Received 01/06/17

Principles of General Fractional Analysis for Banach space valued functions

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Abstract

Here we present a general fractional analysis theory for Banach space valued functions of real domain. A series of general Taylor formulae with Bochner integral remainder is presented. We discuss the continuity of general Riemann-Liouville Bochner fractional integrals and we prove their semigroup property. Then we introduce the right and left generalized Banach space valued fractional derivatives and we establish the corresponding fractional Taylor formulae with Bochner integral remainders. Furthermore we study the iterated generalized left and right fractional derivatives and we establish Taylor formulae for the case, and we find interesting Bochner integral representation formulae for them. We study the differentiation of the left and right Riemann-Liouville fractional Bochner integrals. At the end we give Ostrowski type inequalities on this general setting, plus other interesting applications.

2010 AMS Subject Classification: 26A33, 26D10, 46B25, 46E40.

Key Words and Phrases: Right and left and iterated generalized fractional derivatives, generalized Taylor's formulae, generalized fractional Taylor's formulae, Banach space valued functions and Bochner integral, integral inequalities.

1 Introduction

An account of our work follows: This paper deals with essential aspects of fractional analysis for Banach space valued functions of a real domain. We pursue our results to the greatest possible generality within our setting's limits. The related Fundamental Theorem of Calculus (FTC), by [12], Theorem 2 here, plays a pivotal role in this article, without it would not have been written.

RGMIA Res. Rep. Coll. 20 (2017), Art. 65, 58 pp.

Based on this we produce various very general Taylor formulae with integral remainders, these are with respect to a parameter function, see Theorems 3, 4, 5, 6, 7, and Corollaries 8, 9. In all these the Hausdorff measure is the key to generality. All the above so far belong to section 2, about auxiliary results.

The main results in section 3 unfold by giving first some continuity theorems for parameter function Bochner integrals involving a general fractional kernel and defining functions of the second parameter variable, see Theorems 10, 11. We next define the right and left Riemann-Liouville generalized fractional Bochner integral operators, see Definitions 12, 14, and we prove the semigroup property under composition over continuous Banach space valued functions, see Theorems 13, 15.

Then based on the last we define the Banach valued right and left generalized fractional derivatives, Caputo style, see Definitions 16, 17. The next step is to give related generalized fractional Taylor formulae, see Theorems 18, 19.

We continue with the Canavati style ([7]) generalized fractional Calculus for Banach space valued functions. We introduce the generalized related right and left fractional derivatives and produce Taylor formulae, see Theorems 20, 21. We continue with the Canavati type iterated fractional integrals and derivatives, right and left with a parameter function. The results are right and left iterated fractional Taylor formulae of Canavati type, see Theorems 24, 27.

We continue with right and left iterated fractional Taylor formulae of Caputo type, see Theorems 30, 33. We apply these when the parameter function is the identity map, see Theorems 43, 44.

Then we establish some very important differentiation theorems, see Theorems 34, 35, regarding differentiation of right and left Riemann-Liouville fractional Bochner integrals. Based on the last properties we develop representation formulae for the right and left iterated fractional derivatives of Caputo type for Banach space valued functions, see Theorems 45, 46, 47 and 48, and Theorems 49, 50, 51 and 52. Our results potentially have great applications in the theory of fractional ordinary and partial differential equations, analytic inequalities, approximation theory, and in general computational analysis.

Due to the length of article we only give applications to the well-known Ostrowski inequalities, here at the fractional level, generalized, and for Banach space valued functions. We present the general Theorem 53, then we apply this Ostrowski inequality for specific parameter functions such as e^t, $\cos t$, see Theorems 61, 62.

We also give applications of our major fractional Taylor formulae, see Theorems 55, 56, 57, 58, 59, 60, the parameter functions now are $e^t, \sin t, \tan x$. Applications belong to section 4.

Overall we feel this article opens new research frontiers in the fractional calculus study and many papers can be written based on it.

2 Auxilliary Results

All integrals here are of Bochner type, see [9]. We need

Definition 1 *([12]) A definition of the Hausdorff measure h_α would go as follows: if (T,d) is any metric space, $A \subseteq T$ and $\delta > 0$, let $\Lambda(A,\delta)$ be the set of all arbitrary collections $(C)_i$ of subsets of T, such that $A \subseteq \cup_i C_i$ and diam $(C_i) \leq \delta$ for every i. Now, for every $\alpha > 0$ define*

$$h_\alpha^\delta(A) := \inf\left\{\sum (diam C_i)^\alpha \mid (C_i) \in \Lambda(A,\delta)\right\}. \tag{1}$$

Then there exists $\lim_{\delta\to 0} h_\alpha^\delta(A) = \sup_{\delta>0} h_\alpha^\delta(A)$ and $h_\alpha(A) := \lim_{\delta\to 0} h_\alpha^\delta(A)$ gives an outer measure on power set $\mathcal{P}(T)$ which is countable additive on the σ-field of all Borel subsets of T.

If $T = \mathbb{R}^n$, the Hausdorff measure h_n, restricted to the σ-filed of the Borel subsets of $\mathbb{R}^n$, is identical to the Lebesgue measure on $\mathbb{R}^n$ up to a constant multiple. In particular, $h_1(C) = \mu(C)$ for every Borel set $C \subseteq \mathbb{R}$, where μ is the Lebesgue measure on $\mathbb{R}$.

We also need

Theorem 2 *([12]) (Fundamental Theorem of Calculus) Suppose that for the given*

$$f : [a,b] \to X, \ \ (X, \|\cdot\|) \ \text{ is a Banach space,}$$

there exists $F : [a,b] \to X$, which is continuous, the derivative $F'(t)$ exists and $F'(t) = f(t)$ outside a μ-null Borel set $B \subseteq [a,b]$ such that

$$h_1(F(B)) = 0.$$

Then f is a strongly μ-measurable and if we assume the Bochner integrability of f,

$$F(b) - F(a) = \int_a^b f(t)\,dt. \tag{2}$$

We have

Theorem 3 *Let $n \in \mathbb{N}$ and $f \in C^{n-1}([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, such that $g^{-1} \in C^n(g([a,b]))$. Set*

$$F(x) := \sum_{i=0}^{n-1} \frac{(g(b)-g(x))^i}{i!}\left(f \circ g^{-1}\right)^{(i)}(g(x)), \quad \forall\ x \in [a,b]. \tag{3}$$

Assume that $\left(f \circ g^{-1}\right)^{(n)} \circ g$ exists outside a μ (Lebesgue measure)-null Borel set $B \subseteq [a,b]$ such that $h_1(F(B)) = 0$. We further assume the Bochner integrability of $\left(f \circ g^{-1}\right)^{(n)} \circ g$.

Then

$$f(b) = f(a) + \sum_{i=1}^{n-1} \frac{(g(b) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) + \tag{4}$$

$$\frac{1}{(n-1)!} \int_a^b (g(b) - g(x))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(x)) \, g'(x) \, dx.$$

Proof. We get that $F \in C([a,b], X)$. We get that

$$F'(x) = \frac{(g(b) - g(x))^{n-1}}{(n-1)!} \left(f \circ g^{-1}\right)^{(n)} (g(x)) \, g'(x), \tag{5}$$

$\forall\, x \in [a,b] - B$. Also F' is Bochner integrable.

Notice that $F(b) = f(b)$, and

$$F(a) = \sum_{i=0}^{n-1} \frac{(g(b) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)). \tag{6}$$

We have (by Theorem 2)

$$F(b) - F(a) = \int_a^b F'(t) \, dt. \tag{7}$$

Thus

$$f(b) - \sum_{i=0}^{n-1} \frac{(g(b) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) = \tag{8}$$

$$\frac{1}{(n-1)!} \int_a^b (g(b) - g(x))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(x)) \, g'(x) \, dx,$$

proving the claim. ■

We have

Theorem 4 *Let $n \in \mathbb{N}$ and $f \in C^{n-1}([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, such that $g^{-1} \in C^n(g([a,b]))$. Set*

$$F(x) := \sum_{i=0}^{n-1} \frac{(g(a) - g(x))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(x)), \quad \forall\, x \in [a,b]. \tag{9}$$

Assume that $\left(f \circ g^{-1}\right)^{(n)} \circ g$ exists outside a μ (Lebesgue measure)-null Borel set $B \subseteq [a,b]$ such that $h_1(F(B)) = 0$. We further assume the Bochner integrability of $\left(f \circ g^{-1}\right)^{(n)} \circ g$.

Then

$$f(a) = f(b) + \sum_{i=1}^{n-1} \frac{(g(a) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) + \tag{10}$$

$$\frac{1}{(n-1)!} \int_b^a (g(a) - g(x))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(x)) \, g'(x) \, dx.$$

Proof. We get that $F \in C([a,b], X)$. Then we have

$$F'(x) = \frac{(g(a) - g(x))^{n-1}}{(n-1)!} \left(f \circ g^{-1}\right)^{(n)} (g(x)) \, g'(x), \tag{11}$$

$\forall\, x \in [a,b] - B$. Also F' is Bochner integrable.

Notice here $F(a) = f(a)$, and

$$F(b) = \sum_{i=0}^{n-1} \frac{(g(a) - g(b))^{i}}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)). \tag{12}$$

We have (by Theorem 2)

$$F(b) - F(a) = \int_a^b F'(t)\, dt. \tag{13}$$

That is, we obtain

$$\sum_{i=0}^{n-1} \frac{(g(a) - g(b))^{i}}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) - f(a) = \tag{14}$$

$$\frac{1}{(n-1)!} \int_a^b (g(a) - g(x))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(x))\, g'(x)\, dx =$$

$$-\frac{1}{(n-1)!} \int_b^a (g(a) - g(x))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(x))\, g'(x)\, dx.$$

proving (10). ■

In Bochner integrals the change of variable is a questionable matter, a positive answer follows:

Theorem 5 *Let φ be a strictly increasing function in $C^1([a,b])$, and $\varphi : [a,b] \to [\alpha, \beta]$ with $\varphi(a) = \alpha$, $\varphi(b) = \beta$, $a < b$. Assume that $\varphi^{-1} \in AC([\alpha, \beta])$ (absolutely continuous functions). Let $F : [\alpha, \beta] \to X$ be continuous, where $(X, \|\cdot\|)$ is a Banach space. Assume that the derivative F' exists outside a μ (Lebesgue)-null Borel set $W \subseteq [\alpha, \beta]$ such that $h_1(F(W)) = 0$, and F' is Bochner integrable.*

Then

$$\int_{\varphi(a)}^{\varphi(b)} F'(t)\, dt = \int_a^b F'(\varphi(t))\, \varphi'(t)\, dt. \tag{15}$$

Proof. By Theorem 2, we get that

$$F(\beta) - F(\alpha) = \int_\alpha^\beta F'(t)\, dt. \tag{16}$$

I.e.

$$F(\varphi(b)) - F(\varphi(a)) = \int_{\varphi(a)}^{\varphi(b)} F'(t)\,dt. \tag{17}$$

Consider the function

$$H(t) := F(\varphi(t)), \quad a \le t \le b.$$

Then $H : [a,b] \to X$ is a continuous function.

