Received 31/05/17

POMPEIU TYPE INEQUALITIES USING CONFORMABLE FRACTIONAL CALCULUS AND ITS APPLICATIONS

SAMET ERDEN AND MEHMET ZEKI SARIKAYA

Abstract. We establish Pompeiu's mean value theorem for α-fractional differentiable mappings. Then, some Pompeiu type inequalities using conformable fractional integrals are obtained. In addition, the weighted versions of this Pompeiu type inequalities are presented. Finally, some applications of obtained inequalities in numerical integration and for special means are given.

1. Introduction

In 1938, the classical integral inequality established by Ostrowski [13] as follows:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'}:(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, the inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

Inequality (1.1) has wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some midpoint, trapezoid and Simpson rules and quadrature rules, etc. Hence inequality (1.1) has attracted considerable attention and interest from mathematicans and researchers.

In 1946, Pompeiu [15] derived a variant of Lagrange's mean value theorem, now known as *Pompeiu's mean value theorem.*

Theorem 2. *For every real valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1\neq x_2$ in $[a,b]$, there exist a point ξ between x_1 and x_2 such that*

$$\frac{x_1f(x_2)-x_2f(x_1)}{x_1-x_2}=f(\xi)-\xi f'(\xi).$$

The following Pompeiu type inequality is proved by Dragomir in [6].

2000 *Mathematics Subject Classification.* 26D10, 26D15, 26A33.

Key words and phrases. Pompeiu's mean value theorem, conformable fractional integral, fractal space, numerical integration, special means.

RGMIA Res. Rep. Coll. 20 (2017), Art. 63, 12 pp

Theorem 3. *Let $f:[a,b]\rightarrow\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $[a,b]$ not containing 0. Then for any $x\in[a,b]$, we have the inequality*

$$\begin{aligned}&\left|\frac{a+b}{2}\frac{f(x)}{x}+\frac{1}{b-a}\int_a^b f(t)dt\right|\\ \leq\ &\frac{b-a}{|x|}\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right]\|f-lf'\|_\infty .\end{aligned}$$

where $l(t)=t$ for all $t\in[a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In recent years, many authors have studied the Pompeiu type integral inequalities. For example, authors presented some Ostrowski type inequalities by using mean value theorem in [4] and [16]. In [14], Pečarić and Ungar proved an inequality of Ostrowski type for p-norm, generalizing a result of Dragomir [6]. Dragomir provided some power Pompeiu's type and exponential Pompeiu's type inequalities for complex-valued absolutely continuous functions in [8] and [9]. Also, Dragomir gave generalizations of Pompeiu's inequality and they are applied to obtain some new Ostrowski type results in [7]. In [10], Erden and Sarikaya estblished generalized Pompeiu mean value theorem and Pompeiu type inequalities for local fractional calculus. In [21], Sarikaya obtained some new Pompeiu type inequalities for twice differentiable mappings. Researchers examined some new Ostrowski and Grüss type inequalities via variant Pompeiu' mean value theorem for two variable functions in [18]-[20].

2. Definitions and properties of conformable fractional derivative and integral

Recently, the authors introduced a new simple well-behaved definition of the fractional derivative called the "conformable fractional derivative" depending just on the basic limit definition of the derivative in [12]. Namely, for given a function $f:[0,\infty)\rightarrow\mathbb{R}$ the conformable fractional derivative of order $0<\alpha\leq 1$ of f at $t>0$ was defined by

$$D_\alpha(f)(t)=\lim_{\epsilon\rightarrow 0}\frac{f\left(t+\epsilon t^{1-\alpha}\right)-f(t)}{\epsilon} \tag{2.1}$$

If f is $\alpha-$differentiable in some $(0,a)$, $\alpha>0$, $\lim\limits_{t\rightarrow 0^+}f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0)=\lim_{t\rightarrow 0^+}f^{(\alpha)}(t). \tag{2.2}$$

Also, note that if f is differentiable, then

$$D_\alpha(f)(t)=t^{1-\alpha}f'(t) \tag{2.3}$$

where

$$f'(t)=\lim_{\epsilon\rightarrow 0}\frac{f(t+\epsilon)-f(t)}{\epsilon}.$$

We can write $f^{(\alpha)}(t)$ for $D_{\alpha}(f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is $\alpha-$differentiable.

In order to prove main result, we use the mean value theorem for conformable fractional derivatives. This theorem is established by Iyiola and Nwaeze [11] as the following.

