)\bigvee_t^b(f)\,dt}{[g(t)-g(x)]^{1-\alpha}}\right] \\
&\le \frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}\bigvee_a^x(f)+(g(b)-g(x))^{\alpha}\bigvee_x^b(f)\right] \\
&\le \frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_a^b(f); \\ ((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p})^{1/p}\left((\bigvee_a^x(f))^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ ((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha})\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
&\left|I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)-\frac{[g(x)-g(a)]^{\alpha}f(a)+[g(b)-g(x)]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \\
&\le \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)\bigvee_a^t(f)\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_x^b \frac{g'(t)\bigvee_t^b(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}\right] \\
&\le \frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}\bigvee_a^x(f)+(g(b)-g(x))^{\alpha}\bigvee_x^b(f)\right] \\
&\le \frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_a^b(f); \\ ((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p})^{1/p}\left((\bigvee_a^x(f))^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ ((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha})\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}
\end{aligned} \tag{4.2}$$

for any $x\in(a,b)$

(ii) We also have

$$\begin{aligned}
&\left|\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\frac{f(b)+f(a)}{2}\right| \\
&\le \frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)\bigvee_t^b(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}+\int_a^b \frac{g'(t)\bigvee_a^t(f)\,dt}{[g(t)-g(a)]^{1-\alpha}}\right] \\
&\le \frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\bigvee_a^b(f).
\end{aligned} \tag{4.3}$$

Proof. Using the identity (2.15) and the properties of the modulus, we have

$$\begin{aligned}&\left|I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)-\frac{\left[g\left(x\right)-g\left(a\right)\right]^{\alpha}f\left(a\right)+\left[g\left(b\right)-g\left(x\right)\right]^{\alpha}f\left(b\right)}{\Gamma\left(\alpha+1\right)}\right|\\&\leq\frac{1}{\Gamma\left(\alpha\right)}\left[\int_{a}^{x}\frac{g'\left(t\right)\left|f\left(t\right)-f\left(a\right)\right|dt}{\left[g\left(x\right)-g\left(t\right)\right]^{1-\alpha}}+\int_{x}^{b}\frac{g'\left(t\right)\left|f\left(t\right)-f\left(b\right)\right|dt}{\left[g\left(t\right)-g\left(x\right)\right]^{1-\alpha}}\right]=:B\left(x\right)\end{aligned}$$

for any $x\in(a,b)$.

Since f is of bounded variation on $[a,b]$, then we have

$$\left|f\left(t\right)-f\left(a\right)\right|\leq\bigvee_{a}^{t}\left(f\right)\leq\bigvee_{a}^{x}\left(f\right)\text{ for }a\leq t\leq x$$

and

$$\left|f\left(t\right)-f\left(b\right)\right|\leq\bigvee_{t}^{b}\left(f\right)\leq\bigvee_{x}^{b}\left(f\right)\text{ for }x\leq t\leq b.$$

Therefore

$$\begin{aligned}B\left(x\right)&\leq\frac{1}{\Gamma\left(\alpha\right)}\left[\int_{a}^{x}\frac{g'\left(t\right)\bigvee_{a}^{t}\left(f\right)dt}{\left[g\left(x\right)-g\left(t\right)\right]^{1-\alpha}}+\int_{x}^{b}\frac{g'\left(t\right)\bigvee_{t}^{b}\left(f\right)dt}{\left[g\left(t\right)-g\left(x\right)\right]^{1-\alpha}}\right]\\&\leq\frac{1}{\Gamma\left(\alpha\right)}\left[\bigvee_{a}^{x}\left(f\right)\int_{a}^{x}\frac{g'\left(t\right)dt}{\left[g\left(x\right)-g\left(t\right)\right]^{1-\alpha}}+\bigvee_{x}^{b}\left(f\right)\int_{x}^{b}\frac{g'\left(t\right)\ dt}{\left[g\left(t\right)-g\left(x\right)\right]^{1-\alpha}}\right]\\&=\frac{1}{\Gamma\left(\alpha\right)}\left[\frac{\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}}{\alpha}\bigvee_{a}^{x}\left(f\right)+\frac{\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}}{\alpha}\bigvee_{x}^{b}\left(f\right)\right]\\&=\frac{1}{\Gamma\left(\alpha+1\right)}\left[\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}\bigvee_{a}^{x}\left(f\right)+\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}\bigvee_{x}^{b}\left(f\right)\right],\end{aligned}$$

which proves the first two inequalities in (4.1).