Then the derivative $H'(t) = F'(\varphi(t))\,\varphi'(t)$ exists outside the μ-null Borel set (see [11], p. 108, exercise 14) $B := \varphi^{-1}(W) \subseteq [a,b]$, such that

$$h_1(H(B)) = h_1\left(H\left(\varphi^{-1}(W)\right)\right) = h_1\left(F\left(\varphi\left(\varphi^{-1}(W)\right)\right)\right) = h_1(F(W)) = 0.$$

That is $h_1(H(B)) = 0$.

Here H' is Bochner integrable by:

$$\left\| \int_a^b H'(t)\,dt \right\| \le \int_a^b \|H'(t)\|\,dt = \int_a^b \|F'(\varphi(t))\|\,\varphi'(t)\,dt \quad \text{(by [6], [10])} \tag{18}$$

$$= \int_\alpha^\beta \|F'(t)\|\,dt < \infty.$$

Again by Theorem 2, we get that

$$F(\beta) - F(\alpha) = F(\varphi(b)) - F(\varphi(a)) = H(b) - H(a) = \tag{19}$$

$$\int_a^b H'(t)\,dt = \int_a^b F'(\varphi(t))\,\varphi'(t)\,dt,$$

proving the claim. ■

Using the methods of Theorem 5 we get

Theorem 6 *Let $n \in \mathbb{N}$ and $f \in C^{n-1}([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. Set*

$$G(z) := \sum_{i=0}^{n-1} \frac{(g(b)-z)^i}{i!} \left(f \circ g^{-1}\right)^{(i)}(z), \quad \forall\, z \in [g(a), g(b)]. \tag{20}$$

Assume that $\left(f \circ g^{-1}\right)^{(n)}$ exists outside a μ-null Borel set $W \subseteq [g(a), g(b)]$ such that $h_1(G(W)) = 0$, and $\left(f \circ g^{-1}\right)^{(n)}$ is Bochner integrable.

Then

$$f(b) = f(a) + \sum_{i=1}^{n-1} \frac{(g(b)-g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)}(g(a)) + R_n(a,b,g), \tag{21}$$

where

$$R_n(a,b,g) := \frac{1}{(n-1)!}\int_a^b (g(b)-g(x))^{n-1}\left(f\circ g^{-1}\right)^{(n)}(g(x))\,g'(x)\,dx$$

$$= \frac{1}{(n-1)!}\int_{g(a)}^{g(b)} (g(b)-z)^{n-1}\left(f\circ g^{-1}\right)^{(n)}(z)\,dz. \tag{22}$$

Proof. Notice that $G \in C([g(a),g(b)],X)$. We also notice the existence of

$$G'(z) = \frac{(g(b)-z)^{n-1}}{(n-1)!}\left(f\circ g^{-1}\right)^{(n)}(z), \tag{23}$$

$\forall\, z \in [g(a),g(b)]-W$, and furthermore G' is Bochner integrable over $[g(a),g(b)]$.

By Theorem 2 now we get that

$$G(g(b)) - G(g(a)) = \int_{g(a)}^{g(b)} G'(t)\,dt = \tag{24}$$

$$\frac{1}{(n-1)!}\int_{g(a)}^{g(b)} (g(b)-z)^{n-1}\left(f\circ g^{-1}\right)^{(n)}(z)\,dz.$$

Here we have

$$G(g(b)) = \left(f\circ g^{-1}\right)(g(b)) = f(b),$$

and

$$G(g(a)) = \sum_{i=0}^{n-1}\frac{(g(b)-g(a))^i}{i!}\left(f\circ g^{-1}\right)^{(i)}(g(a)). \tag{25}$$

We have proved that

$$f(b) = f(a) + \sum_{i=1}^{n-1}\frac{(g(b)-g(a))^i}{i!}\left(f\circ g^{-1}\right)^{(i)}(g(a)) + \tag{26}$$

$$\frac{1}{(n-1)!}\int_{g(a)}^{g(b)} (g(b)-z)^{n-1}\left(f\circ g^{-1}\right)^{(n)}(z)\,dz.$$

Consider the function

$$H(x) := G(g(x)), \quad a \le x \le b,$$

then $H : [a,b] \to X$ is a continuous function.

Then the derivative $H'(x) = G'(g(x))\,g'(x)$, exists outside the μ-null Borel set (see [11], p. 108, exercise 14) $B := g^{-1}(W) \subseteq [a,b]$, such that

$$h_1(H(B)) = h_1\left(H\left(g^{-1}(W)\right)\right) = h_1\left(G\left(g\left(g^{-1}(W)\right)\right)\right) = h_1(G(W)) = 0. \tag{27}$$

That is $h_1(H(B)) = 0$.

Here H' is Bochner integrable by:

$$\left\| \int_a^b H'(t)\, dt \right\| \leq \int_a^b \|H'(t)\|\, dt = \int_a^b \|G'(g(x))\|\, g'(x)\, dx \quad \text{(by [10])} \tag{28}$$

$$= \int_{g(a)}^{g(b)} \|G'(t)\|\, dt < \infty.$$

Again by Theorem 2, we get that

$$G(g(b)) - G(g(a)) = H(b) - H(a) = \int_a^b H'(t)\, dt = \int_a^b G'(g(t))\, g'(t)\, dt. \tag{29}$$

Finally we see that

$$f(b) - \sum_{i=1}^{n-1} \frac{(g(b) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) = \tag{30}$$

$$\frac{1}{(n-1)!} \int_a^b (g(b) - g(x))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(x))\, g'(x)\, dx,$$

proving the claim. ■

Using methods from Theorem 5 we get

Theorem 7 *Let $n \in \mathbb{N}$ and $f \in C^{n-1}([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. Set*

$$G(z) := \sum_{i=0}^{n-1} \frac{(g(a) - z)^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (z), \quad \forall\, z \in [g(a), g(b)]. \tag{31}$$

Assume that $\left(f \circ g^{-1}\right)^{(n)}$ exists outside a μ-null Borel set $W \subseteq [g(a), g(b)]$ such that $h_1(G(W)) = 0$, and $\left(f \circ g^{-1}\right)^{(n)}$ is Bochner integrable.

Then

$$f(a) = f(b) + \sum_{i=1}^{n-1} \frac{(g(a) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) + R_n(b, a, g), \tag{32}$$

where

$$R_n(b, a, g) := \frac{1}{(n-1)!} \int_b^a (g(a) - g(x))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(x))\, g'(x)\, dx$$

$$= \frac{1}{(n-1)!} \int_{g(b)}^{g(a)} (g(a) - z)^{n-1} \left(f \circ g^{-1}\right)^{(n)} (z)\, dz. \tag{33}$$

Proof. Notice that $G \in C([g(a), g(b)], X)$. We also notice the existence of

$$G'(z) = \frac{(g(a) - z)^{n-1}}{(n-1)!} \left(f \circ g^{-1}\right)^{(n)}(z), \tag{34}$$

$\forall\, z \in [g(a), g(b)] - W$, and furthermore G' is Bochner integrable over $[g(a), g(b)]$.

By Theorem 2 now we get that

$$G(g(b)) - G(g(a)) = \int_{g(a)}^{g(b)} G'(t)\, dt = \tag{35}$$

$$\frac{1}{(n-1)!} \int_{g(a)}^{g(b)} (g(a) - z)^{n-1} \left(f \circ g^{-1}\right)^{(n)}(z)\, dz =$$

$$\frac{-1}{(n-1)!} \int_{g(b)}^{g(a)} (g(a) - z)^{n-1} \left(f \circ g^{-1}\right)^{(n)}(z)\, dz.$$

Here we have

$$G(g(b)) = \sum_{i=0}^{n-1} \frac{(g(a) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)}(g(b)), \tag{36}$$

and

$$G(g(a)) = f(a).$$

We have proved that

$$\sum_{i=0}^{n-1} \frac{(g(a) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)}(g(b)) - f(a) = \tag{37}$$

$$-\frac{1}{(n-1)!} \int_{g(b)}^{g(a)} (g(a) - z)^{n-1} \left(f \circ g^{-1}\right)^{(n)}(z)\, dz.$$

That is, it holds

$$f(a) = f(b) + \sum_{i=1}^{n-1} \frac{(g(a) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)}(g(b)) + \tag{38}$$

$$\frac{1}{(n-1)!} \int_{g(b)}^{g(a)} (g(a) - z)^{n-1} \left(f \circ g^{-1}\right)^{(n)}(z)\, dz.$$

Consider the function

$$H(x) := G(g(x)), \quad a \le x \le b,$$

then $H : [a, b] \to X$ is a continuous function.

Then the derivative $H'(x) = G'(g(x))\, g'(x)$, exists outside the μ-null Borel set (see [11], p. 108, exercise 14) $B := g^{-1}(W) \subseteq [a,b]$, such that

$$h_1(H(B)) = h_1\left(H\left(g^{-1}(W)\right)\right) = h_1\left(G\left(g\left(g^{-1}(W)\right)\right)\right) = h_1(G(W)) = 0.$$

That is $h_1(H(B)) = 0$.