Theorem 4 (Mean value theorem for conformable fractional differentiable functions)**.** *Let $\alpha \in (0,1]$ and $f:[a,b] \to \mathbb{R}$ be a continuous on $[a,b]$ and an α-fractional differentiable mapping on (a,b) with $0 \leq a < b$. Then, there exists $c \in (a,b)$, such that*

$$D_{\alpha}(f)(c) = \frac{f(b)-f(a)}{\frac{b^{\alpha}}{\alpha}-\frac{a^{\alpha}}{\alpha}}.$$

The following definitions and theorems related to conformable fractional derivative and integral were referred in [1]-[3], [5], [11] amd [12].

Theorem 5. *Let $\alpha \in (0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then*

i. $D_{\alpha}(af+bg) = aD_{\alpha}(f)+bD_{\alpha}(g)$, for all $a,b \in \mathbb{R}$,

ii. $D_{\alpha}(\lambda) = 0$, for all constant functions $f(t)=\lambda$,

iii. $D_{\alpha}(fg) = fD_{\alpha}(g)+gD_{\alpha}(f)$,

iv. $D_{\alpha}\left(\frac{f}{g}\right) = \frac{fD_{\alpha}(g)-gD_{\alpha}(f)}{g^2}$.

Definition 1 (Conformable fractional integral)**.** *Let $\alpha \in (0,1]$ and $0 \leq a < b$. A function $f:[a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)\,d_{\alpha}x := \int_a^b f(x)\,x^{\alpha-1}dx \tag{2.4}$$

exists and is finite.

Remark 1.

$$I_{\alpha}^{a}(f)(t) = I_1^{a}\left(t^{\alpha-1}f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

Theorem 6. *Let $f:(a,b) \to \mathbb{R}$ be differentiable and $0<\alpha\leq 1$. Then, for all $t>a$ we have*

$$I_{\alpha}^{a}D_{\alpha}^{a}f(t) = f(t)-f(a). \tag{2.5}$$

Theorem 7. ***(Integration by parts)*** *Let $f,g:[a,b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\int_a^b f(x)\,D_{\alpha}^{a}(g)(x)\,d_{\alpha}x = fg|_a^b - \int_a^b g(x)\,D_{\alpha}^{a}(f)(x)\,d_{\alpha}x. \tag{2.6}$$

Theorem 8. *Assume that $f:[a,\infty) \to \mathbb{R}$ such that $f^{(n)}(t)$ is continuous and $\alpha \in (n,n+1]$. Then, for all $t>a$ we have*

$$D_{\alpha}^{a}I_{\alpha}^{a}f(t) = f(t).$$

Theorem 9. *Let $\alpha \in (0,1]$ and $f:[a,b]\rightarrow \mathbb{R}$ be a continuous on $[a,b]$ with $0\leq a<b$. Then,*

$$\left|I_{\alpha}^{a}\left(f\right)\left(x\right)\right| \leq I_{\alpha}^{a}\left|f\right|\left(x\right).$$

In[5], Anderson prove Ostrowski's α-fractional inequality using a Motgomery identity as follows:

Theorem 10. *Let $a,b,s,t\in\mathbb{R}$ with $0\leq a<b$, and $f:[a,b]\rightarrow\mathbb{R}$ be α-fractional differentiable for $\alpha\in(0,1]$. Then,*

$$\left|f(t)-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b}f(t)d_{\alpha}t\right| \leq \frac{M}{2\alpha\left(b^{\alpha}-a^{\alpha}\right)}\left[\left(t^{\alpha}-a^{\alpha}\right)^{2}+\left(b^{\alpha}-t^{\alpha}\right)\right]$$

where

$$M=\sup_{t\in(a,b)}\left|D_{\alpha}f(t)\right|<\infty.$$

In this study, Pompeiu's mean value theorem for conformable fractional derivatives is obtained. Then, we present pompeiu's type inequalities involving conformable fractional integrals with applications Ostrowski's inequalities. In addition, some applications of obtained inequalities in numerical integration are given. Finally, some special means for conformable calculus are introduced and some applications of given inequalities for these special means are deduced.

3. Main Results

We prove Pompeiu's mean value theorem for conformable fractional differentiable functions.