The last part of (4.1) is obvious by making use of the elementary Hölder type inequalities for positive real numbers c, d, m, $n\geq0$

$$mc+nd\leq\begin{cases}\max\left\{m,n\right\}\left(c+d\right);\\ \\ \left(m^{p}+n^{p}\right)^{1/p}\left(c^{q}+d^{q}\right)^{1/q}\text{ with }p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1.\end{cases}$$

The inequality (4.2) follows in a similar way by utilising the equality (2.16).

From the equality (2.17) we have

$$\begin{aligned}
&\left|\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\frac{f(b)+f(a)}{2}\right| \\
&\leq \frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)\left|f(t)-f(b)\right|dt}{\left[g(b)-g(t)\right]^{1-\alpha}}+\int_a^b \frac{g'(t)\left|f(t)-f(a)\right|dt}{\left[g(t)-g(a)\right]^{1-\alpha}}\right] \\
&\leq \frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)\bigvee_t^b(f)\,dt}{\left[g(b)-g(t)\right]^{1-\alpha}}+\int_a^b \frac{g'(t)\bigvee_a^t(f)\,dt}{\left[g(t)-g(a)\right]^{1-\alpha}}\right] \\
&\leq \frac{1}{2\Gamma(\alpha)}\left[\bigvee_a^b(f)\int_a^b \frac{g'(t)\,dt}{\left[g(b)-g(t)\right]^{1-\alpha}}+\bigvee_a^b(f)\int_a^b \frac{g'(t)\,dt}{\left[g(t)-g(a)\right]^{1-\alpha}}\right] \\
&= \frac{1}{2\Gamma(\alpha)}\left[\bigvee_a^b(f)\frac{\left[g(b)-g(a)\right]^{\alpha}}{\alpha}+\bigvee_a^b(f)\frac{\left[g(b)-g(a)\right]^{\alpha}}{\alpha}\right] \\
&= \frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\bigvee_a^b(f),
\end{aligned}$$

which proves (4.3). □

Corollary 6. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}
&\left|I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))-\frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\right| \qquad (4.4)\\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)} \frac{g'(t)\bigvee_a^t(f)\,dt}{\left[g(M_g(a,b))-g(t)\right]^{1-\alpha}}+\int_{M_g(a,b)}^b \frac{g'(t)\bigvee_t^b(f)\,dt}{\left[g(t)-g(M_g(a,b))\right]^{1-\alpha}}\right] \\
&\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)}\left(g(b)-g(a)\right)^{\alpha}\bigvee_a^b(f)
\end{aligned}$$

and

$$\begin{aligned}
&\left|I_{M_g(a,b)-,g}^{\alpha}f(a)+I_{M_g(a,b)+,g}^{\alpha}f(b)-\frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\right| \qquad (4.5)\\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)} \frac{g'(t)\bigvee_a^t(f)\,dt}{\left[g(t)-g(a)\right]^{1-\alpha}}+\int_{M_g(a,b)}^b \frac{g'(t)\bigvee_t^b(f)\,dt}{\left[g(b)-g(t)\right]^{1-\alpha}}\right] \\
&\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)}\left(g(b)-g(a)\right)^{\alpha}\bigvee_a^b(f).
\end{aligned}$$

5. Inequalities for Hölder's Continuous Functions

We say that the function $f:[a,b]\to\mathbb{C}$ is *r-H-Hölder continuous* on $[a,b]$ with $r\in(0,1]$ and $H>0$ if

$$\left|f(t)-f(s)\right|\leq H\left|t-s\right|^{r} \qquad (5.1)$$

for any $t,\ s\in[a,b]$. If $r=1$ and $H=L$ we call the function *L-Lipschitzian* on $[a,b]$.