Here H' is Bochner integrable by:

$$\left\| \int_a^b H'(t)\, dt \right\| \leq \int_a^b \|H'(t)\|\, dt = \int_a^b \|G'(g(x))\|\, g'(x)\, dx \quad \text{(by [10])} \tag{39}$$

$$= \int_{g(a)}^{g(b)} \|G'(t)\|\, dt < \infty.$$

Again by Theorem 2, we get that

$$G(g(b)) - G(g(a)) = H(b) - H(a) = \int_a^b H'(t)\, dt = \int_a^b G'(g(t))\, g'(t)\, dt. \tag{40}$$

Finally, we see that

$$\sum_{i=0}^{n-1} \frac{(g(a) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) - f(a) =$$

$$\frac{1}{(n-1)!} \int_a^b (g(a) - g(x))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(x))\, g'(x)\, dx = \tag{41}$$

$$-\frac{1}{(n-1)!} \int_b^a (g(a) - g(x))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(x))\, g'(x)\, dx.$$

Equivalently we have

$$f(a) = f(b) + \sum_{i=1}^{n-1} \frac{(g(a) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) + \tag{42}$$

$$\frac{1}{(n-1)!} \int_b^a (g(a) - g(x))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(x))\, g'(x)\, dx,$$

proving the claim. ■

Based on the Theorems 3, 4 we give

Corollary 8 *Let $n \in \mathbb{N}$ and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, such that $g^{-1} \in C^n(g([a,b]))$. Let any $x, y \in [a,b]$.*

Then

$$f(x) = f(y) + \sum_{i=1}^{n-1} \frac{(g(x) - g(y))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(y)) + \tag{43}$$

$$\frac{1}{(n-1)!} \int_y^x (g(x) - g(t))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(t))\, g'(t)\, dt.$$

Proof. Notice in Theorems 3, 4 that $F(\varnothing) = \varnothing$, where $B = \varnothing$, and $h_1(\varnothing) = \mu(\varnothing) = 0$, where μ is the Lebesgue measure, and $\left(f \circ g^{-1}\right)^{(n)} \circ g$ is continuous over $[a, b]$. ■

We also have

Corollary 9 *Let $n \in \mathbb{N}$ and $f \in C^n([a, b], X)$, where $[a, b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a, b])$,strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. Let any $x, y \in [a, b]$. Then*

$$f(x) = f(y) + \sum_{i=1}^{n-1} \frac{(g(x) - g(y))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(y)) + R_n(y, x, g), \tag{44}$$

where

$$R_n(y, x, g) = \frac{1}{(n-1)!} \int_y^x (g(x) - g(t))^{n-1} \left(f \circ g^{-1}\right)^{(n)} (g(t))\, g'(t)\, dt = \tag{45}$$

$$\frac{1}{(n-1)!} \int_{g(y)}^{g(x)} (g(x) - z)^{n-1} \left(f \circ g^{-1}\right)^{(n)} (z)\, dz.$$

Proof. By Theorems 6, 7. ■

3 Main Results

We need

Theorem 10 *Here $[a, b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space, $F : [a, b] \to X$, $g \in C^1([a, b])$ and increasing. Let $r > 0$ and $F \in L_\infty([a, b], X)$, and the Bochner integral*

$$G(s) := \int_a^s (g(s) - g(t))^{r-1} g'(t)\, F(t)\, dt, \tag{46}$$

all $s \in [a, b]$. Then $G \in AC([a, b], X)$ (absolutely continuous) for $r \geq 1$ and $G \in C([a, b], X)$ for $r \in (0, 1)$.

Proof. Denote by $\|F\|_\infty := \|F\|_{L_\infty([a,b],X)} := \underset{t\in[a,b]}{es\sup} \|F(t)\|_X < +\infty$. Hence $F \in L_1([a, b], X)$. By Theorem 5.4, p. 101, [9], $(g(s) - g(t))^{r-1} g'(t)\, F(t)$ is a strongly measurable function in t, $t \in [a, s]$, $s \in [a, b]$.

So that $(g(s) - g(t))^{r-1} g'(t)\, F(t) \in L_1([a, s], X)$, see [6]. Notice for above that we used

$$\int_a^s (g(s) - g(t))^{r-1} g'(t)\, dt = \frac{(g(s) - g(a))^r}{r}, \tag{47}$$

by [11], p. 107, exercise 13d.

1) Case $r \geq 1$. We use the definition of absolute continuity. So for every $\varepsilon > 0$ we need $\delta > 0$: whenever (a_i, b_i), $i = 1, ..., n$, are disjoint open subintervals of $[a, b]$, then

$$\sum_{i=1}^{n} (b_i - a_i) < \delta \Rightarrow \sum_{i=1}^{n} \|G(b_i) - G(a_i)\| < \varepsilon. \tag{48}$$

If $\|F\|_\infty = 0$, then $G(s) = 0$, for all $s \in [a, b]$, the trivial case and all fulfilled. So we assume $\|F\|_\infty \neq 0$. Hence we have (see [5])

$$\begin{gathered} G(b_i) - G(a_i) = \int_a^b (g(b_i) - g(t))^{r-1} g'(t) F(t) \, dt - \\ \int_a^{a_i} (g(a_i) - g(t))^{r-1} g'(t) F(t) \, dt = \\ \int_a^{a_i} (g(b_i) - g(t))^{r-1} g'(t) F(t) \, dt - \int_a^{a_i} (g(a_i) - g(t))^{r-1} g'(t) F(t) \, dt + \\ \int_{a_i}^{b_i} (g(b_i) - g(t))^{r-1} g'(t) F(t) \, dt = \end{gathered} \tag{49}$$

(see [1], p. 426, Theorem 11.43)

$$\int_a^{a_i} \left((g(b_i) - g(t))^{r-1} - (g(a_i) - g(t))^{r-1} \right) g'(t) F(t) \, dt +$$

$$\int_{a_i}^{b_i} (g(b_i) - g(t))^{r-1} g'(t) F(t) \, dt.$$

Call

$$I_i := \int_a^{a_i} \left| (g(b_i) - g(t))^{r-1} - (g(a_i) - g(t))^{r-1} \right| g'(t) \, dt. \tag{50}$$

Thus

$$\|G(b_i) - G(a_i)\| \leq \left[I_i + \frac{(g(b_i) - g(a_i))^r}{r} \right] \|F\|_\infty := T_i. \tag{51}$$

If $r = 1$, then $I_i = 0$, and

$$\|G(b_i) - G(a_i)\| \leq \|F\|_\infty (g(b_i) - g(a_i)) \leq \|F\|_\infty \|g'\|_\infty (b_i - a_i), \tag{52}$$

for all $i = 1, ..., n$.

If $r > 1$, then since

$$\left[(g(b_i) - g(t))^{r-1} - (g(a_i) - g(t))^{r-1} \right] \geq 0,$$

for all $t \in [a, a_i]$, we find

$$I_i = \int_a^{a_i} \left((g(b_i) - g(t))^{r-1} - (g(a_i) - g(t))^{r-1} \right) g'(t)\, dt = \tag{53}$$

$$\int_{g(a)}^{g(a_i)} \left((g(b_i) - z)^{r-1} - (g(a_i) - z)^{r-1} \right) dz =$$

$$\frac{(g(b_i) - g(a))^r - (g(a_i) - g(a))^r - (g(b_i) - g(a_i))^r}{r} =$$

$$\frac{r(\xi - g(a))^{r-1}(g(b_i) - g(a_i)) - (g(b_i) - g(a_i))^r}{r},$$

for some $\xi \in (g(a_i), g(b_i))$.

Therefore, it holds

$$I_i \leq \frac{r(g(b) - g(a))^{r-1}(g(b_i) - g(a_i)) - (g(b_i) - g(a_i))^r}{r}, \tag{54}$$

and

$$\left(I_i + \frac{(g(b_i) - g(a_i))^r}{r} \right) \leq (g(b) - g(a))^{r-1}(g(b_i) - g(a_i)). \tag{55}$$

That is

$$T_i \leq \|F\|_\infty (g(b) - g(a))^{r-1}(g(b_i) - g(a_i)), \tag{56}$$

so that

$$\|G(b_i) - G(a_i)\| \leq \|F\|_\infty (g(b) - g(a))^{r-1} \|g'\|_\infty (b_i - a_i), \tag{57}$$

for all $i = 1, ..., n$.

So in the case of $r = 1$, and by choosing $\delta := \frac{\varepsilon}{\|g'\|_\infty \|F\|_\infty}$, we get

$$\sum_{i=1}^n \|G(b_i) - G(a_i)\| \overset{(52)}{\leq} \|F\|_\infty \|g'\|_\infty \left(\sum_{i=1}^n (b_i - a_i) \right) \tag{58}$$

$$\leq \|F\|_\infty \|g'\|_\infty \delta = \varepsilon,$$

proving for $r = 1$ that G is absolutely continuous. In the case of $r > 1$, and by choosing $\delta := \frac{\varepsilon}{\|g'\|_\infty \|F\|_\infty (g(b)-g(a))^{r-1}}$, we get

$$\sum_{i=1}^n \|G(b_i) - G(a_i)\| \overset{(57)}{\leq} \|F\|_\infty \|g'\|_\infty (g(b) - g(a))^{r-1} \sum_{i=1}^n (b_i - a_i) \tag{59}$$

$$< \|F\|_\infty \|g'\|_\infty (g(b) - g(a))^{r-1} \delta = \varepsilon,$$

proving for $r > 1$ that G is absolutely continuous again.

2) Case of $0 < r < 1$. Let $a_{i^*}, b_{i^*} \in [a,b] : a_{i^*} \leq b_{i^*}$ and then $g(a_{i^*}) \leq g(b_{i^*})$. Then $(g(a_{i^*}) - g(t))^{r-1} \geq (g(b_{i^*}) - g(t))^{r-1}$, for all $t \in [a, a_{i^*})$. Hence

$$I_{i^*} = \int_a^{a_{i^*}} \left((g(a_{i^*}) - g(t))^{r-1} - (g(b_{i^*}) - g(t))^{r-1} \right) g'(t)\, dt =$$

$$\frac{\left((g(a_{i^*}) - g(a))^r - (g(b_{i^*}) - g(a))^r \right)}{r} + \frac{(g(b_{i^*}) - g(a_{i^*}))^r}{r} \leq \tag{60}$$

$$\frac{(g(b_{i^*}) - g(a_{i^*}))^r}{r} \leq \frac{\|g'\|_\infty^r}{r} (b_{i^*} - a_{i^*})^r,$$

by $[(g(a_{i^*}) - g(a))^r - (g(b_{i^*}) - g(a))^r] < 0$.

Therefore

$$I_{i^*} \leq \frac{\|g'\|_\infty^r}{r} (b_{i^*} - a_{i^*})^r \tag{61}$$

and

$$T_{i^*} \leq 2 \|F\|_\infty \frac{\|g'\|_\infty^r}{r} (b_{i^*} - a_{i^*})^r, \tag{62}$$

proving that

$$\|G(b_{i^*}) - G(a_{i^*})\| \leq \left(\frac{2 \|F\|_\infty \|g'\|_\infty^r}{r} \right) (b_{i^*} - a_{i^*})^r. \tag{63}$$

The last inequality proves that G is continuous for $r \in (0,1)$. The theorem is proved. ■

We also need

Theorem 11 *Here $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space, $F : [a,b] \to X$, $g \in C^1([a,b])$ and increasing. Let $r > 0$ and $F \in L_\infty([a,b], X)$, and the Bochner integral*

$$G(s) := \int_s^b (g(t) - g(s))^{r-1} g'(t) F(t)\, dt, \tag{64}$$

all $s \in [a,b]$. Then $G \in AC([a,b], X)$ (absolutely continuous) for $r \geq 1$ and $G \in C([a,b], X)$ for $r \in (0,1)$.