Theorem 11. *Let $\alpha\in(0,1]$ and $f:[a,b]\subseteq\mathbb{R}\rightarrow\mathbb{R}$ be an α-fractional differentiable mapping on (a,b), with $0<a<b$ and for all pairs $x_1\neq x_2$ in $[a,b]$, there exist a point ξ in (x_1,x_2) such that the following equality holds:*

$$\frac{x_1^{\alpha}f(x_2)-x_2^{\alpha}f(x_1)}{\frac{x_1^{\alpha}}{\alpha}-\frac{x_2^{\alpha}}{\alpha}}=\alpha f(\xi)-\xi^{2-\alpha}D_{\alpha}(f)(\xi). \tag{3.1}$$

Proof. Define function F on $\left[\frac{1}{b},\frac{1}{a}\right]$ by

$$F\left(t\right)=t^{\alpha}f(\frac{1}{t}). \tag{3.2}$$

Using the third item of Theorem 5, we get

$$D_{\alpha}(F)\left(t\right)=\alpha f(\frac{1}{t})-\frac{1}{t^{2-\alpha}}D_{\alpha}(f)(\frac{1}{t}). \tag{3.3}$$

In addition, applying the mean value theorem given for conformable fractional differentiable functions to F on the interval $[x,y]\subset\left[\frac{1}{b},\frac{1}{a}\right]$, we obtain

$$\frac{F(x)-F(y)}{\frac{x^{\alpha}}{\alpha}-\frac{y^{\alpha}}{\alpha}}=D_{\alpha}(F)\left(c\right) \tag{3.4}$$

for all $c\in(x,y)$.
Now, using (3.2) and (3.3) on (3.4), we obtain

$$\frac{x^{\alpha}f(\frac{1}{x})-y^{\alpha}f(\frac{1}{y})}{\frac{x^{\alpha}}{\alpha}-\frac{y^{\alpha}}{\alpha}}=\alpha f(\frac{1}{c})-\frac{1}{c^{2-\alpha}}D_{\alpha}(f)(\frac{1}{c}).$$

Let $x_2 = \frac{1}{x}$, $x_1 = \frac{1}{y}$ and $\xi = \frac{1}{c}$. Then, since $c \in (x, y)$, we have

$$x_1 < \xi < x_2$$

and we write

$$\frac{x_1^\alpha f(x_2) - x_2^\alpha f(x_1)}{\frac{x_1^\alpha}{\alpha} - \frac{x_2^\alpha}{\alpha}} = \alpha f(\xi) - \xi^{2-\alpha} D_\alpha(f)(\xi)$$

which completes the proof. □

Now, we give an Ostrowski type ineqaulity using Pompeiu's mean value theorem given for conformable fractional differentiable functions.

Theorem 12. *Let $\alpha \in (0, 1]$ and $f : [a, b] \to \mathbb{R}$ be a continuous on $[a, b]$ and an α-fractional differentiable mapping on (a, b) with $0 < a < b$. Then, for any $x \in [a, b]$, we have the inequality*

$$\left| \frac{a^\alpha + b^\alpha}{2\alpha} \frac{f(x)}{x^\alpha} - \frac{1}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t \right| \tag{3.5}$$

$$\leq \frac{(b^\alpha - a^\alpha)}{\alpha x^\alpha} \left[\frac{1}{4} + \left(\frac{x^\alpha - \frac{a^\alpha + b^\alpha}{2}}{b^\alpha - a^\alpha} \right)^2 \right] \|f - uD_\alpha(f)\|_\infty$$

where $u(t) = \frac{t^{2-\alpha}}{\alpha}$, $t \in [a, b]$, and $\|f - uD_\alpha(f)\|_\infty = \sup_{\xi \in (a,b)} |f(\xi) - uD_\alpha(f)(\xi)| < \infty$.

Proof. Applying Pompeiu's mean value theorem for conformable fractional differentiable functions, for any $x, t \in [a, b]$, there is ξ beyween x and t such that

$$t^\alpha f(x) - x^\alpha f(t) = \left[f(\xi) - \frac{\xi^{2-\alpha}}{\alpha} D_\alpha(f)(\xi) \right] (t^\alpha - x^\alpha). \tag{3.6}$$

Because of the equality (3.6) and the inequality

$$\begin{aligned} \left| f(\xi) - \frac{\xi^{2-\alpha}}{\alpha} D_\alpha(f)(\xi) \right| &\leq \sup_{\xi \in (a,b)} \left| f(\xi) - \frac{\xi^{2-\alpha}}{\alpha} D_\alpha(f)(\xi) \right| \\ &= \|f - uD_\alpha(f)\|_\infty, \end{aligned}$$

we have the inequality

$$|t^\alpha f(x) - x^\alpha f(t)| \leq \|f - uD_\alpha(f)\|_\infty |t^\alpha - x^\alpha|. \tag{3.7}$$

