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SOME WEIGHTED INEQUALITIES FOR HIGHER-ORDER PARTIAL DERIVATIVES IN TWO DIMENSIONS AND ITS APPLICATIONS

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Abstract. We establish some Ostrowski type inequalities involving higher-order partial derivatives for two dimensional integrals on Lebesgue spaces (L_∞, L_p and L_1). In addition, we obtain some inequalities for double integrals of functions whose higher-order partial derivatives in absolute value are convex on the co- ordinates on rectangle from the plane. Some applications in Numerical Analysis in connection with cubature formula are given. Finally, with the help of obtained inequality, we establish applications for the kth moment of random variables.

1. Introduction

Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'} : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, the inequality holds:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$ [25]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality*.

In a recent paper [2], Barnett and Dragomir proved the following Ostrowski type inequality for double integrals:

Theorem 1. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be continuous on $[a,b] \times [c,d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ exists on $(a,b) \times (c,d)$ and is bounded, i.e.,*

$$\left\| f''_{x,y} \right\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty.$$

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Then, we have the inequality:

$$
\begin{aligned}
(1.2) \qquad & \left| \int_a^b \int_c^d f(s,t)dtds - (d-c)(b-a)f(x,y) \right. \\
& \left. - \left[(b-a) \int_c^d f(x,t)dt + (d-c) \int_a^b f(s,y)ds \right] \right| \\
\leq & \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{d+c}{2} \right)^2 \right] \left\| f''_{x,y} \right\|_\infty
\end{aligned}
$$

for all $(x,y) \in [a,b] \times [c,d]$.

In [2], the inequality (1.2) is established by the use of integral identity involving Peano kernels. In [28], Pecarić and Vukelić gave weighted Montgomery's identities for two variables functions. Recently, many authors have worked on the Ostrowski type inequalities for double integrals. For example, Pachpatte obtained a new inequality in the view (1.2) by using elementary analysis in [26] and [27]. In [8], [10] and [13], some Ostrowski type inequalities for double integrals and applications in numerical analysis in connection with cubature formula are given by researchers. Authors deduced weighted inequality of Ostrowski type for two dimensional integrals in [31] and [35]. Some researchers established some Ostrowski type inequalities for n-time differentiable mappings in [1], [6] and [15]. In [14], weighted integral inequalities for one variable mappings which are $n-$times differentiable are obtained by Erden and Sarıkaya. The researchers established some Ostrowski type inequalities involving higher order partial derivatives for double integrals.in [4], [16] and [36].

Let us now consider a bidimensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$
f\left(tx + (1-t)z, ty + (1-t)w\right) \leq tf(x,y) + (1-t) f(z,w)
$$

holds, for all $(x,y), (z,w) \in \Delta$ and $t \in [0,1]$. A function $f : \Delta \to \mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$ and $y \in [c,d]$ (see, [9]).

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. *A function* $f : \Delta \to \mathbb{R}$ *will be called co-ordinated canvex on* Δ*, for all* $t, s \in [0,1]$ *and* $(x,y), (u,v) \in \Delta$*, if the following inequality holds:*

$$
\begin{aligned}
& f(tx + (1-t)y, su + (1-s)v) \\
\leq \ & tsf(x,u) + s(1-t)f(y,u) + t(1-s)f(x,v) + (1-t)(1-s)f(y,v).
\end{aligned}
$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [9]).

Also, in [9], Dragomir established the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 2. *Suppose that $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$
\begin{aligned}
(1.3)\qquad & f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
\le\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
\le\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\
\le\ & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\
& \left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\
\le\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}
$$

The above inequalities are sharp.

In recent years, researchers have studied some integral inequalities by using some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$. For example, authors gave some Hadamard's type inequalities involving Riemann-Liouville fractional integrals for convex and s-convex functions on the co-ordinates in [7] and [33]. In [22], several new inequalities for differentiable co-ordinated convex functions in two variables which are related to the left side of Hermite- Hadamard type inequality for co-ordinated convex functions in two variables are proved by Latif and Dragomir. Erden and Sarıkaya gave some generalized weighted integral inequalities for functions whose partial derivatives in absolute value are convex on the co- ordinates on rectangle from the plane in [12] and [13]. In [32], Sarikaya et al. proved some new inequalities that give estimate of the deference between the middle and the right most terms in (1.3) for differentiable co-ordinated convex functions. Researchers deduced some integral inequalities for differentiable co-ordinated convex mappings in [17], [20], [23] and [34]. In [21], [24] and [30], some Hermite-Hadamard type inequalities for veriaty co-ordinated convex functions are developed.

In this study, first of all, we establish a new integral inequality involving higher-order partial derivatives. Then, some inequalities of Ostrowski type for two dimensional integrals is gotten by using this identity. Also, some integral inequalities for convex mappings on the co-ordinates on the rectangle from the plane are obtained. Finally, some applications of the Ostrowski type inequality developed in this work for cubature formula and the kth moment of random variables are given.

2. Integral identity

In order to prove generalized weighted integral inequalities for double integrals, we need the following lemma:

Lemma 1. *Let $f:[a,b]\times[c,d]=:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a continuous function such that the partial derivatives $\frac{\partial^{k+l}f(t,s)}{\partial t^k\partial s^l}$, $k=0,1,2,...,n-1$, $l=0,1,2,...,m-1$ exists and are continuous on Δ, and assume that the functions $g:[a,b]\to[0,\infty)$*

and $h:[c,d]\rightarrow[0,\infty)$ *are integrable. In addition,* $P_{n-1}(x,t)$ *and* $Q_{m-1}(y,s)$ *are defined by*

$$P_{n-1}(x,t):=\begin{cases}\frac{1}{(n-1)!}\int\limits_a^t (u-t)^{n-1}g(u)\,du, & a\leq t<x\\ \\ \frac{1}{(n-1)!}\int\limits_b^t (u-t)^{n-1}g(u)\,du, & x\leq t\leq b\end{cases}$$

and

$$Q_{m-1}(y,s):=\begin{cases}\frac{1}{(m-1)!}\int\limits_c^s (u-s)^{m-1}h(u)\,dv, & c\leq s<y\\ \\ \frac{1}{(m-1)!}\int\limits_d^s (u-s)^{m-1}h(u)\,dv, & y\leq s\leq d\end{cases}$$

where $n,m\in\mathbb{N}\backslash\{0\}$. *Then, for all* $(x,y)\in[a,b]\times[c,d]$, *we have the identity*

$$\begin{aligned}&\int\limits_a^b\int\limits_c^d P_{n-1}(x,t)\,Q_{m-1}(y,s)\,\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}dsdt \quad (2.1)\\ =\;&\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int\limits_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\\ &-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int\limits_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int\limits_a^b\int\limits_c^d h(s)g(t)f(t,s)dsdt\end{aligned}$$

where $M_k(x)$ *and* $M_l(y)$ *are defined by*

$$M_k(x)=\int\limits_a^b (u-x)^k g(u)\,du,\quad k=0,1,2,...$$

$$M_l(y)=\int\limits_c^d (u-y)^l h(u)\,du,\quad l=0,1,2,...$$

Proof. We have the equality

$$\begin{aligned}&\int\limits_a^b\int\limits_c^d P_{n-1}(x,t)\,Q_{m-1}(y,s)\,\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}dsdt\\ =\;&\int\limits_a^b P_{n-1}(x,t)\left\{\int\limits_c^d Q_{m-1}(y,s)\,\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}ds\right\}dt.\end{aligned}$$

Applying integration by parts for partial derivatives $\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}$ on $[c,d]$, we obtain

$$\begin{aligned}
&\int_c^d Q_{m-1}(y,s)\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}ds \\
=\;& \frac{1}{(m-1)!}\int_c^y \int_c^s (u-s)^{m-1}h(u)\,du\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}ds \\
&+\frac{1}{(m-1)!}\int_y^d \int_d^s (u-s)^{m-1}h(u)\,du\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}ds \\
=\;& \frac{M_{m-1}(y)}{(m-1)!}\frac{\partial^{n+m-1}f(t,y)}{\partial t^n \partial y^{m-1}}+\int_c^d Q_{m-2}(y,s)\frac{\partial^{n+m-1}f(t,s)}{\partial t^n \partial s^{m-1}}ds.
\end{aligned}$$

As we progress by this method, we get

$$\int_c^d Q_{m-1}(y,s)\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}ds=\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\frac{\partial^{n+l}f(t,y)}{\partial t^n \partial y^l}-\int_c^d h(s)\frac{\partial^n f(t,s)}{\partial t^n}ds.$$

Then, we have

$$\begin{aligned}
(2.2)\quad &\int_a^b \int_c^d P_{n-1}(x,t)\,Q_{m-1}(y,s)\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}dsdt \\
=\;& \sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b P_{n-1}(x,t)\frac{\partial^{n+l}f(t,y)}{\partial t^n \partial y^l}dt-\int_c^d h(s)\int_a^b P_{n-1}(x,t)\frac{\partial^n f(t,s)}{\partial t^n}dtds.
\end{aligned}$$

Similarly, applying integration by parts for partial derivatives $\frac{\partial^{n+l}f(t,y)}{\partial t^n \partial y^l}$ and $\frac{\partial^n f(t,s)}{\partial t^n}$ on $[a,b]$, we can write

$$\begin{aligned}
(2.3)\quad &\int_a^b P_{n-1}(x,t)\frac{\partial^{n+l}f(t,y)}{\partial t^n \partial y^l}dt \\
=\;& \sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\frac{\partial^{k+l}f(x,y)}{\partial x^k \partial y^l}-\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt
\end{aligned}$$

and

$$(2.4)\quad \int_a^b P_{n-1}(x,t)\frac{\partial^n f(t,s)}{\partial t^n}dt=\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\frac{\partial^k f(x,s)}{\partial x^k}-\int_a^b g(t)f(t,s)dt.$$

Substituting the identity (2.3) and (2.4) in (2.2), we deduce desired identity (2.1), and thus the theorem is proved. □

3. Some inequalities for $\frac{\partial^{n+m}f}{\partial t^n \partial s^m}$ belongs to Lebesgue space

We give some results for functions whose $n+m$.th partial derivatives are bounded. We start with the following result.