Proof. Denote by $\|F\|_\infty := \|F\|_{L_\infty([a,b],X)} := es\sup_{t \in [a,b]} \|F(t)\|_X < +\infty$. Hence $F \in L_1([a,b], X)$. By Theorem 5.4, p. 101, [9], $(g(t) - g(s))^{r-1} g'(t) F(t)$ is a strongly measurable function in t, $t \in [s,b]$, $s \in [a,b]$.

So that $(g(t) - g(s))^{r-1} g'(t) F(t) \in L_1([s,b], X)$, see [6].

Notice for above that we used

$$\int_s^b (g(t) - g(s))^{r-1} g'(t)\, dt = \frac{(g(b) - g(s))^r}{r}, \tag{65}$$

by [11], p. 107, exercise 13d.

1) Case $r \geq 1$. We use the definition of absolute continuity. So for every $\varepsilon > 0$ we need $\delta > 0$: whenever (a_i, b_i), $i = 1, ..., n$, are disjoint open subintervals of $[a, b]$, then

$$\sum_{i=1}^{n} (b_i - a_i) < \delta \Rightarrow \sum_{i=1}^{n} \| G(b_i) - G(a_i) \| < \varepsilon.$$

If $\|F\|_\infty = 0$, then $G(s) = 0$, for all $s \in [a, b]$, the trivial case and all fulfilled. So we assume $\|F\|_\infty \neq 0$. Hence we have (see [5])

$$G(b_i) - G(a_i) = \int_{b_i}^{b} (g(t) - g(b_i))^{r-1} g'(t) F(t) dt -$$

$$\int_{a_i}^{b} (g(t) - g(a_i))^{r-1} g'(t) F(t) dt =$$

$$\int_{b_i}^{b} (g(t) - g(b_i))^{r-1} g'(t) F(t) dt - \int_{a_i}^{b_i} (g(t) - g(a_i))^{r-1} g'(t) F(t) dt - \tag{66}$$

$$\int_{b_i}^{b} (g(t) - g(a_i))^{r-1} g'(t) F(t) dt =$$

(see [1], p. 426, Theorem 11.43)

$$\int_{b_i}^{b} \left((g(t) - g(b_i))^{r-1} - (g(t) - g(a_i))^{r-1} \right) g'(t) F(t) dt -$$

$$\int_{a_i}^{b_i} (g(t) - g(a_i))^{r-1} g'(t) F(t) dt.$$

Call

$$I_i := \int_{b_i}^{b} \left| (g(t) - g(b_i))^{r-1} - (g(t) - g(a_i))^{r-1} \right| g'(t) dt. \tag{67}$$

Thus

$$\| G(b_i) - G(a_i) \| \leq \left[I_i + \frac{(g(b_i) - g(a_i))^r}{r} \right] \|F\|_\infty := T_i. \tag{68}$$

If $r = 1$, then $I_i = 0$, and

$$\| G(b_i) - G(a_i) \| \leq \|F\|_\infty (g(b_i) - g(a_i)) \leq \|F\|_\infty \|g'\|_\infty (b_i - a_i), \tag{69}$$

for all $i = 1, ..., n$.

If $r > 1$, then because

$$\left[(g(t) - g(a_i))^{r-1} - (g(t) - g(b_i))^{r-1} \right] \geq 0,$$

for all $t \in [b_i, b]$, we find

$$I_i = \int_{b_i}^{b} \left((g(t) - g(a_i))^{r-1} - (g(t) - g(b_i))^{r-1} \right) g'(t)\, dt = \tag{70}$$

$$\frac{(g(b) - g(a_i))^r - (g(b_i) - g(a_i))^r - (g(b) - g(b_i))^r}{r} =$$

$$\frac{r(g(b) - \xi)^{r-1}(g(b_i) - g(a_i)) - (g(b_i) - g(a_i))^r}{r},$$

for some $\xi \in (g(a_i), g(b_i))$.

Therefore, it holds

$$I_i \leq \frac{r(g(b) - g(a))^{r-1}(g(b_i) - g(a_i)) - (g(b_i) - g(a_i))^r}{r}, \tag{71}$$

and

$$\left(I_i + \frac{(g(b_i) - g(a_i))^r}{r} \right) \leq (g(b) - g(a))^{r-1}(g(b_i) - g(a_i)). \tag{72}$$

That is

$$T_i \leq \|F\|_\infty (g(b) - g(a))^{r-1}(g(b_i) - g(a_i)), \tag{73}$$

so that

$$\|G(b_i) - G(a_i)\| \leq \|F\|_\infty (g(b) - g(a))^{r-1}(g(b_i) - g(a_i)) \leq$$

$$\|F\|_\infty (g(b) - g(a))^{r-1} \|g'\|_\infty (b_i - a_i), \tag{74}$$

for all $i = 1, ..., n$.

So in the case of $r = 1$, and by choosing $\delta := \frac{\varepsilon}{\|g'\|_\infty \|F\|_\infty}$, we get

$$\sum_{i=1}^{n} \|G(b_i) - G(a_i)\| \overset{(69)}{\leq} \|F\|_\infty \|g'\|_\infty \left(\sum_{i=1}^{n} (b_i - a_i) \right) \tag{75}$$

$$\leq \|F\|_\infty \|g'\|_\infty \delta = \varepsilon,$$

proving for $r = 1$ that G is absolutely continuous. In the case of $r > 1$, and by choosing $\delta := \frac{\varepsilon}{\|g'\|_\infty \|F\|_\infty (g(b) - g(a))^{r-1}}$, we get

$$\sum_{i=1}^{n} \|G(b_i) - G(a_i)\| \overset{(74)}{\leq} \|F\|_\infty \|g'\|_\infty (g(b) - g(a))^{r-1} \left(\sum_{i=1}^{n} (b_i - a_i) \right) \tag{76}$$

$$< \|F\|_\infty \|g'\|_\infty (g(b) - g(a))^{r-1} \delta = \varepsilon,$$

proving for $r > 1$ that G is absolutely continuous again.

2) Case of $0 < r < 1$. Let $a_{i^*}, b_{i^*} \in [a,b] : a_{i^*} \leq b_{i^*}$ and $g(a_{i^*}) \leq g(b_{i^*})$. Then $(g(t) - g(a_{i^*}))^{r-1} \leq (g(t) - g(b_{i^*}))^{r-1}$, for all $t \in (b_{i^*}, b]$. Then

$$I_{i^*} = \int_{b_{i^*}}^{b} \left((g(t) - g(b_{i^*}))^{r-1} - (g(t) - g(a_{i^*}))^{r-1} \right) g'(t)\, dt =$$

$$\frac{(g(b) - g(b_{i^*}))^r}{r} - \left(\frac{(g(b) - g(a_{i^*}))^r - (g(b_{i^*}) - g(a_{i^*}))^r}{r} \right) \leq \tag{77}$$

$$\frac{(g(b_{i^*}) - g(a_{i^*}))^r}{r} \leq \frac{\|g'\|_\infty^r (b_{i^*} - a_{i^*})^r}{r},$$

by $\left[(g(b) - g(b_{i^*}))^r - (g(b) - g(a_{i^*}))^r \right] < 0$.

Therefore

$$I_{i^*} \leq \frac{\|g'\|_\infty^r (b_{i^*} - a_{i^*})^r}{r}, \tag{78}$$

and

$$T_{i^*} \leq \frac{2 \|F\|_\infty \|g'\|_\infty^r (b_{i^*} - a_{i^*})^r}{r}, \tag{79}$$

proving that

$$\|G(b_{i^*}) - G(a_{i^*})\| \leq \frac{2 \|F\|_\infty \|g'\|_\infty^r}{r} (b_{i^*} - a_{i^*})^r. \tag{80}$$

The last inequality proves that G is continuous for $r \in (0,1)$. The theorem is proved. ∎

We need

Definition 12 *Let $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ a Banach space, $g \in C^1([a,b])$ and increasing, $f \in C([a,b], X)$, $\nu > 0$.*

We define the left Riemann-Liouville generalized fractional Bochner integral operator

$$\left(J_{a;g}^{\nu} f \right)(x) := \frac{1}{\Gamma(\nu)} \int_a^x (g(x) - g(z))^{\nu-1} g'(z) f(z)\, dz, \tag{81}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function.

The last integral is of Bochner type. Since $f \in C([a,b], X)$, then $f \in L_\infty([a,b], X)$. By Theorem 10 we get that $\left(J_{a;g}^{\nu} f \right) \in C([a,b], X)$. Above we set $J_{a;g}^{0} f := f$ and see that $\left(J_{a;g}^{\nu} f \right)(a) = 0$.

We derive

Theorem 13 *Let all as in Definition 12. Let $m, n > 0$ and $f \in C([a,b], X)$. Then*

$$J_{a;g}^{m} J_{a;g}^{n} f = J_{a;g}^{m+n} f = J_{a;g}^{n} J_{a;g}^{m} f. \tag{82}$$

Proof. Here $a \leq x \leq b$. We have

$$\left(J_{a;g}^{m} J_{a;g}^{n} f\right)(x) = \frac{1}{\Gamma(m)\Gamma(n)} \cdot \tag{83}$$

$$\int_a^x (g(x) - g(t))^{m-1} g'(t) \left(\int_a^t (g(t) - g(\tau))^{n-1} g'(\tau) f(\tau) d\tau \right) dt =$$

$$\frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \int_a^x \chi_{[a,t]}(\tau) (g(x) - g(t))^{m-1} (g(t) - g(\tau))^{n-1} g'(t) g'(\tau) f(\tau) d\tau dt$$

(here $\chi_{[a,t]}$ is the characteristic function, we use Fubini's theorem from [9], p. 93, Theorem 2)

$$= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \int_a^x \chi_{[\tau,x]}(t) (g(x) - g(t))^{m-1} (g(t) - g(\tau))^{n-1} g'(t) g'(\tau) f(\tau) d\tau dt \tag{84}$$

$$= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x f(\tau) g'(\tau) \left(\int_\tau^x (g(x) - g(t))^{m-1} (g(t) - g(\tau))^{n-1} g'(t) dt \right) d\tau$$

(by [10])

$$= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x f(\tau) g'(\tau) \left(\int_{g(\tau)}^{g(x)} (g(x) - z)^{m-1} (z - g(\tau))^{n-1} dz \right) d\tau$$

$$= \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x f(\tau) g'(\tau) \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} (g(x) - g(\tau))^{m+n-1} d\tau$$

$$= \frac{1}{\Gamma(m+n)} \int_a^x (g(x) - g(\tau))^{m+n-1} g'(\tau) f(\tau) d\tau = \left(J_{a;g}^{m+n} f\right)(x), \tag{85}$$

proving the claim. ■

We need

Definition 14 *Let $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ a Banach space, $g \in C^1([a,b])$ and increasing, $f \in C([a,b], X)$, $\nu > 0$.*

We define the right Riemann-Liouville generalized fractional Bochner integral operator

$$\left(J_{b-;g}^{\nu} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_x^b (g(z) - g(x))^{\nu-1} g'(z) f(z) dz, \tag{86}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function.