Integrating both sides of (3.7) with respect to t from a to b for conformable fractional integrals, we get

$$\left| f(x) \int_a^b t^\alpha d_\alpha t - x^\alpha \int_a^b f(t) d_\alpha t \right| \leq \|f - uD_\alpha(f)\|_\infty \int_a^b |t^\alpha - x^\alpha| d_\alpha t \tag{3.8}$$

$$= \|f - uD_\alpha(f)\|_\infty \left(\int_a^x (x^\alpha - t^\alpha) d_\alpha t + \int_x^b (t^\alpha - x^\alpha) d_\alpha t \right).$$

Using the definition 1 and the inequality (3.8), we obtain

$$\begin{aligned}(3.9)\qquad & \left|\frac{b^{2\alpha}-a^{2\alpha}}{2\alpha}f(x)-x^{\alpha}\int_a^b f(t)d_\alpha t\right| \\ \leq\ & \|f-uD_\alpha(f)\|_\infty\left[\frac{(x^\alpha-a^\alpha)^2+(b^\alpha-x^\alpha)^2}{2\alpha}\right].\end{aligned}$$

If we divide the inequality (3.9) with $x^\alpha\left(b^\alpha-a^\alpha\right)$, we easily deduce required result (3.5). □

Corollary 1. *Under the same assumptions of Theorem 12 with* $x^\alpha=\frac{a^\alpha+b^\alpha}{2}$. *Then, we have*

$$\begin{aligned}& \left|\frac{1}{\alpha}f\left(\left(\frac{a^\alpha+b^\alpha}{2}\right)^{\frac{1}{\alpha}}\right)-\frac{1}{b^\alpha-a^\alpha}\int_a^b f(t)d_\alpha t\right| \\ \leq\ & \frac{(b^\alpha-a^\alpha)}{2\alpha\left(a^\alpha+b^\alpha\right)}\|f-uD_\alpha(f)\|_\infty.\end{aligned}$$

We consider now the weighted case of the inequality (3.5).

Theorem 13. *Let* $\alpha\in(0,1]$ *and* $f:[a,b]\to\mathbb{R}$ *be a continuous on* $[a,b]$ *and an* α*-fractional differentiable mapping on* (a,b), *with* $0<a<b$. *If* $w:[a,b]\to\mathbb{R}$ *is nonnegative and* α*-fractional integrable on* $[a,b]$, *then the following inequality given for conformable integrals holds:*

$$\begin{aligned}(3.10)\qquad & \left|\frac{f(x)}{x^\alpha}\int_a^b t^\alpha w(t)d_\alpha t-\int_a^b f(t)w(t)d_\alpha t\right| \\ \leq\ & \|f-uD_\alpha(f)\|_\infty\left[\int_a^x w(t)d_\alpha t-\int_x^b w(t)d_\alpha t\right. \\ & \left.+\frac{1}{x^\alpha}\left(\int_x^b t^\alpha w(t)d_\alpha t-\int_a^x t^\alpha w(t)d_\alpha t\right)\right]\end{aligned}$$

for each $x\in[a,b]$ *and where* $u(t)=\frac{t^{2-\alpha}}{\alpha}$, $t\in[a,b]$, *and* $\|f-uD_\alpha(f)\|_\infty=\sup_{\xi\in(a,b)}|f(\xi)-uD_\alpha(f)(\xi)|<\infty$.

Proof. Multiplying both sides of the inequality (3.7) with $w(t)$ and then integrating both sides of the result with respect to t from a to b for conformable fractional integrals, we have

$$\begin{aligned}
&\left| f(x)\int_a^b t^{\alpha}w(t)d_{\alpha}t - x^{\alpha}\int_a^b f(t)w(t)d_{\alpha}t \right| \\
\leq\ & \left\| f-uD_{\alpha}(f)\right\|_{\infty}\int_a^b w(t)\left| t^{\alpha}-x^{\alpha}\right| d_{\alpha}t \\
=\ & \left\| f-uD_{\alpha}(f)\right\|_{\infty} x^{\alpha}\left(\int_a^x w(t)d_{\alpha}t - \int_x^b w(t)d_{\alpha}t \right) \\
& +\left\| f-uD_{\alpha}(f)\right\|_{\infty}\left(\int_x^b t^{\alpha}w(t)d_{\alpha}t - \int_a^x t^{\alpha}w(t)d_{\alpha}t \right)
\end{aligned}$$

from where we obtain required inequality (3.10). □

We deduce an Ostrowski type ineqaulity using the identity (3.1) in the following theorem.