Theorem 3. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a continuous on Δ such that $\frac{\partial^{n+m}f}{\partial t^n \partial s^m}$ exist on $(a,b)\times(c,d)$ and assume that the functions $g:[a,b]\to[0,\infty)$ and $h:[c,d]\to[0,\infty)$ are integrable. If $\frac{\partial^{n+m}f}{\partial t^n \partial s^m}\in L_\infty(\Delta)$, then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right.\\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
&\leq\frac{1}{n!m!}\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty\begin{cases}
M_n(x)M_m(y) \\ \qquad\qquad\text{if } m \text{ and } n \text{ are even numbers}\\
M_n(x)\left[M_m(y)-2\int_c^y(u-y)^m h(u)du\right] \\ \qquad\qquad\text{if } m \text{ is odd number and } n \text{ is even numbers}\\
M_m(y)\left[M_n(x)-2\int_a^x(u-x)^n g(u)du\right] \\ \qquad\qquad\text{if } m \text{ is even number and } n \text{ is odd number}\\
\left[M_n(x)-2\int_a^x(u-x)^n g(u)du\right] \\ \times\left[M_m(y)-2\int_c^y(u-y)^m h(u)du\right] \\ \qquad\qquad\text{if } m \text{ and } n \text{ are odd numbers}
\end{cases}
\end{aligned}$$

for all $(x,y)\in[a,b]\times[c,d]$, where

$$\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty=\sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|<\infty.$$

Proof. If we take absolute value of both sides of the equality (2.1), because $\frac{\partial^{n+m}f}{\partial t^n \partial s^m}$ is a bounded mapping, we can write

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right.\\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
&\leq\int_a^b\int_c^d\left|P_{n-1}(x,t)\right|\left|Q_{m-1}(y,s)\right|\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|dsdt\\
&\leq\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty\int_a^b\int_c^d\left|P_{n-1}(x,t)\right|\left|Q_{m-1}(y,s)\right|dsdt.
\end{aligned}\tag{3.1}$$

By definitions of $P_{n-1}(x,t)$ and $Q_{m-1}(y,s)$, we get

$$\begin{aligned}
&\int_a^b\int_c^d |P_{n-1}(x,t)|\,|Q_{m-1}(y,s)|\,dsdt \\
= & \left[\int_a^x\left|\int_a^t \frac{(u-t)^{n-1}}{(n-1)!}g(u)du\right|dt+\int_x^b\left|\int_b^t \frac{(u-t)^{n-1}}{(n-1)!}g(u)du\right|dt\right] \\
&\times\left[\int_c^y\left|\int_c^s \frac{(u-s)^{m-1}}{(m-1)!}h(u)\,du\right|ds+\int_y^d\left|\int_d^s \frac{(u-s)^{m-1}}{(m-1)!}h(u)\,du\right|ds\right].
\end{aligned}$$

By using the change of order of integration, we obtain

$$\begin{aligned}
&\int_a^b\int_c^d |P_{n-1}(x,t)|\,|Q_{m-1}(y,s)|\,dsdt \\
= & \left[\int_a^x \frac{(x-u)^{n}}{n!}g(u)du+\int_x^b \frac{(u-x)^{n}}{n!}g(u)du\right] \\
&\times\left[\int_c^y \frac{(y-u)^{m}}{m!}h(u)du+\int_y^d \frac{(u-y)^{m}}{m!}h(u)du\right]
\end{aligned}$$

which completes the proof. $\square$

Remark 1. *Under the same assumptions of Theorem 3 with $n=m=1$, then the following inequality holds:*

$$\begin{aligned}
(3.2)\qquad &\left|M_0(x)M_0(y)f(x,y)-M_0(y)\int_a^b g(t)f(t,y)dt\right. \\
&\left.-M_0(x)\int_c^d h(s)f(x,s)ds+\int_a^b\int_c^d g(t)h(s)f(t,s)dsdt\right| \\
\leq\ & \left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty\left[M_1(x)-2\int_a^x (x-u)\,g(u)du\right]\left[M_1(y)-2\int_c^y (y-u)\,h(u)du\right]
\end{aligned}$$

*which is "**weighted Ostrowski**" type inequality for $\|\|_\infty$ −norm. This inequality was deduced by Sarikaya and Ogunmez in* [31].

Remark 2. *If we take $g(u)=h(u)=1$ in (3.2), then the inequality (3.2) reduce to the inequality (1.2).*

Remark 3. *Taking $g(u) = h(u) = 1$, $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (3.2), then we have the inequality*

$$\begin{aligned}
(3.3)\qquad & \left| (b-a)(d-c) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - (d-c)\int_a^b f\left(t, \frac{c+d}{2}\right) dt \right. \\
& \left. - (b-a)\int_c^d f\left(\frac{a+b}{2}, s\right) ds + \int_a^b \int_c^d f(t,s)\, ds\, dt \right| \\
\leq\ & \frac{(b-a)^2 (d-c)^2}{16} \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty
\end{aligned}$$

which was given by Barnett and Dragomir in [2].

Remark 4. *Under the same assumptions of Theorem 3 with $g(u) = h(u) = 1$, then we have the inequality*

$$\begin{aligned}
(3.4)\qquad & \left| \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{X_k(x)}{k!} \frac{Y_l(y)}{l!} \frac{\partial^{k+l} f(x,y)}{\partial x^k \partial y^l} - \sum_{l=0}^{m-1} \frac{Y_l(y)}{l!} \int_a^b \frac{\partial^l f(t,y)}{\partial y^l} dt \right. \\
& \left. - \sum_{k=0}^{n-1} \frac{X_k(x)}{k!} \int_c^d \frac{\partial^k f(x,s)}{\partial x^k} ds + \int_a^b \int_c^d f(t,s)\, ds\, dt \right| \\
\leq\ & \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty \left[\frac{(b-x)^{n+1} + (x-a)^{n+1}}{(n+1)!} \right] \left[\frac{(d-y)^{m+1} + (y-c)^{m+1}}{(m+1)!} \right]
\end{aligned}$$

where

$$(3.5)\qquad X_k(x) = \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)}$$

and

$$(3.6)\qquad Y_l(y) = \frac{(d-y)^{l+1} + (-1)^l (y-c)^{l+1}}{(l+1)}.$$

This inequality (3.4) was proved by Hanna et al. in [16].

Theorem 4. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous on Δ such that $\frac{\partial^{n+m} f}{\partial t^n \partial s^m}$ exist on $(a,b) \times (c,d)$ and assume that the functions $g : [a,b] \to [0,\infty)$ and $h : [c,d] \to [0,\infty)$ are integrable. If $\frac{\partial^{n+m} f}{\partial t^n \partial s^m} \in L_\infty(\Delta)$, then we have the inequality*

$$\begin{aligned}
(3.7)\qquad & \left| \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \frac{M_k(x)}{k!} \frac{M_l(y)}{l!} \frac{\partial^{k+l} f(x,y)}{\partial x^k \partial y^l} - \sum_{l=0}^{m-1} \frac{M_l(y)}{l!} \int_a^b g(t) \frac{\partial^l f(t,y)}{\partial y^l} dt \right. \\
& \left. - \sum_{k=0}^{n-1} \frac{M_k(x)}{k!} \int_c^d h(s) \frac{\partial^k f(x,s)}{\partial x^k} ds + \int_a^b \int_c^d h(s) g(t) f(t,s)\, ds\, dt \right| \\
\leq\ & \left\| \frac{\partial^{n+m} f}{\partial t^n \partial s^m} \right\|_\infty \frac{\|g\|_{[a,b],\infty}}{(n+1)!} \frac{\|h\|_{[c,d]\infty}}{(m+1)!} \\
& \times \left[(b-x)^{n+1} + (x-a)^{n+1} \right] \left[(d-y)^{m+1} + (y-c)^{m+1} \right]
\end{aligned}$$

for all $(x,y)\in[a,b]\times[c,d]$, *where* $\|g\|_{[a,b],\infty}=\sup_{u\in[a,b]}|g(u)|$, $\|h\|_{[c,d]\infty}=\sup_{u\in[c,d]}|h(u)|$ *and*

$$\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty=\sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|<\infty.$$

Proof. Taking moduls of both sides of the equality (2.1), because $\frac{\partial^{n+m}f}{\partial t^n\partial s^m}$ is a bounded mapping, we have the inequality (3.1). Because of boundedness g and h, and by definitions of $P_{n-1}(x,t)$ and $Q_{m-1}(y,s)$, we get

$$\int_a^b\int_c^d|P_{n-1}(x,t)|\,|Q_{m-1}(y,s)|\,dsdt \tag{3.8}$$
$$\le \frac{\|g\|_{[a,b],\infty}}{(n-1)!}\frac{\|h\|_{[c,d]\infty}}{(m-1)!}\left\{\int_a^x\int_c^y\left|\int_a^t(u-t)^{n-1}du\right|\left|\int_c^s(u-s)^{m-1}du\right|dsdt\right.$$
$$+\int_a^x\int_y^d\left|\int_a^t(u-t)^{n-1}du\right|\left|\int_d^s(u-s)^{m-1}du\right|dsdt$$
$$+\int_x^b\int_c^y\left|\int_b^t(u-t)^{n}du\right|\left|\int_c^s(u-s)^{m-1}du\right|dsdt$$
$$\left.+\int_x^b\int_y^d\left|\int_b^t(u-t)^{n}du\right|\left|\int_d^s(u-s)^{m-1}du\right|dsdt\right\}.$$