The last integral is of Bochner type. Since $f \in C([a,b], X)$, then $f \in L_\infty([a,b], X)$. By Theorem 11 we get that $\left(J_{b-;g}^{\nu} f\right) \in C([a,b], X)$. Above we set $J_{b-;g}^{0} f := f$ and see that $\left(J_{b-;g}^{\nu} f\right)(b) = 0$.

We derive

Theorem 15 *Let all as in Definition 14. Let $\alpha, \beta > 0$ and $f \in C([a,b], X)$. Then*

$$\left(J_{b-;g}^{\alpha} J_{b-;g}^{\beta} f\right)(x) = \left(J_{b-;g}^{\alpha+\beta} f\right)(x) = \left(J_{b-;g}^{\beta} J_{b-;g}^{\alpha} f\right)(x), \tag{87}$$

$\forall\, x \in [a,b]$.

Proof. We have that

$$\left(J_{b-;g}^{\alpha} J_{b-;g}^{\beta} f\right)(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \cdot \tag{88}$$

$$\int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \left(\int_t^b (g(\tau) - g(t))^{\beta-1} g'(\tau) f(\tau) d\tau\right) dt =$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_x^b \chi_{[t,b]}(\tau) (g(t) - g(x))^{\alpha-1} (g(\tau) - g(t))^{\beta-1} g'(t) g'(\tau) f(\tau) d\tau dt$$

(here $\chi_{[t,b]}$ is the characteristic function, we use Fubini's theorem from [9], p. 93, Theorem 2)

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \left(\int_x^\tau (g(t) - g(x))^{\alpha-1} (g(\tau) - g(t))^{\beta-1} g'(t) g'(\tau) f(\tau) dt\right) d\tau$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b f(\tau) g'(\tau) \left(\int_x^\tau (g(\tau) - g(t))^{\beta-1} (g(t) - g(x))^{\alpha-1} g'(t) dt\right) d\tau \tag{89}$$

(by [10])

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b f(\tau) g'(\tau) \left(\int_{g(x)}^{g(\tau)} (g(\tau) - z)^{\beta-1} (z - g(x))^{\alpha-1} dz\right) d\tau$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b f(\tau) g'(\tau) \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} (g(\tau) - g(x))^{\alpha+\beta-1} d\tau$$

$$= \frac{1}{\Gamma(\alpha+\beta)} \int_x^b (g(\tau) - g(x))^{\alpha+\beta-1} g'(\tau) f(\tau) d\tau = \left(J_{b-;g}^{\alpha+\beta} f\right)(x), \tag{90}$$

proving the claim. ■

We need

Definition 16 *Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$.*

We define the left generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{a+;g}^{\alpha} f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x) - g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t)) dt, \tag{91}$$

$\forall\ x \in [a,b]$. *The last integral is of Bochner type.*

If $\alpha \notin \mathbb{N}$, *by Theorem 10, we have that* $\left(D^{\alpha}_{a+;g} f\right) \in C\left([a,b], X\right)$.

We see that

$$\left(J^{n-\alpha}_{a;g}\left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = \left(D^{\alpha}_{a+;g} f\right)(x), \quad \forall\ x \in [a,b]. \tag{92}$$

We set

$$D^{n}_{a+;g} f(x) := \left(\left(f \circ g^{-1}\right)^{n} \circ g\right)(x) \in C\left([a,b], X\right), \quad n \in \mathbb{N}, \tag{93}$$

$$D^{0}_{a+;g} f(x) = f(x), \quad \forall\ x \in [a,b].$$

When $g = id$, *then*

$$D^{\alpha}_{a+;g} f = D^{\alpha}_{a+;id} f = D^{\alpha}_{*a} f, \tag{94}$$

the usual left X*-valued Caputo fractional derivative, see [4].*

We need

Definition 17 *Let* $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ *the ceiling of the number. Let* $f \in C^{n}\left([a,b], X\right)$, *where* $[a,b] \subset \mathbb{R}$, *and* $\left(X, \|\cdot\|\right)$ *is a Banach space. Let* $g \in C^{1}\left([a,b]\right)$, *strictly increasing, such that* $g^{-1} \in C^{n}\left([g(a), g(b)]\right)$.

We define the right generalized g*-fractional derivative* X*-valued of* f *of order* α *as follows:*

$$\left(D^{\alpha}_{b-;g} f\right)(x) := \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{x}^{b} \left(g(t) - g(x)\right)^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} \left(g(t)\right) dt, \tag{95}$$

$\forall\ x \in [a,b]$. *The last integral is of Bochner type.*

If $\alpha \notin \mathbb{N}$, *by Theorem 11, we have that* $\left(D^{\alpha}_{b-;g} f\right) \in C\left([a,b], X\right)$.

We see that

$$J^{n-\alpha}_{b-;g}\left((-1)^{n}\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x) = \left(D^{\alpha}_{b-;g} f\right)(x), \quad a \leq x \leq b. \tag{96}$$

We set

$$D^{n}_{b-;g} f(x) := (-1)^{n}\left(\left(f \circ g^{-1}\right)^{n} \circ g\right)(x) \in C\left([a,b], X\right), \quad n \in \mathbb{N}, \tag{97}$$

$$D^{0}_{b-;g} f(x) := f(x), \quad \forall\ x \in [a,b].$$

When $g = id$, *then*

$$D^{\alpha}_{b-;g} f(x) = D^{\alpha}_{b-;id} f(x) = D^{\alpha}_{b-} f, \tag{98}$$

the usual right X*-valued Caputo fractional derivative, see [2]*

We give

Theorem 18 *Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, $a \leq x \leq b$. Then*

$$f(x) = f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) +$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D_{a+;g}^{\alpha} f\right)(t) \, dt =$$

$$f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) + \tag{99}$$

$$\frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} \left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z) \, dz.$$

Proof. We have that

$$\left(J_{a;g}^{\alpha} D_{a+;g}^{\alpha} f\right)(x) = \left(J_{a;g}^{\alpha} \left(J_{a;g}^{n-\alpha} \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)\right)(x) =$$

$$\left(J_{a;g}^{\alpha+n-\alpha} \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = \left(J_{a;g}^{n} \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = \tag{100}$$

$$\frac{1}{(n-1)!} \int_a^x (g(x) - g(t))^{n-1} g'(t) \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(t) \, dt.$$

We have proved that

$$\left(J_{a;g}^{\alpha} D_{a+;g}^{\alpha} f\right)(x) = \frac{1}{(n-1)!} \int_a^x (g(x) - g(t))^{n-1} g'(t) \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(t) \, dt \tag{101}$$

$$= R_n(a, x, g), \quad \text{all } a \leq x \leq b.$$

But also it holds

$$R_n(a, x, g) = \left(J_{a;g}^{\alpha} D_{a+;g}^{\alpha} f\right)(x) = \tag{102}$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D_{a+;g}^{\alpha} f\right)(t) \, dt,$$

all $a \leq x \leq b$, proving the claim. ∎

We give

Theorem 19 *Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, $a \leq x \leq b$. Then*

$$f(x) = f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) +$$

$$\frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1} g'(t)\left(D_{b-;g}^{\alpha}f\right)(t)\,dt =$$

$$f(b)+\sum_{i=1}^{n-1}\frac{(g(x)-g(b))^i}{i!}\left(f\circ g^{-1}\right)^{(i)}(g(b))+ \tag{103}$$

$$\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)} (z-g(x))^{\alpha-1}\left(\left(D_{b-;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz.$$

Proof. We have that

$$\left(J_{b-;g}^{\alpha}D_{b-;g}^{\alpha}f\right)(x)=(-1)^n\left(J_{b-;g}^{\alpha}\left(J_{b-;g}^{n-\alpha}\left(\left(f\circ g^{-1}\right)^{(n)}\circ g\right)\right)\right)(x)= \tag{104}$$

$$(-1)^n\left(J_{b-;g}^{\alpha+n-\alpha}\left(\left(f\circ g^{-1}\right)^{(n)}\circ g\right)\right)(x)=(-1)^n\left(J_{b-;g}^{n}\left(\left(f\circ g^{-1}\right)^{(n)}\circ g\right)\right)(x)=$$

$$(-1)^n\frac{1}{(n-1)!}\int_x^b (g(t)-g(x))^{n-1} g'(t)\left(\left(f\circ g^{-1}\right)^{(n)}\circ g\right)(t)\,dt=$$

$$\frac{(-1)^{2n}}{(n-1)!}\int_b^x (g(x)-g(t))^{n-1} g'(t)\left(\left(f\circ g^{-1}\right)^{(n)}\circ g\right)(t)\,dt=$$

$$\frac{1}{(n-1)!}\int_b^x (g(x)-g(t))^{n-1} g'(t)\left(\left(f\circ g^{-1}\right)^{(n)}\circ g\right)(t)\,dt=R_n(b,x,g).$$

That is

$$R_n(b,x,g)=\left(J_{b-;g}^{\alpha}D_{b-;g}^{\alpha}f\right)(x)= \tag{105}$$

$$\frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1} g'(t)\left(D_{b-;g}^{\alpha}f\right)(t)\,dt,$$

all $a\leq x\leq b$, proving the claim. ■

Let $g:[a,b]\to\mathbb{R}$ be a strictly increasing function. Let $f\in C^n([a,b],X)$, $(X,\|\cdot\|)$ is a Banach space, $n\in\mathbb{N}$. Assume that $g\in C^1([a,b])$, and $g^{-1}\in C^n([g(a),g(b)])$. Call $l:=f\circ g^{-1}:[g(a),g(b)]\to X$. It is clear that $l,l',...,l^{(n)}$ are continuous functions from $[g(a),g(b)]$ into $f([a,b])\subseteq X$.