Theorem 14. *Let $\alpha \in (0,1]$ and $f:[a,b]\to\mathbb{R}$ be an α-fractional differentiable mapping on (a,b) with $0<a<b$. Then, for any $x\in[a,b]$, we have the inequality*

$$\begin{aligned}
(3.11)\qquad & \left| \frac{f(x)}{\alpha x^{\alpha}} - \frac{1}{b^{\alpha}-a^{\alpha}}\int_a^b \frac{f(t)}{t^{\alpha}}d_{\alpha}t \right| \\
\leq\ & \frac{2}{\alpha\left(b^{\alpha}-a^{\alpha}\right)}\left(\ln\frac{x^{\alpha}}{\sqrt{a^{\alpha}b^{\alpha}}} + \frac{\frac{a^{\alpha}+b^{\alpha}}{2}-x^{\alpha}}{x^{\alpha}} \right)\left\| f-uD_{\alpha}(f)\right\|_{\infty}
\end{aligned}$$

where $u(t)=\frac{t^{2-\alpha}}{\alpha}$, $t\in[a,b]$, and $\left\| f-uD_{\alpha}(f)\right\|_{\infty}=\sup_{\xi\in(a,b)}\left| f(\xi)-uD_{\alpha}(f)(\xi)\right| <\infty$.

Proof. If we divide both sides of (3.6) with $t^{\alpha}x^{\alpha}$, we obtain the inequality

$$(3.12)\qquad \left| \frac{f(x)}{x^{\alpha}} - \frac{f(t)}{t^{\alpha}} \right| \leq \left\| f-uD_{\alpha}(f)\right\|_{\infty}\left| \frac{1}{x^{\alpha}}-\frac{1}{t^{\alpha}}\right|$$

for any $t,x\in[a,b]$.

Integrating over $t\in[a,b]$ for conformable fractional integrals, we get

$$\begin{aligned}
(3.13)\qquad & \left| \frac{f(x)}{x^{\alpha}}\frac{b^{\alpha}-a^{\alpha}}{\alpha} - \int_a^b \frac{f(t)}{t^{\alpha}}d_{\alpha}t \right| \\
\leq\ & \int_a^b \left| \frac{f(x)}{x^{\alpha}} - \frac{f(t)}{t^{\alpha}} \right| d_{\alpha}t \\
\leq\ & \left\| f-uD_{\alpha}(f)\right\|_{\infty}\int_a^b \left| \frac{1}{x^{\alpha}}-\frac{1}{t^{\alpha}}\right| d_{\alpha}t.
\end{aligned}$$

We observe that

$$\begin{aligned}
(3.14)\qquad \int_a^b \left|\frac{1}{x^\alpha}-\frac{1}{t^\alpha}\right| d_\alpha t &= \int_a^b \left(\frac{1}{t^\alpha}-\frac{1}{x^\alpha}\right) d_\alpha t + \int_a^b \left(\frac{1}{x^\alpha}-\frac{1}{t^\alpha}\right) d_\alpha t \\
&= \ln\frac{x}{a} - \frac{x^\alpha - a^\alpha}{\alpha x^\alpha} + \frac{b^\alpha - x^\alpha}{\alpha x^\alpha} - \ln\frac{b}{x} \\
&= \frac{2}{\alpha}\left(\ln\frac{x^\alpha}{\sqrt{a^\alpha b^\alpha}} + \frac{\frac{a^\alpha+b^\alpha}{2} - x^\alpha}{x^\alpha}\right)
\end{aligned}$$

for any $x \in [a,b]$. If we sabstitute (3.14) in (3.13), then we deduce desired result (3.11). $\square$

Corollary 2. *Under the same assumptions of Theorem 14 with $x^\alpha = \frac{a^\alpha+b^\alpha}{2}$. Then, we have*

$$\begin{aligned}
&\left|\frac{f\left(\left(\frac{a^\alpha+b^\alpha}{2}\right)^{\frac{1}{\alpha}}\right)}{\alpha\frac{a^\alpha+b^\alpha}{2}} - \frac{1}{b^\alpha - a^\alpha}\int_a^b \frac{f(t)}{t^\alpha} d_\alpha t\right| \\
\leq\ & \frac{2}{\alpha\left(b^\alpha - a^\alpha\right)}\left(\ln\frac{a^\alpha+b^\alpha}{2} - \ln\sqrt{a^\alpha b^\alpha}\right)\left\|f - uD_\alpha(f)\right\|_\infty .
\end{aligned}$$

We consider now the weighted integral case of the inequality (3.11).