If we calculate the above four integrals and also substitute the results in (3.8), we obtain desired inequality (3.7) which completes the proof. □

Corollary 1. *Under the same assumptions of Theorem 4 with* $n=m=1$*, then the following inequality holds:*

$$\left|M_0(x)M_0(y)f(x,y)-M_0(y)\int_a^b g(t)f(t,y)dt\right. \tag{3.9}$$
$$\left.-M_0(x)\int_c^d h(s)f(x,s)ds+\int_a^b\int_c^d g(t)h(s)f(t,s)dsdt\right|$$
$$\le\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty\|g\|_{[a,b],\infty}\|h\|_{[c,d]\infty}$$
$$\times\left[\frac14(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac14(d-c)^2+\left(y-\frac{d+c}{2}\right)^2\right]$$

*which is "**weighted Ostrowski**" type inequality for* $\|\|_\infty$ *−norm.*

Remark 5. *If we take* $g(u)=h(u)=1$ *in (3.9), then the inequality (3.9) reduce to the inequality (1.2).*

Remark 6. *If we choose* $g(u)=h(u)=1$ *in theorem 4, then the inequality (3.7) becomes (3.4).*

Corollary 2. *Under the same assumptions of Theorem 4 with $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, then we have the inequality*

$$
\begin{aligned}
(3.10)\qquad & \left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\frac{a+b}{2})}{k!}\frac{M_l(\frac{c+d}{2})}{l!}\frac{\partial^{k+l}f(\frac{a+b}{2},\frac{c+d}{2})}{\partial x^k\partial y^l}\right. \\
& -\sum_{l=0}^{m-1}\frac{M_l(\frac{c+d}{2})}{l!}\int_a^b g(t)\frac{\partial^l f(t,\frac{c+d}{2})}{\partial y^l}dt \\
& \left.-\sum_{k=0}^{n-1}\frac{M_k(\frac{a+b}{2})}{k!}\int_c^d h(s)\frac{\partial^k f(\frac{a+b}{2},s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right| \\
\leq\; & \left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty\frac{\|g\|_{[a,b],\infty}}{(n+1)!}\frac{\|h\|_{[c,d]\infty}}{(m+1)!}\frac{(b-a)^{n+1}}{2^n}\frac{(d-c)^{m+1}}{2^m}
\end{aligned}
$$

*which is Ostrowski type inequality for double integrals. Thus, (3.10) is a higher degree "**weighted mid-point**" inequality for $\|\|_\infty$ $-$norm.*

Corollary 3. *Choosing $n=m=1$ in (3.10), we obtain*

$$
\begin{aligned}
& \left|M_0(x)M_0(y)f(\frac{a+b}{2},\frac{c+d}{2})-M_0(y)\int_a^b g(t)f(t,\frac{c+d}{2})dt\right. \\
& \left.-M_0(x)\int_c^d h(s)f(\frac{a+b}{2},s)ds+\int_a^b\int_c^d g(t)h(s)f(t,s)dsdt\right| \\
\leq\; & \left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty\|g\|_{[a,b],\infty}\|h\|_{[c,d]\infty}\frac{(b-a)^2(d-c)^2}{16}
\end{aligned}
$$

*which is "**weighted mid-point**" inequality for double integrals.*

Now, we deduce some inequalities for mappings whose higher order partial derivatives belongs to $L_p(\Delta)$ and $L_1(\Delta)$.

Theorem 5. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a continuous on Δ such that $\frac{\partial^{n+m}f}{\partial t^n\partial s^m}$ exist on $(a,b)\times(c,d)$ and assume that the functions $g:[a,b]\to[0,\infty)$ and $h:[c,d]\to[0,\infty)$ are integrable. If $\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_p(\Delta)$, $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$, then we have the inequality*

$$
\begin{aligned}
& \left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right. \\
& \left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right| \\
\leq\; & \left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p\frac{\|g\|_{[a,b],\infty}}{n!(nq+1)^{\frac{1}{q}}}\frac{\|h\|_{[c,d]\infty}}{m!(mq+1)^{\frac{1}{q}}} \\
& \times\left[(x-a)^{nq+1}+(b-x)^{nq+1}\right]^{\frac{1}{q}}\left[(y-c)^{mq+1}+(d-y)^{mq+1}\right]^{\frac{1}{q}}
\end{aligned}
$$

for all $(x,y)\in[a,b]\times[c,d]$, *where* $\|g\|_{[a,b],\infty}=\sup\limits_{u\in[a,b]}|g(u)|$, $\|h\|_{[c,d]\infty}=\sup\limits_{u\in[c,d]}|h(u)|$ *and*

$$\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p=\left(\int\limits_a^b\int\limits_c^d\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|^p dsdt\right)^{\frac{1}{p}}.$$

Proof. From (2.1), using the properties of modulus and from Hölder's inequality, we get

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int\limits_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right.\\&\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int\limits_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int\limits_a^b\int\limits_c^d h(s)g(t)f(t,s)dsdt\right|\\ \leq\;&\left[\int\limits_a^b\int\limits_c^d|P_{n-1}(x,t)|^q|Q_{m-1}(y,s)|^q dsdt\right]^{\frac{1}{q}}\left[\int\limits_a^b\int\limits_c^d\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|^p dsdt\right]^{\frac{1}{p}}.\end{aligned}$$

Because of boundedness g and h, and by definitions of $P_{n-1}(x,t)$ and $Q_{m-1}(y,s)$, we can write

$$\begin{aligned}&\left[\int\limits_a^b\int\limits_c^d|P_{n-1}(x,t)|^q|Q_{m-1}(y,s)|^q dsdt\right]^{\frac{1}{q}}\\ \leq\;&\frac{\|g\|_{[a,b],\infty}}{(n-1)!}\frac{\|h\|_{[c,d]\infty}}{(m-1)!}\left(\int\limits_a^x\left|\int\limits_a^t(u-t)^{n-1}du\right|^q dt+\int\limits_x^b\left|\int\limits_b^t(u-t)^n du\right|^q dt\right)^{\frac{1}{q}}\\&\times\left(\int\limits_c^y\left|\int\limits_c^s(u-s)^{m-1}du\right|^q ds+\int\limits_y^d\left|\int\limits_d^s(u-s)^{m-1}du\right|^q ds\right)^{\frac{1}{q}}.\end{aligned}$$

By simple calculations, we easily deduced required inequality, and thus the theorem is proved. ∎

Corollary 4. *Under the same assumptions of Theorem 5 with $n = m = 1$, then the following inequality holds:*

$$\begin{aligned}
(3.11)\qquad & \left| M_0(x)M_0(y)f(x,y) - M_0(y)\int_a^b g(t)f(t,y)dt \right. \\
& \left. -M_0(x)\int_c^d h(s)f(x,s)ds + \int_a^b\int_c^d g(t)h(s)f(t,s)dsdt \right| \\
\leq\ & \left\| \frac{\partial^2 f}{\partial t\partial s} \right\|_p \|g\|_{[a,b],\infty} \|h\|_{[c,d]\infty} \\
& \times \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \left[\frac{(y-c)^{q+1} + (d-y)^{q+1}}{q+1} \right]^{\frac{1}{q}}
\end{aligned}$$

*which is "**weighted Ostrowski**" type inequality for $\|\|_p$ −norm.*

Corollary 5. *If we choose $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (3.11), then we have the inequality*

$$\begin{aligned}
& \left| M_0(x)M_0(y)f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - M_0(y)\int_a^b g(t)f\left(t, \frac{c+d}{2}\right)dt \right. \\
& \left. -M_0(x)\int_c^d h(s)f\left(\frac{a+b}{2}, s\right)ds + \int_a^b\int_c^d g(t)h(s)f(t,s)dsdt \right| \\
\leq\ & \left\| \frac{\partial^2 f}{\partial t\partial s} \right\|_p \|g\|_{[a,b],\infty} \|h\|_{[c,d]\infty} \frac{(b-a)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}} \frac{(d-c)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}
\end{aligned}$$

*which is "**weighted mid-point**" inequality for two dimensional integrals. This inequality is a weighted Ostrowski type inequality for $\|\|_p$ −norm.*

Remark 7. *If we take $g(u) = h(u) = 1$ in (3.11), then we get*

$$\begin{aligned}
& \left| (b-a)(d-c)f(x,y) - (d-c)\int_a^b f(t,y)dt \right. \\
& \left. -(b-a)\int_c^d f(x,s)ds + \int_a^b\int_c^d f(t,s)dsdt \right| \\
\leq\ & \left\| \frac{\partial^2 f}{\partial t\partial s} \right\|_p \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \\
& \times \left[\frac{(y-c)^{q+1} + (d-y)^{q+1}}{q+1} \right]^{\frac{1}{q}}
\end{aligned}$$

which was proved by Dragomir et al. in [8].

Remark 8. *Under the same assumptions of Theorem 5 with $g(u) = h(u) = 1$, then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{X_k(x)}{k!}\frac{Y_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{Y_l(y)}{l!}\int_a^b\frac{\partial^l f(t,y)}{\partial y^l}dt\right.\\
&\left.-\sum_{k=0}^{n-1}\frac{X_k(x)}{k!}\int_c^d\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d f(t,s)dsdt\right|\\
\le\ &\frac{1}{n!m!}\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p\left[\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}}\\
&\times\left[\frac{(y-c)^{mq+1}+(d-y)^{mq+1}}{mq+1}\right]^{\frac{1}{q}}
\end{aligned}\tag{3.12}$$

where $X_k(x)$ and $Y_l(y)$ are defined as in (3.5) and (3.6), respectively. This inequality (3.12) was deduced by Hanna in [16].