Theorem 4. *Assume that $f:[a,b]\to\mathbb{C}$ is r-H-Hölder continuous on $[a,b]$ with $r\in(0,1]$ and $H>0$, and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then*

$$\left|I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)-\frac{[g(x)-g(a)]^{\alpha}f(a)+[g(b)-g(x)]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \tag{5.2}$$
$$\leq\frac{H}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)(t-a)^r dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b\frac{g'(t)(b-t)^r dt}{[g(t)-g(x)]^{1-\alpha}}\right]$$
$$\leq\frac{H}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}(x-a)^r+(g(b)-g(x))^{\alpha}(b-x)^r\right]$$
$$\leq\frac{H}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}[(x-a)^r+(b-x)^r];\\((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p})^{1/p}((x-a)^{rq}+(b-x)^{rq})^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha})\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\end{cases}$$

and

$$\left|I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)-\frac{[g(x)-g(a)]^{\alpha}f(a)+[g(b)-g(x)]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \tag{5.3}$$
$$\leq\frac{H}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)(t-a)^r dt}{[g(t)-g(a)]^{1-\alpha}}+\int_x^b\frac{g'(t)(b-t)^r dt}{[g(b)-g(t)]^{1-\alpha}}\right]$$
$$\leq\frac{H}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}(x-a)^r+(g(b)-g(x))^{\alpha}(b-x)^r\right]$$
$$\leq\frac{H}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}[(x-a)^r+(b-x)^r];\\((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p})^{1/p}((x-a)^{rq}+(b-x)^{rq})^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha})\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\end{cases}$$

for any $x\in(a,b)$

(ii) We also have

$$\left|\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\frac{f(b)+f(a)}{2}\right| \tag{5.4}$$
$$\leq\frac{H}{2\Gamma(\alpha)}\left[\int_a^b\frac{g'(t)(b-t)^r dt}{[g(b)-g(t)]^{1-\alpha}}+\int_a^b\frac{g'(t)(t-a)^r dt}{[g(t)-g(a)]^{1-\alpha}}\right]$$
$$\leq\frac{H}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}(b-a)^r.$$

Proof. Using the identity (2.15) and the properties of the modulus, we have

$$\left|I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)-\frac{[g(x)-g(a)]^{\alpha}f(a)+[g(b)-g(x)]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)|f(t)-f(a)|dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b\frac{g'(t)|f(t)-f(b)|dt}{[g(t)-g(x)]^{1-\alpha}}\right]=:C(x)$$

for any $x\in(a,b)$.

Since $f : [a,b] \to \mathbb{C}$ is r-H-*Hölder continuous* on $[a,b]$ with $r \in (0,1]$ and $H > 0$, hence

$$\begin{aligned} C(x) &\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{g'(t)(t-a)^r \, dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)(b-t)^r \, dt}{[g(t)-g(x)]^{1-\alpha}} \right] \\ &\leq \frac{H}{\Gamma(\alpha)} \left[(x-a)^r \int_a^x \frac{g'(t) \, dt}{[g(x)-g(t)]^{1-\alpha}} + (b-x)^r \int_x^b \frac{g'(t) \, dt}{[g(t)-g(x)]^{1-\alpha}} \right] \\ &= \frac{H}{\Gamma(\alpha)} \left[(x-a)^r \frac{(g(x)-g(a))^\alpha}{\alpha} + (b-x)^r \frac{(g(b)-g(x))^\alpha}{\alpha} \right] \\ &= \frac{H}{\Gamma(\alpha+1)} \left[(x-a)^r (g(x)-g(a))^\alpha + (b-x)^r (g(b)-g(x))^\alpha \right], \end{aligned}$$

for any $x \in (a,b)$, which proves the first two inequalities in (5.2). The rest is obvious.

The inequality (5.3) follows in a similar way by utilising the equality (2.16).