Theorem 15. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$ and an α-fractional differentiable mapping on (a,b), with $0 < a < b$. If $w : [a,b] \to \mathbb{R}$ is nonnegative and α-fractional integrable on $[a,b]$, then the following inequality given for conformable integrals holds:*

$$\begin{aligned}
&\left|\frac{f(x)}{x^\alpha}\int_a^b w(t) d_\alpha t - \int_a^b \frac{f(t)}{t^\alpha} w(t) d_\alpha t\right| \\
\leq\ & \left\|f - uD_\alpha(f)\right\|_\infty \left[\int_a^x \frac{w(t)}{t^\alpha} d_\alpha t - \int_x^b \frac{w(t)}{t^\alpha} d_\alpha t \right. \\
& \left. + \frac{1}{x^\alpha}\left(\int_x^b w(t) d_\alpha t - \int_a^x w(t) d_\alpha t\right)\right]
\end{aligned}$$

for each $x \in [a,b]$ and where $u(t) = \frac{t^{2-\alpha}}{\alpha}$, $t \in [a,b]$ and $\left\|f - uD_\alpha(f)\right\|_\infty = \sup_{\xi\in(a,b)} \left|f(\xi) - uD_\alpha(f)(\xi)\right| < \infty$.

Proof. Using the inequality (3.12), we have

$$\begin{aligned} & \left| \frac{f(x)}{x^{\alpha}} \int_a^b w(t) d_{\alpha} t - \int_a^b \frac{f(t)}{t^{\alpha}} w(t) d_{\alpha} t \right| \\ & \int_a^b \left| \frac{f(x)}{x^{\alpha}} - \frac{f(t)}{t^{\alpha}} \right| w(t) d_{\alpha} t \\ \leq \quad & \| f - u D_{\alpha}(f) \|_{\infty} \int_a^b \left| \frac{1}{x^{\alpha}} - \frac{1}{t^{\alpha}} \right| w(t) d_{\alpha} t. \end{aligned}$$

By simple calculations, we easily deduced required inequality, and thus the theorem is proved. □

4. Applications to Numerical Integration

We now deal with applications of the integral inequalities involving conformable fractional integral, to obtain estimates of composite quadrature rules which, it turns out have a markedly smaller error than that which may be obtained by the classical results.

Consider the partition of the inteval $[a, b]$, $0 < a < b$, given by

$$I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$$

and $\xi_i \in [x_i, x_{i+1}]$, $i = 0, ..., n-1$ a sequance of intermediate points. Define the quadrature

$$S(f, I_n, \xi) := \frac{1}{2\alpha} \sum_{i=0}^{n-1} \frac{f(\xi_i)}{\xi_i^{\alpha}} \left(x_{i+1}^{\alpha} + x_i^{\alpha} \right) h_i \tag{4.1}$$

where $h_i = (x_{i+1}^{\alpha} - x_i^{\alpha})$, $i = 0, ..., n-1$.

Theorem 16. *Let $\alpha \in (0, 1]$ and $f : [a, b] \to \mathbb{R}$ be a continuous on $[a, b]$ and an α-fractional differentiable mapping on (a, b) with $0 < a < b$. Then we have the representation*

$$\int_a^b f(t) d_{\alpha} t = S(f, I_n, \xi) + R(f, I_n, \xi)$$

where $S(f, I_n, \xi)$ is as defined in (4.1) and the remainder satisfies the astimation:

$$\begin{aligned} & |R(f, I_n, \xi)| \\ \leq \quad & \frac{1}{\alpha} \| f - u D_{\alpha}(f) \|_{\infty} \sum_{i=0}^{n-1} \frac{h_i^2}{\xi_i^{\alpha}} \left[\frac{1}{4} + \left(\frac{\xi_i^{\alpha} - \frac{x_i^{\alpha} + x_{i+1}^{\alpha}}{2}}{h_i} \right)^2 \right]. \end{aligned} \tag{4.2}$$