Let $\nu\geq 1$ such that $[\nu]=n$, $n\in\mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0<\nu<1$, $[\nu]=0$. Next we follow [3].

I) Let $h\in C([g(a),g(b)],X)$, we define the left Riemann-Liouville Bochner fractional integral as

$$\left(J_{\nu}^{z_0}h\right)(z):=\frac{1}{\Gamma(\nu)}\int_{z_0}^{z}(z-t)^{\nu-1}h(t)\,dt, \tag{106}$$

for $g(a)\leq z_0\leq z\leq g(b)$, where Γ is the gamma function; $\Gamma(\nu)=\int_0^{\infty}e^{-t}t^{\nu-1}dt$.

We set $J_0^{z_0}h=h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C^{\nu}_{g(x_0)}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a,b]$: $C^{\nu}_{g(x_0)}([g(a), g(b)], X) =$
$\left\{ h \in C^{[\nu]}([g(a), g(b)], X) : J^{g(x_0)}_{1-\alpha} h^{([\nu])} \in C^1([g(x_0), g(b)], X) \right\}$.

So let $h \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, we define the left g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(x_0), g(b)]$ as

$$D^{\nu}_{g(x_0)} h := \left(J^{g(x_0)}_{1-\alpha} h^{([\nu])} \right)'. \tag{107}$$

Clearly, for $h \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, there exists

$$\left(D^{\nu}_{g(x_0)} h \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z-t)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{108}$$

for all $g(x_0) \leq z \leq g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, we have that

$$\left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z-t)^{-\alpha} \left(f \circ g^{-1} \right)^{([\nu])}(t)\, dt, \tag{109}$$

for all $g(x_0) \leq z \leq g(b)$. We have that $D^{n}_{g(x_0)}\left(f \circ g^{-1} \right) = \left(f \circ g^{-1} \right)^{(n)}$ and $D^{0}_{g(x_0)}\left(f \circ g^{-1} \right) = f \circ g^{-1}$, see [3].

By [3], we have for $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ is fixed, that

(i) if $\nu \geq 1$, then

$$\left(f \circ g^{-1} \right)(z) = \sum_{k=0}^{[\nu]-1} \frac{\left(f \circ g^{-1} \right)^{(k)}(g(x_0))}{k!} (z - g(x_0))^k +$$
$$\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{z} (z-t)^{\nu-1} \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(t)\, dt, \tag{110}$$

all $z \in [g(a), g(b)] : z \geq g(x_0)$,

(ii) if $0 < \nu < 1$, we get

$$\left(f \circ g^{-1} \right)(z) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{z} (z-t)^{\nu-1} \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(t)\, dt, \tag{111}$$

all $z \in [g(a), g(b)] : z \geq g(x_0)$.

We have proved the following left generalized g-fractional, of Canavati type, X-valued Taylor's formula:

Theorem 20 *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ is fixed.*

(i) if $\nu \geq 1$*, then*

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!}(g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^{\nu}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{112}$$

all $x \in [a, b] : x \geq x_0$*,*

(ii) if $0 < \nu < 1$*, we get*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^{\nu}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{113}$$

all $x \in [a, b] : x \geq x_0$*.*

II) Let $h \in C([g(a), g(b)], X)$, we define the right Riemann-Liouville Bochner fractional integral as

$$\left(J_{z_0-}^{\nu} h\right)(z) := \frac{1}{\Gamma(\nu)} \int_z^{z_0} (t - z)^{\nu-1} h(t)\, dt, \tag{114}$$

for $g(a) \leq z \leq z_0 \leq g(b)$. We set $J_{z_0-}^{0} h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a, b]$:

$$C_{g(x_0)-}^{\nu}([g(a), g(b)], X) :=$$

$$\left\{h \in C^{[\nu]}([g(a), g(b)], X) : J_{g(x_0)-}^{1-\alpha} h^{([\nu])} \in C^1([g(a), g(x_0)], X)\right\}. \tag{115}$$

So let $h \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, we define the right g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(a), g(x_0)]$ as

$$D_{g(x_0)-}^{\nu} h := (-1)^{n-1} \left(J_{g(x_0)-}^{1-\alpha} h^{([\nu])}\right)'. \tag{116}$$

Clearly, for $h \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, there exists

$$\left(D_{g(x_0)-}^{\nu} h\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^{g(x_0)} (t - z)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{117}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, we have that

$$\left(D_{g(x_0)-}^{\nu}\left(f \circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^{g(x_0)} (t - z)^{-\alpha} \left(f \circ g^{-1}\right)^{([\nu])}(t)\, dt, \tag{118}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

We get that

$$\left(D_{g(x_0)-}^{n}\left(f \circ g^{-1}\right)\right)(z) = (-1)^n \left(f \circ g^{-1}\right)^{(n)}(z) \tag{119}$$

and $\left(D_{g(x_0)-}^{0}\left(f \circ g^{-1}\right)\right)(z) = \left(f \circ g^{-1}\right)(z)$, all $z \in [g(a), g(b)]$, see [3].

By [3], we have for $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ is fixed, that

(i) if $\nu \geq 1$, then

$$\left(f \circ g^{-1}\right)(z) = \sum_{k=0}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!}(z - g(x_0))^k + \tag{120}$$

$$\frac{1}{\Gamma(\nu)} \int_z^{g(x_0)} (t-z)^{\nu-1} \left(D_{g(x_0)-}^{\nu}\left(f \circ g^{-1}\right)\right)(t)\, dt,$$

all $z \in [g(a), g(b)] : z \leq g(x_0)$,

(ii) if $0 < \nu < 1$, we get

$$\left(f \circ g^{-1}\right)(z) = \frac{1}{\Gamma(\nu)} \int_z^{g(x_0)} (t-z)^{\nu-1} \left(D_{g(x_0)-}^{\nu}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{121}$$

all $z \in [g(a), g(b)] : z \leq g(x_0)$.

We have proved the following right generalized g-fractional, of Canavati type, X-valued Taylor's formula:

Theorem 21 *Let $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ is fixed.*

(i) if $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!}(g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D_{g(x_0)-}^{\nu}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{122}$$

all $a \leq x \leq x_0$,

(ii) if $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D_{g(x_0)-}^{\nu}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{123}$$

all $a \leq x \leq x_0$.

III) Denote by

$$D_{g(x_0)}^{m\nu} = D_{g(x_0)}^{\nu} D_{g(x_0)}^{\nu} ... D_{g(x_0)}^{\nu} \quad (m\text{-times}),\ m \in \mathbb{N}. \tag{124}$$

Also denote by

$$J_{m\nu}^{g(x_0)} = J_{\nu}^{g(x_0)} J_{\nu}^{g(x_0)} ... J_{\nu}^{g(x_0)} \quad (m\text{-times}),\ m \in \mathbb{N}. \tag{125}$$

We need

Theorem 22 *Here* $0 < \nu < 1$. *Assume that* $\left(D_{g(x_0)}^{m\nu}\left(f \circ g^{-1}\right)\right) \in C_{g(x_0)}^{\nu}\left(\left[g\left(a\right), g\left(b\right)\right], X\right)$, *where* $x_0 \in [a, b]$ *is fixed. Then*

$$\left(J_{m\nu}^{g(x_0)} D_{g(x_0)}^{m\nu}\left(f \circ g^{-1}\right)\right)\left(g\left(x\right)\right) - \left(J_{(m+1)\nu}^{g(x_0)} D_{g(x_0)}^{(m+1)\nu}\left(f \circ g^{-1}\right)\right)\left(g\left(x\right)\right) = 0, \tag{126}$$

for all $x_0 \leq x \leq b$.

Proof. We observe that ($l := f \circ g^{-1}$)

$$\left(J_{m\nu}^{g(x_0)} D_{g(x_0)}^{m\nu}\left(l\right)\right)\left(g\left(x\right)\right) - \left(J_{(m+1)\nu}^{g(x_0)} D_{g(x_0)}^{(m+1)\nu}\left(l\right)\right)\left(g\left(x\right)\right) =$$

$$\left(J_{m\nu}^{g(x_0)}\left(D_{g(x_0)}^{m\nu}\left(l\right) - J_{\nu}^{g(x_0)} D_{g(x_0)}^{(m+1)\nu}\left(l\right)\right)\right)\left(g\left(x\right)\right) = \tag{127}$$

$$\left(J_{m\nu}^{g(x_0)}\left(D_{g(x_0)}^{m\nu}\left(l\right) - \left(J_{\nu}^{g(x_0)} D_{g(x_0)}^{\nu}\right)\left(\left(D_{g(x_0)}^{m\nu}\left(l\right)\right) \circ g \circ g^{-1}\right)\right)\right)\left(g\left(x\right)\right) =$$

$$\left(J_{m\nu}^{g(x_0)}\left(D_{g(x_0)}^{m\nu}\left(l\right) - \left(D_{g(x_0)}^{m\nu}\left(l\right)\right)\right)\right)\left(g\left(x\right)\right) = \left(J_{m\nu}^{g(x_0)}\left(0\right)\right)\left(g\left(x\right)\right) = 0.$$

■

We make

Remark 23 *Let* $0 < \nu < 1$. *Assume that* $\left(D_{g(x_0)}^{i\nu}\left(f \circ g^{-1}\right)\right) \in C_{g(x_0)}^{\nu}\left(\left[g\left(a\right), g\left(b\right)\right], X\right)$, $x_0 \in [a, b]$, *for all* $i = 0, 1, ..., m$. *We have that*

$$\sum_{i=0}^{m}\left[\left(J_{i\nu}^{g(x_0)} D_{g(x_0)}^{i\nu}\left(f \circ g^{-1}\right)\right)\left(g\left(x\right)\right) - \left(J_{(i+1)\nu}^{g(x_0)} D_{g(x_0)}^{(i+1)\nu}\left(f \circ g^{-1}\right)\right)\left(g\left(x\right)\right)\right] = 0. \tag{128}$$

Hence it holds

$$f\left(x\right) - \left(J_{(m+1)\nu}^{g(x_0)} D_{g(x_0)}^{(m+1)\nu}\left(f \circ g^{-1}\right)\right)\left(g\left(x\right)\right) = 0, \tag{129}$$

for all $x_0 \leq x \leq b$.

That is

$$f\left(x\right) = \left(J_{(m+1)\nu}^{g(x_0)} D_{g(x_0)}^{(m+1)\nu}\left(f \circ g^{-1}\right)\right)\left(g\left(x\right)\right), \tag{130}$$

for all $x_0 \leq x \leq b$.