Corollary 6. *Under the same assumptions of Theorem 5 with $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\frac{a+b}{2})}{k!}\frac{M_l(\frac{c+d}{2})}{l!}\frac{\partial^{k+l}f(\frac{a+b}{2},\frac{c+d}{2})}{\partial x^k\partial y^l}\right.\\
&-\sum_{l=0}^{m-1}\frac{M_l(\frac{c+d}{2})}{l!}\int_a^b g(t)\frac{\partial^l f(t,\frac{c+d}{2})}{\partial y^l}dt\\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(\frac{a+b}{2})}{k!}\int_c^d h(s)\frac{\partial^k f(\frac{a+b}{2},s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
\le\ &\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_p\frac{\|g\|_{[a,b],\infty}}{n!\,(nq+1)^{\frac{1}{q}}}\frac{\|h\|_{[c,d]\infty}}{m!\,(mq+1)^{\frac{1}{q}}}\frac{(b-a)^{n+\frac{1}{q}}}{2^n}\frac{(d-c)^{m+\frac{1}{q}}}{2^m}
\end{aligned}$$

*which is "**weighted mid-point**" inequality for double integrals. This inequality is a higher degree weighted Ostrowski type for $\|\|_p$ −norm.*

Theorem 6. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous on Δ such that $\frac{\partial^{n+m}f}{\partial t^n\partial s^m}$ exist on $(a, b)\times(c, d)$ and assume that the functions $g : [a, b] \to [0, \infty)$ and $h : [c, d] \to [0, \infty)$*

are integrable. If $\frac{\partial^{n+m}f}{\partial t^n \partial s^m} \in L_1(\Delta)$, *then we have*

$$\begin{aligned}
(3.13)\qquad & \left| \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k \partial y^l} - \sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt \right. \\
& \left. -\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds + \int_a^b\int_c^d h(s)g(t)f(t,s)dsdt \right| \\
\leq\ & \frac{\|g\|_{[a,b],\infty}}{n!}\frac{\|h\|_{[c,d]\infty}}{m!}\left[\frac{(x-a)^n+(b-x)^n}{2}+\left|\frac{(b-x)^n-(x-a)^n}{2}\right|\right] \\
& \times\left[\frac{(y-c)^n+(d-y)^n}{2}+\left|\frac{(d-y)^n-(y-c)^n}{2}\right|\right]\left\|\frac{\partial^{n+m}f}{\partial t^n \partial s^m}\right\|_1
\end{aligned}$$

for all $(x,y) \in [a,b]\times[c,d]$, *where* $\|g\|_{[a,b],\infty} = \sup_{u\in[a,b]}|g(u)|$, $\|h\|_{[c,d]\infty} = \sup_{u\in[\ c,d]}|h(u)|$ *and*

$$\left\|\frac{\partial^{n+m}}{\partial t^n \partial s^m}\right\|_1 = \int_a^b\int_c^d \left|\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}\right| dsdt.$$

Proof. By taking absolute value of (2.1), we find that

$$\begin{aligned}
& \left| \sum_{k=0}^{n-1}\sum_{l=0}^{m-1} \frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k \partial y^l} - \sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt \right. \\
& \left. -\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds + \int_a^b\int_c^d h(s)g(t)f(t,s)dsdt \right| \\
\leq\ & \int_a^b\int_c^d |P_{n-1}(x,t)|\,|Q_{m-1}(y,s)|\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}\right| dsdt \\
\leq\ & \sup_{(t,s)\in(a,b)\times(c,d)}|P_{n-1}(x,t)|\,|Q_{m-1}(y,s)|\int_a^b\int_c^d\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}\right| dsdt.
\end{aligned}$$

By boundedness g and h, and because of definitions of $P_{n-1}(x,t)$ and $Q_{m-1}(y,s)$, we have

$$\begin{aligned}
& \sup_{(t,s)\in(a,b)\times(c,d)}|P_{n-1}(x,t)|\,|Q_{m-1}(y,s)| \\
\leq\ & \frac{\|g\|_{[a,b],\infty}}{n!}\frac{\|h\|_{[c,d]\infty}}{m!}\max\left\{(x-a)^n,(b-x)^n\right\}\max\left\{(y-c)^m,(d-y)^m\right\}.
\end{aligned}$$

We obtain desired inequality (3.13) using the identity

$$\max\{X,Y\} = \frac{X+Y}{2}+\left|\frac{Y-X}{2}\right|.$$

The proof is thus completed. □

Corollary 7. *Under the same assumptions of Theorem 6 with $n = m = 1$, then the following inequality holds:*

$$\begin{aligned}
&\left| M_0(x)M_0(y)f(x,y) - M_0(y)\int_a^b g(t)f(t,y)dt \right. \\
&\left. -M_0(x)\int_c^d h(s)f(x,s)ds + \int_a^b\int_c^d g(t)h(s)f(t,s)dsdt \right| \\
\leq\ & \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_1 \|g\|_{[a,b],\infty} \|h\|_{[c,d]\infty} \\
&\times \left[\frac{(b-a)}{2} + \left| \frac{a+b}{2} - x \right| \right] \left[\frac{(d-c)}{2} + \left| \frac{c+d}{2} - y \right| \right]
\end{aligned} \tag{3.14}$$

*which is "**weighted Ostrowski**" inequality for double integrals of the Ostrowski type inequality for $\|\|_1$ −norm.*

Corollary 8. *If we choose $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (3.14), then we have the inequality*

$$\begin{aligned}
&\left| M_0(x)M_0(y)f(\frac{a+b}{2}, \frac{c+d}{2}) - M_0(y)\int_a^b g(t)f(t, \frac{c+d}{2})dt \right. \\
&\left. -M_0(x)\int_c^d h(s)f(\frac{a+b}{2}, s)ds + \int_a^b\int_c^d g(t)h(s)f(t,s)dsdt \right| \\
\leq\ & \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_1 \|g\|_{[a,b],\infty} \|h\|_{[c,d]\infty} \frac{(b-a)(d-c)}{4}
\end{aligned}$$

*which is "**weighted mid-point**" inequality for the two dimensional integrals of the Ostrowski type inequality for $\|\|_1$ −norm.*

Remark 9. *If we take $g(u) = h(u) = 1$ in (3.14), then we get*

$$\begin{aligned}
&\left| (b-a)(d-c)f(x,y) - (d-c)\int_a^b f(t,y)dt \right. \\
&\left. -(b-a)\int_c^d f(x,s)ds + \int_a^b\int_c^d f(t,s)dsdt \right| \\
\leq\ & \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_1 \left[\frac{(b-a)}{2} + \left| \frac{a+b}{2} - x \right| \right] \left[\frac{(d-c)}{2} + \left| \frac{c+d}{2} - y \right| \right]
\end{aligned} \tag{3.15}$$

which is Ostrowski type inequality for $\|\|_1$ −norm.

Remark 10. *Taking $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (3.15), we get*

$$\begin{aligned}
&\left|(b-a)(d-c)f(\frac{a+b}{2},\frac{c+d}{2})-(d-c)\int\limits_a^b f(t,\frac{c+d}{2})dt\right. \\
&\left.-(b-a)\int\limits_c^d f(\frac{a+b}{2},s)ds+\int\limits_a^b\int\limits_c^d f(t,s)dsdt\right| \\
\leq\ &\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_1\frac{(b-a)(d-c)}{4}
\end{aligned}$$

*which is "**mid-point**" inequality for double integrals of the Ostrowski type inequality for $\|\|_1$ −norm.*

Remark 11. *Under the same assumptions of Theorem 6 with $g(u)=h(u)=1$, then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{X_k(x)}{k!}\frac{Y_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{Y_l(y)}{l!}\int\limits_a^b\frac{\partial^l f(t,y)}{\partial y^l}dt\right. \\
&\left.-\sum_{k=0}^{n-1}\frac{X_k(x)}{k!}\int\limits_c^d\frac{\partial^k f(x,s)}{\partial x^k}ds+\int\limits_a^b\int\limits_c^d f(t,s)dsdt\right| \\
\leq\ &\frac{1}{n!m!}\left[\frac{(x-a)^n+(b-x)^n}{2}+\left|\frac{(b-x)^n-(x-a)^n}{2}\right|\right] \\
&\times\left[\frac{(y-c)^n+(d-y)^n}{2}+\left|\frac{(d-y)^n-(y-c)^n}{2}\right|\right]\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_1
\end{aligned}\tag{3.16}$$

where $X_k(x)$ and $Y_l(y)$ are defined as in (3.5) and (3.6), respectively. This inequality (3.16) was proved by Hanna et al. in [16].

Corollary 9. *Under the same assumptions of Theorem 6 with $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\frac{a+b}{2})}{k!}\frac{M_l(\frac{c+d}{2})}{l!}\frac{\partial^{k+l}f(\frac{a+b}{2},\frac{c+d}{2})}{\partial x^k\partial y^l}\right. \\
&-\sum_{l=0}^{m-1}\frac{M_l(\frac{c+d}{2})}{l!}\int\limits_a^b g(t)\frac{\partial^l f(t,\frac{c+d}{2})}{\partial y^l}dt \\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(\frac{a+b}{2})}{k!}\int\limits_c^d h(s)\frac{\partial^k f(\frac{a+b}{2},s)}{\partial x^k}ds+\int\limits_a^b\int\limits_c^d h(s)g(t)f(t,s)dsdt\right| \\
\leq\ &\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_1\frac{\|g\|_{[a,b],\infty}}{n!}\frac{\|h\|_{[c,d]\infty}}{m!}\frac{(b-a)^n}{2^n}\frac{(d-c)^m}{2^m}
\end{aligned}\tag{3.17}$$

*which is "**weighted mid-point**" inequality for double integrals. Thus, (3.17) is a heigher degree weighted Ostrowski type inequality for $\|\|_1$ −norm.*

4. Some inequalities for co-ordinated convex mappings

For convenience, we give the following notations used to simplify the details of some results given in this section;

$$A_n(x) = (b-a)\,\frac{(x-a)^{n+1}}{n+1} + \frac{(b-x)^{n+2}-(x-a)^{n+2}}{n+2},$$

$$B_n(x) = (b-a)\,\frac{(b-x)^{n+1}}{n+1} + \frac{(x-a)^{n+2}-(b-x)^{n+2}}{n+2},$$

$$C_m(y) = (d-c)\,\frac{(y-c)^{m+1}}{m+1} + \frac{(d-y)^{m+2}-(y-c)^{m+2}}{m+2}$$

and

$$D_m(y) = (d-c)\,\frac{(d-y)^{m+1}}{m+1} + \frac{(y-c)^{m+2}-(d-y)^{m+2}}{m+2}.$$

We start with the following result.