The inequality (5.4) follows by utilising the equality (2.17). □

Corollary 7. *With the assumptions of Theorem 4 we have*

$$\begin{aligned} &\left| I_{a+,g}^\alpha f(M_g(a,b)) + I_{b-,g}^\alpha f(M_g(a,b)) - \frac{f(a)+f(b)}{2^\alpha \Gamma(\alpha+1)} [g(b)-g(a)]^\alpha \right| \\ &\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t)(t-a)^r \, dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)(b-t)^r \, dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}} \right] \\ &\qquad \leq \frac{H}{2^\alpha \Gamma(\alpha+1)} (g(b)-g(a))^\alpha \left[(M_g(a,b)-a)^r + (b-M_g(a,b))^r \right] \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} &\left| I_{M_g(a,b)-,g}^\alpha f(a) + I_{M_g(a,b)+,g}^\alpha f(b) - \frac{f(a)+f(b)}{2^\alpha \Gamma(\alpha+1)} [g(b)-g(a)]^\alpha \right| \\ &\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^{M_g(a,b)} \frac{g'(t)(t-a)^r \, dt}{[g(t)-g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)(b-t)^r \, dt}{[g(b)-g(t)]^{1-\alpha}} \right] \\ &\qquad \leq \frac{H}{2^\alpha \Gamma(\alpha+1)} (g(b)-g(a))^\alpha \left[(M_g(a,b)-a)^r + (b-M_g(a,b))^r \right]. \end{aligned} \tag{5.6}$$

6. Applications for Hadamard Fractional Integrals

If we take $g(t) = \ln t$ and $0 \leq a < x \leq b$, then by Theorem 3 for Hadamard fractional integrals H_{a+}^α and H_{b-}^α we have for $f : [a,b] \to \mathbb{C}$, a function of bounded variation on $[a,b]$ that

$$
\begin{aligned}
(6.1)\quad & \left| H_{a+}^{\alpha} f(x) + H_{b-}^{\alpha} f(x) - \frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} f(a) + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \bigvee_a^t (f)\, dt}{t} + \int_x^b \frac{\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \bigvee_t^b (f)\, dt}{t} \right] \\
& \leq \frac{1}{\Gamma(\alpha+1)} \left[\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} \bigvee_a^x (f) + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha} \bigvee_x^b (f) \right] \\
& \leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha} \bigvee_a^b (f); \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p} \left(\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right) \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
(6.2)\quad & \left| H_{x-}^{\alpha} f(a) + H_{x+}^{\alpha} f(b) - \frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} f(a) + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1} \bigvee_a^t (f)\, dt}{t} + \int_x^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1} \bigvee_t^b (f)\, dt}{t} \right] \\
& \leq \frac{1}{\Gamma(\alpha+1)} \left[\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha} \bigvee_a^x (f) + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha} \bigvee_x^b (f) \right] \\
& \leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha} \bigvee_a^b (f); \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p} \left(\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right) \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \end{cases}
\end{aligned}
$$

for any $x \in (a,b)$

We also have

$$
\begin{aligned}
(6.3)\quad & \left| \frac{H_{b-}^{\alpha} f(a) + H_{a+}^{\alpha} f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \frac{f(b)+f(a)}{2} \right| \\
& \leq \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1} \bigvee_t^b (f)\, dt}{t} + \int_a^b \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1} g'(t) \bigvee_a^t (f)\, dt}{t} \right] \\
& \leq \frac{1}{\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \bigvee_a^b (f).
\end{aligned}
$$