We have proved the following modified and generalized left X-valued fractional Taylor's formula of Canavati type:

Theorem 24 *Let $0 < \nu < 1$. Assume that $\left(D^{i\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)}\left(\left[g\left(a\right), g\left(b\right)\right], X\right)$, $x_0 \in [a, b]$, for $i = 0, 1, ..., m$. Then*

$$f\left(x\right) = \frac{1}{\Gamma\left(\left(m+1\right)\nu\right)} \int_{g(x_0)}^{g(x)} \left(g\left(x\right) - z\right)^{(m+1)\nu - 1} \left(D^{(m+1)\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)\left(z\right) dz, \tag{131}$$

all $x_0 \leq x \leq b$.

IV) Denote by

$$D^{m\nu}_{g(x_0)-} = D^{\nu}_{g(x_0)-} D^{\nu}_{g(x_0)-} ... D^{\nu}_{g(x_0)-} \quad (m \text{ times}),\ m \in \mathbb{N}. \tag{132}$$

Also denote by

$$J^{m\nu}_{g(x_0)-} = J^{\nu}_{g(x_0)-} J^{\nu}_{g(x_0)-} ... J^{\nu}_{g(x_0)-} \quad (m \text{ times}),\ m \in \mathbb{N}. \tag{133}$$

We need

Theorem 25 *Let $0 < \nu < 1$. Assume that $\left(D^{m\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)-}\left(\left[g\left(a\right), g\left(b\right)\right], X\right)$, where $x_0 \in [a, b]$ is fixed. Then*

$$\left(J^{m\nu}_{g(x_0)-} D^{m\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)\left(g\left(x\right)\right) - \left(J^{(m+1)\nu}_{g(x_0)-} D^{(m+1)\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)\left(g\left(x\right)\right) = 0, \tag{134}$$

for all $a \leq x \leq x_0$.

Proof. We observe that $\left(l := f \circ g^{-1}\right)$

$$\left(J^{m\nu}_{g(x_0)-} D^{m\nu}_{g(x_0)-}\left(l\right)\right)\left(g\left(x\right)\right) - \left(J^{(m+1)\nu}_{g(x_0)-} D^{(m+1)\nu}_{g(x_0)-}\left(l\right)\right)\left(g\left(x\right)\right) =$$

$$\left(J^{m\nu}_{g(x_0)-}\left(D^{m\nu}_{g(x_0)-}\left(l\right)\right) - J^{\nu}_{g(x_0)-} D^{(m+1)\nu}_{g(x_0)-}\left(l\right)\right)\left(g\left(x\right)\right) =$$

$$\left(J^{m\nu}_{g(x_0)-}\left(D^{m\nu}_{g(x_0)-}\left(l\right) - \left(J^{\nu}_{g(x_0)-} D^{\nu}_{g(x_0)-}\right)\left(\left(D^{m\nu}_{g(x_0)-}\left(l\right)\right) \circ g \circ g^{-1}\right)\right)\right)\left(g\left(x\right)\right) = \tag{135}$$

$$\left(J^{m\nu}_{g(x_0)-}\left(D^{m\nu}_{g(x_0)-}\left(l\right) - D^{m\nu}_{g(x_0)-}\left(l\right)\right)\right)\left(g\left(x\right)\right) = J^{m\nu}_{g(x_0)-}\left(0\right)\left(g\left(x\right)\right) = 0.$$

■

We make

Remark 26 *Let $0 < \nu < 1$. Assume that $\left(D^{i\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)-}\left(\left[g\left(a\right), g\left(b\right)\right], X\right)$, $x_0 \in [a, b]$, for all $i = 0, 1, ..., m$. We have that (by (134))*

$$\sum_{i=0}^{m} \left[\left(J^{i\nu}_{g(x_0)-} D^{i\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)\left(g\left(x\right)\right) - \left(J^{(i+1)\nu}_{g(x_0)-} D^{(i+1)\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)\left(g\left(x\right)\right)\right] = 0. \tag{136}$$

Hence it holds

$$f(x)-\left(J_{g(x_0)-}^{(m+1)\nu}D_{g(x_0)-}^{(m+1)\nu}\left(f\circ g^{-1}\right)\right)(g(x))=0, \tag{137}$$

for all $a\leq x\leq x_0\leq b$.

That is

$$f(x)=\left(J_{g(x_0)-}^{(m+1)\nu}D_{g(x_0)-}^{(m+1)\nu}\left(f\circ g^{-1}\right)\right)(g(x)), \tag{138}$$

for all $a\leq x\leq x_0\leq b$.

We have proved the following modified and generalized right X-valued fractional Taylor's formula of Canavati type:

Theorem 27 *Let* $0<\nu<1$. *Assume that* $\left(D_{g(x_0)-}^{i\nu}\left(f\circ g^{-1}\right)\right)\in C_{g(x_0)-}^{\nu}([g(a),g(b)],X)$, $x_0\in[a,b]$, *for all* $i=0,1,...,m$. *Then*

$$f(x)=\frac{1}{\Gamma((m+1)\nu)}\int_{g(x)}^{g(x_0)}(z-g(x))^{(m+1)\nu-1}\left(D_{g(x_0)-}^{(m+1)\nu}\left(f\circ g^{-1}\right)\right)(z)\,dz, \tag{139}$$

all $a\leq x\leq x_0\leq b$.

From Theorem 18 when $0<\alpha\leq 1$, we get that

$$\begin{gathered}\left(I_{a+;g}^{\alpha}D_{a+;g}^{\alpha}f\right)(x)=f(x)-f(a)=\\ \frac{1}{\Gamma(\alpha)}\int_a^x(g(x)-g(t))^{\alpha-1}g'(t)\left(D_{a+;g}^{\alpha}f\right)(t)\,dt=\\ \frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D_{a+;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz,\end{gathered} \tag{140}$$

and by Theorem 19 when $0<\alpha\leq 1$ we get

$$\begin{gathered}\left(I_{b-;g}^{\alpha}D_{b-;g}^{\alpha}f\right)(x)=f(x)-f(b)=\\ \frac{1}{\Gamma(\alpha)}\int_x^b(g(t)-g(x))^{\alpha-1}g'(t)\left(D_{b-;g}^{\alpha}f\right)(t)\,dt=\\ \frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)}(z-g(x))^{\alpha-1}\left(\left(D_{b-;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz,\end{gathered} \tag{141}$$

all $a\leq x\leq b$.

Above we considered $f\in C^1([a,b],X)$, $g\in C^1([a,b])$, strictly increasing, such that $g^{-1}\in C^1([g(a),g(b)])$.

Denote by

$$D_{a+;g}^{n\alpha}:=D_{a+;g}^{\alpha}D_{a+;g}^{\alpha}...D_{a+;g}^{\alpha}\quad(n\text{ times}),\ n\in\mathbb{N}. \tag{142}$$

Also denote by

$$I_{a+;g}^{n\alpha} := I_{a+;g}^{\alpha} I_{a+;g}^{\alpha} ... I_{a+;g}^{\alpha} \quad (n \text{ times}). \tag{143}$$

Here to remind

$$\left(I_{a+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) f(t) \, dt, \quad x \geq a. \tag{144}$$

By convention $I_{a+;g}^{0} = D_{a+;g}^{0} = I$ (identity operator).

We need

Theorem 28 *Let $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$. Let $F_k := D_{a+;g}^{k\alpha} f$, $k = n, n+1$, that fulfill $F_k \in C([a,b], X)$, and $F_n \in C^1([a,b], X)$.*

Then

$$\left(I_{a+;g}^{n\alpha} D_{a+;g}^{n\alpha} f\right)(x) - \left(I_{a+;g}^{(n+1)\alpha} D_{a+;g}^{(n+1)\alpha} f\right)(x) = \frac{(g(x) - g(a))^{n\alpha}}{\Gamma(n\alpha + 1)} \left(D_{a+;g}^{n\alpha} f\right)(a), \tag{145}$$

all $x \in [a,b]$.

Proof. By semigroup property of $I_{a+;g}^{\alpha}$, we get

$$\left(I_{a+;g}^{n\alpha} D_{a+;g}^{n\alpha} f\right)(x) - \left(I_{a+;g}^{(n+1)\alpha} D_{a+;g}^{(n+1)\alpha} f\right)(x) =$$

$$\left(I_{a+;g}^{n\alpha} \left(D_{a+;g}^{n\alpha} f - I_{a+;g}^{\alpha} D_{a+;g}^{(n+1)\alpha} f\right)\right)(x) =$$

$$\left(I_{a+;g}^{n\alpha} \left(D_{a+;g}^{n\alpha} f - \left(I_{a+;g}^{\alpha} D_{a+;g}^{\alpha}\right)\left(D_{a+;g}^{n\alpha} f\right)\right)\right)(x) \overset{(140)}{=}$$

$$\left(I_{a+;g}^{n\alpha} \left(D_{a+;g}^{n\alpha} f - D_{a+;g}^{n\alpha} f + D_{a+;g}^{n\alpha} f(a)\right)\right)(x) =$$

$$\left(I_{a+;g}^{n\alpha} \left(D_{a+;g}^{n\alpha} f(a)\right)\right)(x) = \left(D_{a+;g}^{n\alpha} f(a)\right)\left(I_{a+;g}^{n\alpha}(1)\right)(x) = \tag{146}$$

[notice that

$$\left(I_{a+;g}^{\alpha} 1\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \, dt = \frac{(g(x) - g(a))^{\alpha}}{\Gamma(\alpha+1)}. \tag{147}$$

Hence

$$\left(I_{a+;g}^{2\alpha} 1\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \, dt \frac{(g(t) - g(a))^{\alpha}}{\Gamma(\alpha+1)} dt =$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) (g(t) - g(a))^{\alpha} \, dt =$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} (z - g(a))^{(\alpha+1)-1} \, dt = \tag{148}$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)}\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}(g(x)-g(a))^{2\alpha}.$$

That is

$$\left(I_{a+;g}^{2\alpha}1\right)(x)=\frac{(g(x)-g(a))^{2\alpha}}{\Gamma(2\alpha+1)},$$

etc.]

$$=\left(D_{a+;g}^{n\alpha}f(a)\right)\frac{(g(x)-g(a))^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{149}$$

proving the claim. ■

We make

Remark 29 *Suppose that* $F_k=D_{a+;g}^{k\alpha}f$*, for* $k=1,...,n+1$*; are as in Theorem 28,* $0<\alpha\leq 1$*. By (145) we get*

$$\sum_{i=0}^{n}\left(\left(I_{a+;g}^{i\alpha}D_{a+;g}^{i\alpha}f\right)(x)-I_{a+;g}^{(i+1)\alpha}D_{a+;g}^{(i+1)\alpha}f(x)\right)= \tag{150}$$

$$\sum_{i=0}^{n}\frac{(g(x)-g(a))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{a+;g}^{i\alpha}f\right)(a).$$

That is

$$f(x)-\left(I_{a+;g}^{(n+1)\alpha}D_{a+;g}^{(n+1)\alpha}f\right)(x)=\sum_{i=0}^{n}\frac{(g(x)-g(a))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{a+;g}^{i\alpha}f\right)(a). \tag{151}$$

Hence

$$f(x)=\sum_{i=0}^{n}\frac{(g(x)-g(a))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{a+;g}^{i\alpha}f\right)(a)+\left(I_{a+;g}^{(n+1)\alpha}D_{a+;g}^{(n+1)\alpha}f\right)(x)= \tag{152}$$

$$\sum_{i=0}^{n}\frac{(g(x)-g(a))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{a+;g}^{i\alpha}f\right)(a)+R_g(a,x), \tag{153}$$

where

$$R_g(a,x):=\frac{1}{\Gamma((n+1)\alpha)}\int_a^x(g(x)-g(t))^{(n+1)\alpha-1}g'(t)\left(D_{a+;g}^{(n+1)\alpha}f\right)(t)\,dt. \tag{154}$$

(there $D_{a+;g}^{(n+1)\alpha}f$ *is continuous over* $[a,b]$*.)*

We have proved the following g-left generalized modified X-valued Taylor's formula.