Theorem 7. *Suppose that all the assumptions of Lemma 1 hold. If* $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|$ *is a convex function on the co-ordinates on* Δ*, then the following inequality holds:*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l} - \sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right.\\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds + \int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
\leq\;& \frac{\|g\|_{[a,b],\infty}}{(b-a)\,n!}\frac{\|h\|_{[c,d]\infty}}{(d-c)\,m!}\left\{\left|\frac{\partial^{n+m}f(a,c)}{\partial t^n\partial s^m}\right|A_n(x)C_m(y) + \left|\frac{\partial^{n+m}f(a,d)}{\partial t^n\partial s^m}\right|A_n(x)D_m(y)\right.\\
&\left.+\left|\frac{\partial^{n+m}f(b,c)}{\partial t^n\partial s^m}\right|B_n(x)C_m(y) + \left|\frac{\partial^{n+m}f(b,d)}{\partial t^n\partial s^m}\right|B_n(x)D_m(y)\right\}
\end{aligned}\tag{4.1}$$

for all $(x,y)\in[a,b]\times[c,d]$*, where* $\|g\|_{[a,b],\infty} = \sup_{u\in[a,b]}|g(u)|$ *and* $\|h\|_{[c,d]\infty} = \sup_{u\in[c,d]}|h(u)|$.

Proof. If we take absolute value of both sides of the equality (2.1), we find that

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l} - \sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right.\\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds + \int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
\leq\;& \int_a^b\int_c^d |P_{n-1}(x,t)|\,|Q_{m-1}(y,s)|\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|dsdt.
\end{aligned}$$

Since $\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}\right|$ is a convex function on the co-ordinates on Δ, we have

$$\left|\frac{\partial^{n+m}}{\partial t^n \partial s^m} f\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b, \frac{d-s}{d-c}c+\frac{s-a}{d-c}d\right)\right| \tag{4.2}$$
$$\leq \frac{b-t}{b-a}\frac{d-s}{d-c}\left|\frac{\partial^{n+m}f(a,c)}{\partial t^n \partial s^m}\right| + \frac{b-t}{b-a}\frac{s-c}{d-c}\left|\frac{\partial^{n+m}f(a,d)}{\partial t^n \partial s^m}\right|$$
$$+\frac{t-a}{b-a}\frac{d-s}{d-c}\left|\frac{\partial^{n+m}f(b,c)}{\partial t^n \partial s^m}\right| + \frac{t-a}{b-a}\frac{s-c}{d-c}\left|\frac{\partial^{n+m}f(b,d)}{\partial t^n \partial s^m}\right|.$$

Utilising the inequality (4.2), we can write

$$\int_a^b \int_c^d |P_{n-1}(x,t)|\,|Q_{m-1}(y,s)| \left|\frac{\partial^{n+m}f(t,s)}{\partial t^n \partial s^m}\right| dsdt \tag{4.3}$$
$$\leq \frac{1}{(b-a)(d-c)}$$
$$\times \left\{ \left|\frac{\partial^{n+m}f(a,c)}{\partial t^n \partial s^m}\right| \int_a^b \int_c^d (b-t)\,|P_{n-1}(x,t)|\,(d-s)\,|Q_{m-1}(y,s)|\, dsdt \right.$$
$$+\left|\frac{\partial^{n+m}f(a,d)}{\partial t^n \partial s^m}\right| \int_a^b \int_c^d (b-t)\,|P_{n-1}(x,t)|\,(s-c)\,|Q_{m-1}(y,s)|\, dsdt$$
$$+\left|\frac{\partial^{n+m}f(b,c)}{\partial t^n \partial s^m}\right| \int_a^b \int_c^d (t-a)\,|P_{n-1}(x,t)|\,(d-s)\,|Q_{m-1}(y,s)|\, dsdt$$
$$\left. +\left|\frac{\partial^{n+m}f(b,d)}{\partial t^n \partial s^m}\right| \int_a^b \int_c^d (t-a)\,|P_{n-1}(x,t)|\,(s-c)\,|Q_{m-1}(y,s)|\, dsdt \right\}.$$

If we calculate the above four double inetgrals and also substitute the results in (4.3), because of $\|g\|_{[a,x],\infty}$, $\|g\|_{[x,b],\infty} \leq \|g\|_{[a,b],\infty}$ and $\|h\|_{[c,y]\infty}$, $\|h\|_{[y,d]\infty} \leq \|h\|_{[c,d]\infty}$, we obtain required inequality (4.1) which completes the proof. $\square$

Remark 12. *Under the same assumptions of Theorem 7 with $n = m = 1$, then the following inequality holds:*

$$\left| M_0(x)M_0(y)f(x,y) - M_0(y)\int_a^b g(t)f(t,y)dt \right. \tag{4.4}$$
$$\left. -M_0(x)\int_c^d h(s)f(x,s)ds + \int_a^b \int_c^d g(t)h(s)f(t,s)dsdt \right|$$
$$\leq \frac{\|g\|_{[a,b],\infty}}{(b-a)} \frac{\|h\|_{[c,d]\infty}}{(d-c)} \left\{ \left|\frac{\partial^2 f(a,c)}{\partial t \partial s}\right| A_1(x)C_1(y) + \left|\frac{\partial^2 f(a,d)}{\partial t \partial s}\right| A_1(x)D_1(y) \right.$$
$$\left. + \left|\frac{\partial^2 f(b,c)}{\partial t \partial s}\right| B_1(x)C_1(y) + \left|\frac{\partial^2 f(b,d)}{\partial t \partial s}\right| B_1(x)D_1(y) \right\}$$

which was given by Erden and Sarikaya in [11] *(in case of $\lambda = 0$).*

Remark 13. *If we take $g(u)=h(u)=1$ in (4.4), then we get*

$$\begin{aligned}
(4.5)\qquad &\left|(b-a)(d-c)f(x,y)-(d-c)\int_a^b f(t,y)dt\right.\\
&\left.-(b-a)\int_c^d f(x,s)ds+\int_a^b\int_c^d f(t,s)dsdt\right|\\
\leq\ &\frac{1}{(b-a)(d-c)}\left\{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|A_1(x)C_1(y)+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|A_1(x)D_1(y)\right.\\
&\left.+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|B_1(x)C_1(y)+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|B_1(x)D_1(y)\right\}
\end{aligned}$$

which was given by Erden and Sarıkaya in [13].

Remark 14. *Taking $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (4.5), we get*

$$\begin{aligned}
&\left|(b-a)(d-c)f(\frac{a+b}{2},\frac{c+d}{2})-(d-c)\int_a^b f(t,\frac{c+d}{2})dt\right.\\
&\left.-(b-a)\int_c^d f(\frac{a+b}{2},s)ds+\int_a^b\int_c^d f(t,s)dsdt\right|\\
\leq\ &\frac{(b-a)^2(d-c)^2}{16}\left\{\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|}{4}\right\}
\end{aligned}$$

which was given by Latif and Dragomir in [22].

Corollary 10. *Under the same assumptions of Theorem 7 with $g(u)=h(u)=1$, then we have the inequality*

$$\begin{aligned}
&(4.6)\\
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{X_k(x)}{k!}\frac{Y_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{Y_l(y)}{l!}\int_a^b\frac{\partial^l f(t,y)}{\partial y^l}dt\right.\\
&\left.-\sum_{k=0}^{n-1}\frac{X_k(x)}{k!}\int_c^d\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d f(t,s)dsdt\right|\\
\leq\ &\frac{1}{(b-a)(d-c)}\frac{1}{n!m!}\left\{\left|\frac{\partial^{n+m}f(a,c)}{\partial t^n\partial s^m}\right|A_n(x)C_m(y)+\left|\frac{\partial^{n+m}f(a,d)}{\partial t^n\partial s^m}\right|A_n(x)D_m(y)\right.\\
&\left.+\left|\frac{\partial^{n+m}f(b,c)}{\partial t^n\partial s^m}\right|B_n(x)C_m(y)+\left|\frac{\partial^{n+m}f(b,d)}{\partial t^n\partial s^m}\right|B_n(x)D_m(y)\right\}
\end{aligned}$$

where $X_k(x)$ and $Y_l(y)$ are defined as in (3.5) and (3.6), respectively. This result is a Ostrowski type inequality for mappings whose absolute value of heigher degree partial derivatives are co-ordinated convex.