If we take in (6.1) and (6.2) $x = G(a,b)$, then we get

$$\left| H_{a+}^{\alpha} f(G(a,b)) + H_{b-}^{\alpha} f(G(a,b)) - \frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \right| \tag{6.4}$$
$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{G(a,b)} \frac{\left[\ln\left(\frac{G(a,b)}{t}\right)\right]^{\alpha-1} \bigvee_a^t(f)\,dt}{t} + \int_{G(a,b)}^b \frac{\left[\ln\left(\frac{t}{G(a,b)}\right)\right]^{\alpha-1} \bigvee_t^b(f)\,dt}{t} \right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \bigvee_a^b(f)$$

and

$$\left| H_{G(a,b)-}^{\alpha} f(a) + H_{G(a,b)+}^{\alpha} f(b) - \frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \right| \tag{6.5}$$
$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{G(a,b)} \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1} \bigvee_a^t(f)\,dt}{t} + \int_{G(a,b)}^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1} \bigvee_t^b(f)\,dt}{t} \right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \bigvee_a^b(f).$$

Assume that $f : [a,b] \to \mathbb{C}$ is r-H-*Hölder continuous* on $[a,b]$ with $r \in (0,1]$ and $H > 0$. If we take $g(t) = \ln t$ and $0 \leq a < x \leq b$ in Theorem 4, then we get

$$\left| H_{a+}^{\alpha} f(x) + H_{b-}^{\alpha} f(x) - \frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} f(a) + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \tag{6.6}$$
$$\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} (t-a)^r\,dt}{t} + \int_x^b \frac{\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} (b-t)^r\,dt}{t} \right]$$
$$\leq \frac{H}{\Gamma(\alpha+1)} \left[\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} (x-a)^r + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha} (b-x)^r \right]$$
$$\leq \frac{H}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha} \bigvee_a^b(f); \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p} \left(\left(\bigvee_a^x(f)\right)^q + \left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q > 1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right) \left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right] \end{cases}$$

and

$$\left| H_{x-}^{\alpha} f(a) + H_{x+}^{\alpha} f(b) - \frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} f(a) + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \tag{6.7}$$
$$\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^x \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1} (t-a)^r \, dt}{t} + \int_x^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1} (b-t)^r \, dt}{t} \right]$$
$$\leq \frac{H}{\Gamma(\alpha+1)} \left[\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} (x-a)^r + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha} (b-x)^r \right]$$
$$\leq \frac{H}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha} \bigvee_a^b(f); \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha p} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha p}\right)^{1/p} \left(\left(\bigvee_a^x(f)\right)^q + \left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\left(\ln\left(\frac{x}{a}\right)\right)^{\alpha} + \left(\ln\left(\frac{b}{x}\right)\right)^{\alpha}\right) \left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right] \end{cases}$$

for any $x \in (a,b)$.

We also have

$$\left| \frac{H_{b-}^{\alpha} f(a) + H_{a+}^{\alpha} f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \frac{f(b)+f(a)}{2} \right| \tag{6.8}$$
$$\leq \frac{H}{2\Gamma(\alpha)} \left[\int_a^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1} (b-t)^r \, dt}{t} + \int_a^b \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1} (t-a)^r \, dt}{t} \right]$$
$$\leq \frac{H}{\Gamma(\alpha+1)} (b-a)^r \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}.$$

If we take in (6.7) and (6.8) $x = G(a,b)$, then we get

$$\left| H_{a+}^{\alpha} f(G(a,b)) + H_{b-}^{\alpha} f(G(a,b)) - \frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \right| \tag{6.9}$$
$$\leq \frac{H}{\Gamma(\alpha)} \left[\int_a^{G(a,b)} \frac{\left[\ln\left(\frac{G(a,b)}{t}\right)\right]^{\alpha-1} (t-a)^r \, dt}{t} + \int_{G(a,b)}^b \frac{\left[\ln\left(\frac{t}{G(a,b)}\right)\right]^{\alpha-1} (b-t)^r \, dt}{t} \right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} (b-a)^r$$

and

$$\left| H_{G(a,b)-}^{\alpha} f(a) + H_{G(a,b)+}^{\alpha} f(b) - \frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \right| \tag{6.10}$$
$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^{G(a,b)} \frac{\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1} (t-a)^r \, dt}{t} + \int_{G(a,b)}^b \frac{\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1} (b-t)^r \, dt}{t} \right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} (b-a)^r.$$

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