Proof. Applying Theorem 12 on the interval $[x_i, x_{i+1}]$ for the intermadiate points ξ_i, we obtain

$$\begin{aligned} & \left| \frac{1}{2\alpha} \frac{f(\xi_i)}{\xi_i^\alpha} \left(x_{i+1}^\alpha + x_i^\alpha \right)^\alpha h_i - \int_{x_i}^{x_{i+1}} f(t) d_\alpha t \right| \\ \leq & \frac{1}{\alpha} \frac{h_i^2}{\xi_i^\alpha} \left[\frac{1}{4} + \left(\frac{\xi_i^\alpha - \frac{x_i^\alpha + x_{i+1}^\alpha}{2}}{h_i} \right)^2 \right] \| f - u D_\alpha(f) \|_\infty \end{aligned}$$

for all $i = 0, ..., n-1$. Summing over i from 0 to $n-1$ and using the triangle inequality we obtain the estimation (4.2). □

Now, define the mid-point rule as the following:

$$M(f, I_n) := \frac{1}{\alpha} \sum_{i=0}^{n-1} f \left(\left(\frac{x_i^\alpha + x_{i+1}^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) h_i$$

where $h_i = (x_{i+1}^\alpha - x_i^\alpha)$, $i = 0, ..., n-1$.

Corollary 3. *Under the same assumptions of Theorem 16 with* $\xi_i^\alpha = \frac{x_i^\alpha + x_{i+1}^\alpha}{2}$. *Then, we have*

$$\int_a^b f(t) d_\alpha t = M(f, I_n) + R(f, I_n),$$

where the remainder satisfies the astimation:

$$|R(f, I_n)| \leq \frac{1}{2\alpha} \| f - u D_\alpha(f) \|_\infty \sum_{i=0}^{n-1} \frac{h_i^2}{\left(x_i^\alpha + x_{i+1}^\alpha \right)^\alpha}.$$

5. Applications to Some Special Means

We define conformable arithmatic, geometric and p-logarithmic means, respectively:

$$A_\alpha(a, b) = \frac{a^\alpha + b^\alpha}{2},$$

$$G_\alpha(a, b) = \sqrt{a^\alpha b^\alpha},$$

$$L_p^\alpha(a, b) = \left[\frac{b^{\alpha(p+1)} - a^{\alpha(p+1)}}{\alpha (p+1) (b^\alpha - a^\alpha)} \right]^{\frac{1}{p}}, \ p \in \mathbb{R} \backslash \{-1, 0\}.$$

In order to get the results in this section, we will use the following inequalities obtained in Corollary 1 and Corollary 2.

Consider the mapping $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^{\alpha p}$, $p \in \mathbb{R} \backslash \{-1, 0\}$. Then, $0 < a < b$, we have

$$f \left(\left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{1}{\alpha}} \right) = [A_\alpha(a, b)]^p,$$

$$\frac{1}{b^\alpha - a^\alpha} \int_a^b f(t) d_\alpha t = \left[L_p^\alpha(a, b) \right]^p$$

and

$$\frac{1}{b^\alpha - a^\alpha}\int_a^b \frac{f(t)}{t^\alpha} d_\alpha t = \left[L_{p-1}^\alpha(a,b)\right]^{p-1}.$$

Also, if we use the identity (2.3), then we obtain

$$\begin{aligned}\|f - uD_\alpha(f)\|_\infty &= \delta(a,b)\\ &= \begin{cases} \left(1 - pa^{2-2\alpha}\right) a^{\alpha p}, & \text{if } p \in (-\infty, 0) \setminus \{-1\} \\ \\ \left|1 - pb^{2-2\alpha}\right| b^{\alpha p}, & \text{if } p \in (0,1) \cup (1,\infty). \end{cases}\end{aligned}$$

Finally, utilizing the corollary 1 and corollary 2, we deduce the inequalities

$$\begin{aligned}&\left|\frac{1}{\alpha}\left[A_\alpha(a,b)\right]^p - \left[L_p^\alpha(a,b)\right]^p\right|\\ &\le \frac{(b^\alpha - a^\alpha)}{4\alpha A_\alpha(a,b)}\delta(a,b)\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{1}{\alpha}\left[A_\alpha(a,b)\right]^{p-1} - \left[L_{p-1}^\alpha(a,b)\right]^{p-1}\right|\\ &\le \frac{\delta(a,b)}{\alpha\left(b^\alpha - a^\alpha\right)}\ln\left[\frac{A_\alpha(a,b)}{G_\alpha(a,b)}\right]^2,\end{aligned}$$

respectively.