Corollary 11. *Under the same assumptions of Theorem 7 with* $x = \frac{a+b}{2}$ *and* $y = \frac{c+d}{2}$*, then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(\frac{a+b}{2})}{k!}\frac{M_l(\frac{c+d}{2})}{l!}\frac{\partial^{k+l}f(\frac{a+b}{2},\frac{c+d}{2})}{\partial x^k\partial y^l}\right.\\
&-\sum_{l=0}^{m-1}\frac{M_l(\frac{c+d}{2})}{l!}\int_a^b g(t)\frac{\partial^l f(t,\frac{c+d}{2})}{\partial y^l}dt\\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(\frac{a+b}{2})}{k!}\int_c^d h(s)\frac{\partial^k f(\frac{a+b}{2},s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
\leq\ &\frac{\|g\|_{[a,b],\infty}}{(n+1)!}\frac{\|h\|_{[c,d]\infty}}{(m+1)!}\frac{(b-a)^{n+1}}{2^{n+1}}\frac{(d-c)^{m+1}}{2^{m+1}}\\
&\times\left\{\left|\frac{\partial^{n+m}f(a,c)}{\partial t^n\partial s^m}\right|+\left|\frac{\partial^{n+m}f(a,d)}{\partial t^n\partial s^m}\right|+\left|\frac{\partial^{n+m}f(b,c)}{\partial t^n\partial s^m}\right|+\left|\frac{\partial^{n+m}f(b,d)}{\partial t^n\partial s^m}\right|\right\}
\end{aligned}$$

*which is "**weighted mid-point**" inequality for functions whoose absolute value of heigher degree partial derivatives are co-ordinated convex.*

We establish some weighted integral inequalities by using convexity of $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^q$.

Theorem 8. *Suppose that all the assumptions of Lemma 1 hold. If* $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^q$ *is a convex function on the co-ordinates on* Δ, $\frac{1}{p}+\frac{1}{q}=1$ *and* $q>1$, *then the following inequality holds:*

$$\begin{aligned}
(4.7)\qquad &\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right.\\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
\leq\ &\frac{\|g\|_{[a,b],\infty}}{n!\,(np+1)^{\frac{1}{p}}}\frac{\|h\|_{[c,d]\infty}}{m!\,(mp+1)^{\frac{1}{p}}}(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}\\
&\times\left[(x-a)^{np+1}+(b-x)^{np+1}\right]^{\frac{1}{p}}\left[(y-c)^{mp+1}+(d-y)^{mp+1}\right]^{\frac{1}{p}}\\
&\times\left[\frac{\left|\frac{\partial^{n+m}f(a,c)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}f(a,d)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}f(b,c)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}f(b,d)}{\partial t^n\partial s^m}\right|^q}{4}\right]^{\frac{1}{q}}
\end{aligned}$$

for all $(x,y)\in[a,b]\times[c,d]$, *where* $\|g\|_{[a,b],\infty}=\sup_{u\in[a,b]}|g(u)|$ *and* $\|h\|_{[c,d]\infty}=\sup_{u\in[c,d]}|h(u)|$.

Proof. Taking absolute value of (2.1), from Hölder's inequality, then we get

$$\begin{aligned}
(4.8)\quad & \left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right.\\
& \left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
\leq & \left(\int_a^b\int_c^d |P_{n-1}(x,t)|^p\,|Q_{m-1}(y,s)|^p\,dsdt\right)^{\frac{1}{p}}\left(\int_a^b\int_c^d\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|^q dsdt\right)^{\frac{1}{q}}.
\end{aligned}$$

By similar methods in the proof of Theorem 5, we obtain

$$\begin{aligned}
(4.9)\quad & \left(\int_a^b\int_c^d |P_{n-1}(x,t)|^p\,|Q_{m-1}(y,s)|^p\,dsdt\right)^{\frac{1}{p}}\\
\leq\; & \frac{\|g\|_{[a,b],\infty}}{n!\,(np+1)^{\frac{1}{p}}}\frac{\|h\|_{[c,d]\infty}}{m!\,(mp+1)^{\frac{1}{p}}}\\
& \times\left[(x-a)^{np+1}+(b-x)^{np+1}\right]^{\frac{1}{p}}\left[(y-c)^{mp+1}+(d-y)^{mp+1}\right]^{\frac{1}{p}}.
\end{aligned}$$

Since $\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|^q$ is a convex function on the co-ordinates on Δ, we have

$$\begin{aligned}
(4.10)\quad & \left|\frac{\partial^{n+m}}{\partial t^n\partial s^m}f\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b,\frac{d-s}{d-c}c+\frac{s-a}{d-c}d\right)\right|^q\\
\leq\; & \frac{b-t}{b-a}\frac{d-s}{d-c}\left|\frac{\partial^{n+m}f(a,c)}{\partial t^n\partial s^m}\right|^q+\frac{b-t}{b-a}\frac{s-c}{d-c}\left|\frac{\partial^{n+m}f(a,d)}{\partial t^n\partial s^m}\right|^q\\
& +\frac{t-a}{b-a}\frac{d-s}{d-c}\left|\frac{\partial^{n+m}f(b,c)}{\partial t^n\partial s^m}\right|^q+\frac{t-a}{b-a}\frac{s-c}{d-c}\left|\frac{\partial^{n+m}f(b,d)}{\partial t^n\partial s^m}\right|^q.
\end{aligned}$$

Using the inequality (4.10), it follows that

$$\begin{aligned}
(4.11)\quad & \left(\int_a^b\int_c^d\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|^q dsdt\right)^{\frac{1}{q}}\\
\leq\; & (b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}\\
& \times\left[\frac{\left|\frac{\partial^{n+m}f(a,c)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}f(a,d)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}f(b,c)}{\partial t^n\partial s^m}\right|^q+\left|\frac{\partial^{n+m}f(b,d)}{\partial t^n\partial s^m}\right|^q}{4}\right]^{\frac{1}{q}}.
\end{aligned}$$

Substituting the inequalities (4.9) and (4.11) in (4.8), we deduce the inequality (4.7). Hence, the proof is completed. □

Remark 15. *Under the same assumptions of Theorem 8 with $n = m = 1$, then the following inequality holds:*

$$\begin{aligned}
(4.12)\qquad & \left| M_0(x)M_0(y)f(x,y) - M_0(y)\int_a^b g(t)f(t,y)dt \right. \\
& \left. -M_0(x)\int_c^d h(s)f(x,s)ds + \int_a^b\int_c^d g(t)h(s)f(t,s)dsdt \right| \\
\le\ & \|g\|_{[a,b],\infty}\|h\|_{[c,d]\infty}(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}} \\
& \times\left[\frac{(x-a)^{p+1}+(b-x)^{p+1}}{p+1}\right]^{\frac{1}{p}}\left[\frac{(y-c)^{p+1}+(d-y)^{p+1}}{p+1}\right]^{\frac{1}{p}} \\
& \times\left[\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}
\end{aligned}$$

which was given by Erden and Sarikaya in [11] *(in case of $\lambda = 0$).*

Corollary 12. *Let us substitute $(x,y) = (a,c)$, (a,d), (b,c) and (b,d) in (4.12). Subsequently, if we add the obtained results and use the triangle inequality for the modulus, we get the inequality*

$$\begin{aligned}
(4.13)\ & \left| M_0(x)M_0(y)\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \int_a^b\int_c^d g(t)h(s)f(t,s)dsdt \right. \\
& \left. -\frac{1}{2}\left[M_0(y)\int_a^b g(t)\left[f(t,c)+f(t,d)\right]dt + M_0(x)\int_c^d h(s)\left[f(a,s)+f(b,s)\right]ds\right]\right| \\
\le\ & \|g\|_{[a,b],\infty}\|h\|_{[c,d]\infty}\frac{(b-a)^2(d-c)^2}{4(p+1)^{\frac{1}{p}}} \\
& \times\left[\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}
\end{aligned}$$

which is a weighted Hermite-Hadamard type inequality for double integrals.

Remark 16. *If we take $g(u) = h(u) = 1$ in (4.13), then we have*

$$\begin{aligned}
& \left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right. \\
& \left. -\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left[f(t,c)+f(t,d)\right]dt + \frac{1}{d-c}\int_c^d\left[f(a,s)+f(b,s)\right]ds\right]\right| \\
\le\ & \frac{(b-a)(d-c)}{4(p+1)^{\frac{1}{p}}}\left[\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}
\end{aligned}$$

which was deduced by Sarikaya et al. in [32].

Remark 17. *If we take $g(u)=h(u)=1$ in (4.12), then we get*

$$\begin{aligned}
(4.14)\qquad & \left|(b-a)(d-c)f(x,y)-(d-c)\int_a^b f(t,y)dt\right. \\
& \left.-(b-a)\int_c^d f(x,s)ds+\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\; & (b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}} \\
& \times\left[\frac{(x-a)^{p+1}+(b-x)^{p+1}}{p+1}\right]^{\frac{1}{p}}\left[\frac{(y-c)^{p+1}+(d-y)^{p+1}}{p+1}\right]^{\frac{1}{p}} \\
& \times\left[\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}
\end{aligned}$$

which was given by Erden and Sarıkaya in [13].

Remark 18. *Taking $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (4.14), we get*

$$\begin{aligned}
& \left|(b-a)(d-c)f(\frac{a+b}{2},\frac{c+d}{2})-(d-c)\int_a^b f(t,\frac{c+d}{2})dt\right. \\
& \left.-(b-a)\int_c^d f(\frac{a+b}{2},s)ds+\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\; & \frac{(b-a)^2(d-c)^2}{4(p+1)^{\frac{2}{p}}} \\
& \times\left[\frac{\left|\frac{\partial^2 f(a,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(a,d)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,c)}{\partial t\partial s}\right|^q+\left|\frac{\partial^2 f(b,d)}{\partial t\partial s}\right|^q}{4}\right]^{\frac{1}{q}}
\end{aligned}$$

which was given by Latif and Dragomir in [22].

Similarly, the other reults related to Theorem 8 can be obtained as Corollary 10 and 11.

Theorem 9. *Suppose that all the assumptions of Lemma 1 hold. If $\left|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right|^q$ is a convex function on the co-ordinates on Δ, $\frac{1}{p}+\frac{1}{q}=1$ and $q\geq 1$, then the following* *inequality holds:*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right. \\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right| \\
\leq\ &\frac{1}{\left[(b-a)(d-c)\right]^{\frac{1}{q}}}\frac{\|g\|_{[a,b],\infty}}{n!\,(n+1)^{\frac{1}{p}}}\frac{\|h\|_{[c,d]\infty}}{m!\,(m+1)^{\frac{1}{p}}} \\
&\times\left[(b-x)^{n+1}+(x-a)^{n+1}\right]^{\frac{1}{p}}\left[(d-y)^{m+1}+(y-c)^{m+1}\right]^{\frac{1}{p}} \\
&\times\left\{\left|\frac{\partial^{n+m}f(a,c)}{\partial t^n\partial s^m}\right|^q A_n(x)C_m(y)+\left|\frac{\partial^{n+m}f(a,d)}{\partial t^n\partial s^m}\right|^q A_n(x)D_m(y)\right. \\
&\left.+\left|\frac{\partial^{n+m}f(b,c)}{\partial t^n\partial s^m}\right|^q B_n(x)C_m(y)+\left|\frac{\partial^{n+m}f(b,d)}{\partial t^n\partial s^m}\right|^q B_n(x)D_m(y)\right\}^{\frac{1}{q}}
\end{aligned}\tag{4.15}$$

for all $(x,y)\in[a,b]\times[c,d]$, *where* $\|g\|_{[a,b],\infty}=\sup_{u\in[a,b]}|g(u)|$ *and* $\|h\|_{[c,d]\infty}=\sup_{u\in[c,d]}|h(u)|$.

Proof. We take absolute value of (2.1). Because of $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p}+\frac{1}{q}$ can be written instead of 1. Using Hölder's inequality, we find that

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right. \\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right| \\
\leq\ &\left(\int_a^b\int_c^d\left|P_{n-1}(x,t)\right|\left|Q_{m-1}(y,s)\right|dsdt\right)^{\frac{1}{p}} \\
&\times\left(\int_a^b\int_c^d\left|P_{n-1}(x,t)\right|\left|Q_{m-1}(y,s)\right|\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|^q dsdt\right)^{\frac{1}{q}}.
\end{aligned}\tag{4.16}$$

By simple calculations, we write

$$\begin{aligned}
&\int_a^b\int_c^d\left|P_{n-1}(x,t)\right|\left|Q_{m-1}(y,s)\right|dsdt \\
\leq\ &\frac{\|g\|_{[a,b],\infty}}{(n+1)!}\frac{\|h\|_{[c,d]\infty}}{(m+1)!} \\
&\times\left[(b-x)^{n+1}+(x-a)^{n+1}\right]\left[(d-y)^{m+1}+(y-c)^{m+1}\right].
\end{aligned}\tag{4.17}$$

By similar methods in the proof of Theorem 7, from (4.10), we obtain

$$\begin{aligned}
&\int_a^b \int_c^d |P_{n-1}(x,t)|\,|Q_{m-1}(y,s)| \left|\frac{\partial^{n+m} f(t,s)}{\partial t^n \partial s^m}\right|^q ds dt \\
\leq\ & \frac{\|g\|_{[a,b],\infty}}{(b-a)\,n!} \frac{\|h\|_{[c,d]\infty}}{(d-c)\,m!} \left\{ \left|\frac{\partial^{n+m} f(a,c)}{\partial t^n \partial s^m}\right|^q A_n(x)C_m(y) + \left|\frac{\partial^{n+m} f(a,d)}{\partial t^n \partial s^m}\right|^q A_n(x)D_m(y) \right. \\
&\left. + \left|\frac{\partial^{n+m} f(b,c)}{\partial t^n \partial s^m}\right|^q B_n(x)C_m(y) + \left|\frac{\partial^{n+m} f(b,d)}{\partial t^n \partial s^m}\right|^q B_n(x)D_m(y) \right\}.
\end{aligned} \tag{4.18}$$

Substituting the inequalities (4.17) and (4.18) in (4.16), we easily deduce required inequality (4.15) which completes the proof. □

Remark 19. *In case $(p,q) = (\infty, 1)$, if we take limit as $p \to \infty$ in Theorem 9, then the inequality (4.15) becomes the inequality (4.1). Thus, we obtain all of the results which are similar to Theorem 7.*

5. Applications to Cubature Formulae

We now deal with applications of the integral inequalities developed in the previous section, to obtain estimates of cubature formula which, it turns out have a markedly smaller error than that which may be obtained by the classical results. Thus the following applications in numerical integration is naturel to be considered.

Let $I_\nu : a = x_0 < x_1 < ... < x_{\nu-1} < x_\nu = b$ and $J_\mu : c = y_0 < y_1 < ... < y_{\mu-1} < y_\mu = d$ be divisions of the intervals $[a,b]$ and $[c,d]$, $\xi_i \in [x_i, x_{i+1}]$ and $\eta_j \in [y_j, y_{j+1}]$ with $(i = 0,...,\nu-1;\ j = 0,...,\mu-1)$. Consider the equivalent

$$\begin{aligned}
S(f, I_\nu, J_\mu, \xi, \eta) \ =\ & \sum_{l=0}^{m-1} \sum_{i=0}^{\nu-1} \sum_{j=0}^{\mu-1} \frac{M_l^{(j)}(\eta_j)}{l!} \int_{x_i}^{x_{i+1}} g(t) \frac{\partial^l f(t,\eta_j)}{\partial y^l} dt \\
& + \sum_{k=0}^{n-1} \sum_{i=0}^{\nu-1} \sum_{j=0}^{\mu-1} \frac{M_k^{(i)}(\xi_i)}{k!} \int_{y_j}^{y_{j+1}} h(s) \frac{\partial^k f(\xi_i,s)}{\partial x^k} ds \\
& - \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \sum_{i=0}^{\nu-1} \sum_{j=0}^{\mu-1} \frac{M_k^{(i)}(\xi_i)}{k!} \frac{M_l^{(j)}(\eta_j)}{l!} \frac{\partial^{k+l} f(\xi_i,\eta_j)}{\partial x^k \partial y^l}
\end{aligned} \tag{5.1}$$

where $M_k^{(i)}(\xi_i)$ and $M_l^{(j)}(\eta_j)$ are defined by

$$M_k^{(i)}(\xi_i) = \int_{x_i}^{x_{i+1}} (u-\xi_i)^k g(u)\,du, \quad k = 0,1,2,...;$$

$$M_l^{(j)}(\eta_j) = \int_{y_j}^{y_{j+1}} (u-\eta_j)^l h(u)\,du, \quad l = 0,1,2,...$$

Theorem 10. *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a continuous on Δ such that $\frac{\partial^{n+m} f}{\partial t^n \partial s^m}$ exist on $(a,b) \times (c,d)$ and assume that the functions $g : [a,b] \to [0,\infty)$ and h :*

$[c,d]\to[0,\infty)$ are integrable. If $\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_\infty(\Delta)$, then we have the representation

$$\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt=S(f,I_\nu,J_\mu,\xi,\eta)+R(f,I_\nu,J_\mu,\xi,\eta)$$

where $S(f,I_\nu,J_\mu,\xi,\eta)$ is defined as in (5.1) and the remainder term satisfies the estimations:

$$\begin{aligned}(5.2)\quad &|R(f,I_n,J_m,\xi,\eta)|\\ \leq\ & \left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty\frac{\|g\|_{[a,b],\infty}}{(n+1)!}\frac{\|h\|_{[c,d]\infty}}{(m+1)!}\\ &\times\sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1}\left[(x_{i+1}-\xi_i)^{n+1}+(\xi_i-x_i)^{n+1}\right]\left[\left(y_{j+1}-\eta_j\right)^{m+1}+\left(\eta_j-y_j\right)^{m+1}\right]\end{aligned}$$

for all $(\xi_i,\eta_j)\in[x_i,x_{i+1}]\times[y_j,y_{j+1}]$ with $(i=0,...,\nu-1;\ j=0,...,\mu-1)$, where $\|g\|_{[x_i,x_{i+1}],\infty}=\sup_{u\in[x_i,x_{i+1}]}|g(u)|$, $\|h\|_{[y_j,y_{j+1}],\infty}=\sup_{u\in[y_j,y_{j+1}]}|h(u)|$ and

$$\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty=\sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|<\infty.$$

Proof. Applying Theorem 4 on the interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, $(i=0,...,\nu-1;\ j=0,...,\mu-1)$, we obtain

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k^{(i)}(\xi_i)}{k!}\frac{M_l^{(j)}(\eta_j)}{l!}\frac{\partial^{k+l}f(\xi_i,\eta_j)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l^{(j)}(\eta_j)}{l!}\int_{x_i}^{x_{i+1}}g(t)\frac{\partial^l f(t,\eta_j)}{\partial y^l}dt\right.\\ &\left.-\sum_{k=0}^{n-1}\frac{M_k^{(i)}(\xi_i)}{k!}\int_{y_j}^{y_{j+1}}h(s)\frac{\partial^k f(\xi_i,s)}{\partial x^k}ds+\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}h(s)g(t)f(t,s)dsdt\right|\\ \leq\ & \left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty\frac{\|g\|_{[x_i,x_{i+1}],\infty}}{(n+1)!}\frac{\|h\|_{[y_j,y_{j+1}],\infty}}{(m+1)!}\\ &\times\left[(x_{i+1}-\xi_i)^{n+1}+(\xi_i-x_i)^{n+1}\right]\left[\left(y_{j+1}-\eta_j\right)^{m+1}+\left(\eta_j-y_j\right)^{m+1}\right]\end{aligned}$$

for all $i=0,...,\nu-1;\ j=0,...,\mu-1$.