References

[1] T. Abdeljawad, *On conformable fractional calculus*, Journal of Computational and Applied Mathematics 279 (2015) 57–66.

[2] M. Abu Hammad, R. Khalil, *Conformable fractional heat differential equations*, International Journal of Differential Equations and Applications 13(3), 2014, 177-183.

[3] M. Abu Hammad, R. Khalil, *Abel's formula and wronskian for conformable fractional differential equations*, International Journal of Differential Equations and Applications 13(3), 2014, 177-183.

[4] A. M. Acu, A. Babos and F. D. Sofonea, *The mean value theorems and inequalities of Ostrowski type*. Sci. Stud. Res. Ser. Math. Inform. 21 (2011), no. 1, 5–16.

[5] D. R. Anderson, *Taylor's formula and integral inequalities for conformable fractional derivatives*, Contributions in Mathematics and Engineering, in Honor of Constantin Caratheodory, Springer, to appear.

[6] S.S. Dragomir, *An inequality of Ostrowski type via Pompeiu's mean value theorem*, J. of Inequal. in Pure and Appl. Math., 6(3) (2005), Art. 83.

[7] S.S. Dragomir, *Ostrowski via a two functions Pompeiu's inequality,* RGMIA Research Report Collection, 16(2013), Article 90, 13 pp.

[8] S.S. Dragomir, *Exponential Pompeiu's type inequalities with applications Ostrowski's inequality*, Acta Mathematica Universitatis Comenianae, 184 (1), 2015, 39-50.

[9] S. S. Dragomir, *Power Pompeiu's type inequalities for absolutely continuous functions with applications to Ostrowski's inequality,* RGMIA Research Report Collection, 16(2013), Article 68, 8 pp.

[10] S. Erden, and M. Z. Sarikaya, *Generalized Pompeiu type inequalities for local fractional integrals and Its Applications*. Apllied Math. and Computations, 274 (2016) 282-291.

[11] O.S. Iyiola and E.R.Nwaeze, *Some new results on the new conformable fractional calculus with application using D'Alambert approach*, Progr. Fract. Differ. Appl., 2(2), 115-122, 2016.

[12] R. Khalil, M. Al horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative*, Journal of Computational Apllied Mathematics, 264 (2014), 65-70.

[13] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226-227.

[14] J. Pečarić and Š. Ungar, *On an inequality of Ostrowski type*, J. Ineq. Pure & Appl. Math. 7 (2006). No. 4. Art. 151.

[15] D. Pompeiu, *Sur une proposition analogue au théoréme des accroissements finis*, Mathematica (Cluj, Romania), 22(1946), 143-146.

[16] E. C. Popa, *An inequality of Ostrowski type via a mean value theorem*, General Mathematics, 15 (1), 2007, 93-100.

[17] M. Z. Sarikaya, S. Erden and H. Yaldiz, *On power pompeiu type inequalities for double integrals*, South Bohemia Mathematical Letters, 23 (1), (2015), 13-85.

[18] M. Z. Sarikaya and H. Budak, *On an Inequality of Ostrowski Type via Variant of Pompeiu's Mean Value Theorem*, Turkish Journal of Analysis and Number Theory, 2014, Vol. 2, No. 3, 80-84.

[19] M. Z. Sarikaya and H. Budak, *An inequality of Grüss like via variant of Pompeiu's mean value theorem*, Konuralp J. of Math., 3 (1), 2015, 29-35.

[20] M. Z. Sarikaya , *On an inequality of Grüss type via variant of Pompeiu's mean value theorem*, Pure and Applied Mathematics Letters, 2(2014)26-30.

[21] M. Z. Sarikaya , *Some new integral inequalities via variant of Pompeiu's mean value theorem*, Mathematica Moravica, Vol. 19-2 (2015), 89–95.

Department of Mathematics, Faculty of Science, Bartin University, BARTIN-TURKEY
E-mail address: erdensmt@gmail.com

Department of Mathematics, Faculty of Science and Arts, Düzce University, Konuralp Campus, Düzce-TURKEY
E-mail address: sarikayamz@gmail.com