Summing over i from 0 to $\nu-1$ and over j from 0 to $\mu-1$ using the generalized triangle inequality we deduce the estimations (5.2). □

Remark 20. *If we take $g(u)=h(u)=1$ and $m=n=1$ in Theorem 10, then we recapture the cubature formula*

$$\int_a^b\int_c^d f(t,s)dsdt=S(f,I_\nu,J_\mu,\xi,\eta)+R(f,I_\nu,J_\mu,\xi,\eta)$$

where the remainder $R(f, I_\nu, J_\mu, \xi, \eta)$ *satisfies the estimation:*

$$\begin{aligned} &|R(f,I_n,J_m,\xi,\eta)| \\ \leq\ & \left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty \sum_{i=0}^{\nu-1}\sum_{j=0}^{\mu-1}\left\{\left[\frac{(x_{i+1}-x_i)^2}{4}+\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\right. \\ & \left.\times\left[\frac{(y_{i+1}-y_i)^2}{4}+\left(\eta_j-\frac{y_j+y_{j+1}}{2}\right)^2\right]\right\} \end{aligned} \tag{5.3}$$

which was given by Barnett and Dragomir in [2].

Remark 21. *if we consider the inequality (3.3), then we recapture the midpoint cubature formula*

$$\int_a^b\int_c^d f(t,s)dsdt = S_M(f,I_\nu,J_\mu)+R_M(f,I_\nu,J_\mu)$$

where the remainder $R_M(f, I_\nu, J_\mu)$ *satisfies the estimation:*

$$|R_M(f,I_n,J_m)| \leq \frac{1}{16}\left\|\frac{\partial^2 f}{\partial t\partial s}\right\|_\infty \sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\sum_{j=0}^{m-1}(y_{i+1}-y_i)^2$$

which was proved by Barnett and Dragomir in [2].

A similar process can be undertaken in producing composite rules if we use the other results given in previous sections, but we omit the details.

6. Some applications for the moments

Distribution functions and density functions provide complete descriptions of the distribution of probality for a given random variable. However, they do not allow us to easily make comparisons between two different distributions. The set of moments that uniquely characterizes the distribution under reasonable conditions are useful in making comparisons. Knowing the probability function, we can determine moments if they exist. Applying the mathematical inequalities, some estimations for the moments of random variables were recently studied (see, [3], [5], [18], [19], [29]).

Set X to denote a random variable whose probability density function is $g : [a,b] \to [0,\infty)$ on the interval of real numbers I $(a, b \in I,\ a < b)$ and Y to denote a random variable whose probability density function is $h : [c,d] \to \mathbb{R}$ on the interval of real numbers I $(c, d \in I,\ c < d)$. Denoted by $M_r(x)$ and $M_r(y)$ the r.th central moment of the random variable X and Y, respectively, defined as

$$M_r(x) = \int_a^b (u-E(x))^r\, g(u)\, du, \quad r = 0,1,2,...$$

and

$$M_r(y) = \int_c^d (u-E(y))^r\, h(u)\, du, \quad r = 0,1,2,...$$

where $E(x)$ and $E(y)$ are the mean of the random variables X and Y, respectively. It may be noted that $M_0(x) = M_0(y) = 1$, $M_1(x) = M_1(y) = 0$, $M_2(x) = \sigma^2(X)$ and $M_2(y) = \sigma^2(Y)$ where $\sigma^2(X)$ and $\sigma^2(Y)$ are the variance of the random variables X and Y, respectively.

Now, we reconsider the identity (2.1) by changing conditions given in Lemma 1. Herewith, we deduce an identity involving r.th moment.

Lemma 2. *Let $f:[a,b]\times[c,d]=:\Delta\subset\mathbb{R}^2\rightarrow\mathbb{R}$ be a continuous function such that the partial derivatives $\frac{\partial^{k+l}f(t,s)}{\partial t^k\partial s^l}$, $k=0,1,2,...,n-1$, $l=0,1,2,...,m-1$ exists and are continuous on Δ, and let X and Y be random variables whose p.d.f. are $g:[a,b]\rightarrow[0,\infty)$ and $h:[c,d]\rightarrow[0,\infty)$, respectively. Then, for all $(x,y)\in[a,b]\times[c,d]$, we have the identity*

$$\begin{aligned}
&\int_a^b\int_c^d P_{n-1}(x,t)\,Q_{m-1}(y,s)\,\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}dsdt\\
=\;&\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\\
&-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt
\end{aligned}$$

where $n,m\in\mathbb{N}\backslash\{0\}$, $M_k(x)$ and $M_l(y)$ are the k^{th} moment, and $P_{n-1}(x,t)$ and $Q_{m-1}(y,s)$ are defined as in Lemma 1.

Theorem 11. *Suppose that all the assumptions of Lemma 2 hold. If $\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\in L_\infty(\Delta)$, then we have the inequality*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right.\\
&\left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right|\\
\leq\;&\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty\left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^n\left(\frac{d-c}{2}+\left|y-\frac{c+d}{2}\right|\right)^m
\end{aligned}\tag{6.1}$$

for all $(x,y)\in[a,b]\times[c,d]$, where

$$\left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty=\sup_{(t,s)\in(a,b)\times(c,d)}\left|\frac{\partial^{n+m}f(t,s)}{\partial t^n\partial s^m}\right|<\infty.$$

Proof. By similar methods in the proof of Theorem 3, we obtain

$$
\begin{aligned}
(6.2)\qquad & \left|\sum_{k=0}^{n-1}\sum_{l=0}^{m-1}\frac{M_k(x)}{k!}\frac{M_l(y)}{l!}\frac{\partial^{k+l}f(x,y)}{\partial x^k\partial y^l}-\sum_{l=0}^{m-1}\frac{M_l(y)}{l!}\int_a^b g(t)\frac{\partial^l f(t,y)}{\partial y^l}dt\right. \\
& \left.-\sum_{k=0}^{n-1}\frac{M_k(x)}{k!}\int_c^d h(s)\frac{\partial^k f(x,s)}{\partial x^k}ds+\int_a^b\int_c^d h(s)g(t)f(t,s)dsdt\right| \\
\leq\ & \left\|\frac{\partial^{n+m}f}{\partial t^n\partial s^m}\right\|_\infty\left[\int_a^x\frac{(x-u)^n}{n!}g(u)du+\int_x^b\frac{(u-x)^n}{n!}g(u)dudt\right] \\
& \times\left[\int_c^y\frac{(y-u)^m}{m!}h(u)du+\int_y^d\frac{(u-y)^m}{m!}h(u)du\right].
\end{aligned}
$$

We observe that

$$
\begin{aligned}
& \int_a^x\frac{(x-u)^n}{n!}g(u)du+\int_x^b\frac{(u-x)^n}{n!}g(u)du \\
\leq\ & \frac{1}{n!}\left[\sup_{u\in[a,x]}(x-u)^n\int_a^x g(u)du+\sup_{u\in[x,b]}(u-x)^n\int_x^b g(u)du\right] \\
=\ & \left[(x-a)^n\int_a^x g(u)du+(b-x)^n\int_x^b g(u)du\right] \\
\leq\ & \max\left\{(x-a)^n,(b-x)^n\right\}\int_a^b g(u)du.
\end{aligned}
$$

Because g is a p.d.f., $\int_a^b g(u)du=1$. Using the identity

$$
\max\{X,Y\}=\frac{X+Y}{2}+\left|\frac{Y-X}{2}\right|,
$$

we get

$$
\max\left\{(x-a)^n,(b-x)^n\right\}\int_a^b g(u)du=\left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^n.
$$

Similarly, if we examine the other integral in (6.2), we obtain desired inequality (6.1). Thus, the proof is completed. $\square$

Remark 22. *With the assumptions of theorem 11, then we have the represantation*

$$\begin{aligned}\left| f(x,y) - \int_a^b g(t)f(t,y)dt \right.\\ \left. - \int_c^d h(s)f(x,s)ds + \int_a^b\int_c^d g(t)h(s)f(t,s)dsdt \right| \\ \leq \left\| \frac{\partial^2 f}{\partial t \partial s} \right\|_\infty \left(\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right) \left(\frac{d-c}{2} + \left| y - \frac{c+d}{2} \right| \right).\end{aligned} \tag{6.3}$$

Proof. If we take $n = m = 1$ in (6.1), then we get the inequality (6.3). □

Similarly, using the other integrals in section 3 and section 4, we obtain some results involving r.th central moment of the random variable X and Y.

Theorem 12. *Let $f : [a,b] \times [c,d] =: \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that the partial derivatives $\frac{\partial^{k+l} f(t,s)}{\partial t^k \partial s^l}$, $k = 0,1$, $l = 0,1,2$ exists and are continuous on Δ, and let X and Y be random variables whose p.d.f. are $g : [a,b] \to [0,\infty)$ and $h : [c,d] \to [0,\infty)$, respectively. Then we have*

$$\begin{aligned}\left| f(x,y) - \int_a^b g(t)f(t,y)dt \right.\\ \left. - \int_c^d h(s)f(x,s)ds + \int_a^b\int_c^d g(t)h(s)f(t,s)dsdt \right| \\ \leq \left\| \frac{\partial^4 f}{\partial t^2 \partial s^2} \right\|_\infty \sigma^2(X)\sigma^2(Y)\end{aligned} \tag{6.4}$$

for all $(x,y) \in [a,b] \times [c,d]$, where

$$\left\| \frac{\partial^4 f}{\partial t^2 \partial s^2} \right\|_\infty = \sup_{(t,s)\in(a,b)\times(c,d)} \left| \frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} \right| < \infty.$$

Proof. If we take $n = m = 2$ in (6.2), we obtain desired inequality (6.4). □

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