Theorem 30 *Let $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$. Let $F_k := D^{k\alpha}_{a+;g} f$, $k = 1, ..., n$, that fulfill $F_k \in C^1([a,b], X)$, and $F_{n+1} \in C([a,b], X)$.*

Then

$$f(x) = \sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D^{i\alpha}_{a+;g} f\right)(a) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (g(x) - g(t))^{(n+1)\alpha - 1} g'(t) \left(D^{(n+1)\alpha}_{a+;g} f\right)(t)\, dt, \tag{155}$$

$\forall\ x \in [a,b]$.

Denote by

$$D^{n\alpha}_{b-;g} := D^{\alpha}_{b-;g} D^{\alpha}_{b-;g} ... D^{\alpha}_{b-;g} \quad (n \text{ times}),\ n \in \mathbb{N}. \tag{156}$$

Also denote by

$$I^{n\alpha}_{b-;g} := I^{\alpha}_{b-;g} I^{\alpha}_{b-;g} ... I^{\alpha}_{b-;g} \quad (n \text{ times}). \tag{157}$$

Here to remind

$$\left(I^{\alpha}_{b-;g} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha - 1} g'(t) f(t)\, dt, \quad x \leq b. \tag{158}$$

We need

Theorem 31 *Let $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$. Suppose that $F_k := D^{k\alpha}_{b-;g} f$, $k = n, n+1$, fulfill $F_k \in C([a,b], X)$, and $F_n \in C^1([a,b], X)$, $0 < \alpha \leq 1$, $n \in \mathbb{N}$.*

Then

$$\left(I^{n\alpha}_{b-;g} D^{n\alpha}_{b-;g} f\right)(x) - \left(I^{(n+1)\alpha}_{b-;g} D^{(n+1)\alpha}_{b-;g} f\right)(x) = \frac{(g(b) - g(x))^{n\alpha}}{\Gamma(n\alpha + 1)} \left(D^{n\alpha}_{b-;g} f\right)(b). \tag{159}$$

Proof. By semigroup property of $I^{\alpha}_{b-;g}$, we get

$$\left(I^{n\alpha}_{b-;g} D^{n\alpha}_{b-;g} f\right)(x) - \left(I^{(n+1)\alpha}_{b-;g} D^{(n+1)\alpha}_{b-;g} f\right)(x) =$$

$$\left(I^{n\alpha}_{b-;g} \left(D^{n\alpha}_{b-;g} f - I^{\alpha}_{b-;g} D^{(n+1)\alpha}_{b-;g} f\right)\right)(x) =$$

$$\left(I^{n\alpha}_{b-;g} \left(D^{n\alpha}_{b-;g} f - \left(I^{\alpha}_{b-;g} D^{\alpha}_{b-;g}\right)\left(D^{n\alpha}_{b-;g} f\right)\right)\right)(x) \overset{(141)}{=} \tag{160}$$

$$\left(I^{n\alpha}_{b-;g} \left(D^{n\alpha}_{b-;g} f - D^{n\alpha}_{b-;g} f + D^{n\alpha}_{b-;g} f(b)\right)\right)(x) =$$

$$\left(I^{n\alpha}_{b-;g} \left(D^{n\alpha}_{b-;g} f(b)\right)\right)(x) = \left(D^{n\alpha}_{b-;g} f(b)\right)\left(I^{n\alpha}_{b-;g}(1)\right)(x) = \tag{161}$$

[Notice that

$$\left(I_{b-;g}^{\alpha}1\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(g(t)-g(x))^{\alpha-1}g'(t)\,dt= \tag{162}$$

$$\frac{1}{\Gamma(\alpha)}\frac{(g(b)-g(x))^{\alpha}}{\alpha}=\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha}.$$

Thus we have

$$\left(I_{b-;g}^{\alpha}1\right)(x)=\frac{(g(b)-g(x))^{\alpha}}{\Gamma(\alpha+1)}. \tag{163}$$

Hence it holds

$$\left(I_{b-;g}^{2\alpha}1\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(g(t)-g(x))^{\alpha-1}g'(t)\frac{(g(b)-g(t))^{\alpha}}{\Gamma(\alpha+1)}dt=$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_{x}^{b}(g(b)-g(t))^{\alpha}(g(t)-g(x))^{\alpha-1}g'(t)\,dt=$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)}\int_{g(x)}^{g(b)}(g(b)-z)^{(\alpha+1)-1}(z-g(x))^{\alpha-1}dz=$$

$$\frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)}\frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)}(g(b)-g(x))^{2\alpha}=\frac{1}{\Gamma(2\alpha+1)}(g(b)-g(x))^{2\alpha}, \tag{164}$$

etc.]

$$=\left(D_{b-;g}^{n\alpha}f\right)(b)\frac{(g(b)-g(x))^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{165}$$

proving the claim. ■

We make

Remark 32 *Suppose that* $F_k=D_{b-;g}^{k\alpha}f$, *for* $k=1,...,n+1$; *are as in last Theorem 31,* $0<\alpha\leq 1$. *By (159) we get*

$$\sum_{i=0}^{n}\left(\left(I_{b-;g}^{i\alpha}D_{b-;g}^{i\alpha}f\right)(x)-I_{b-;g}^{(i+1)\alpha}D_{b-;g}^{(i+1)\alpha}f(x)\right)= \tag{166}$$

$$\sum_{i=0}^{n}\frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{b-;g}^{i\alpha}f\right)(b).$$

That is (notice that $I_{b-;g}^{0}f=D_{b-;g}^{0}f=f$*)*

$$f(x)-\left(I_{b-;g}^{(n+1)\alpha}D_{b-;g}^{(n+1)\alpha}f\right)(x)=\sum_{i=0}^{n}\frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{b-;g}^{i\alpha}f\right)(b). \tag{167}$$

Hence

$$f(x)=\sum_{i=0}^{n}\frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{b-;g}^{i\alpha}f\right)(b)+\left(I_{b-;g}^{(n+1)\alpha}D_{b-;g}^{(n+1)\alpha}f\right)(x)= \tag{168}$$

$$\sum_{i=0}^{n}\frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{b-;g}^{i\alpha}f\right)(b)+R_g(x,b), \tag{169}$$

where

$$R_g(x,b):=\frac{1}{\Gamma((n+1)\alpha)}\int_x^b(g(t)-g(x))^{(n+1)\alpha-1}g'(t)\left(D_{b-;g}^{(n+1)\alpha}f\right)(t)\,dt. \tag{170}$$

(there $D_{b-;g}^{(n+1)\alpha}f$ is continuous over $[a,b]$.)

We have proved the following g-right generalized modified X-valued Taylor's formula.

Theorem 33 *Let $f\in C^1([a,b],X)$, $g\in C^1([a,b])$, strictly increasing, such that $g^{-1}\in C^1([g(a),g(b)])$. Suppose that $F_k:=D_{b-;g}^{k\alpha}f$, $k=1,...,n$, fulfill $F_k\in C^1([a,b],X)$, and $F_{n+1}\in C([a,b],X)$, where $0<\alpha\leq 1$, $n\in\mathbb{N}$.*

Then

$$f(x)=\sum_{i=0}^{n}\frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{b-;g}^{i\alpha}f\right)(b)+$$

$$\frac{1}{\Gamma((n+1)\alpha)}\int_x^b(g(t)-g(x))^{(n+1)\alpha-1}g'(t)\left(D_{b-;g}^{(n+1)\alpha}f\right)(t)\,dt, \tag{171}$$

$\forall\ x\in[a,b]$.

For differentiation of functions from real numbers to normed linear spaces the definition is the same as for the real valued functions, however the limit and convergence is in the norm of linear space $(X,\|\cdot\|)$.

We need

Theorem 34 *Let $0\leq s\leq x$ and $f\in L_\infty([0,x],X)$, $r>0$, $(X,\|\cdot\|)$ is a Banach space. Define*

$$F(s):=\int_0^s(s-t)^r f(t)\,dt, \tag{172}$$

the last integral is of Bochner type.

Then there exists

$$F'(s)=r\int_0^s(s-t)^{r-1}f(t)\,dt,\quad \text{all } s\in[0,x]. \tag{173}$$

Proof. Fix $s \in [0, x]$ and notice that

$$F(s_0) = \int_0^{s_0} (s_0 - t)^r f(t)\,dt = \int_0^x \chi_{[0,s_0]}(t)(s_0 - t)^r f(t)\,dt.$$

We call $g(s,t) := \chi_{[0,s]}(t)(s-t)^r f(t)$, which is a Bochner integrable function for every $s \in [0, x]$, χ is the indicator function.

That is, $g(s_0, t) = \chi_{[0,s_0]}(t)(s_0 - t)^r f(t)$, all $t \in [0, x]$, and $F(s_0) = \int_0^x g(s_0, t)\,dt$.

We would like to study if there exists

$$\frac{\partial g(s_0, t)}{\partial s} = f(t)\left(\lim_{h\to 0} \frac{\chi_{[0,s_0+h]}(t)(s_0 + h - t)^r - \chi_{[0,s_0]}(t)(s_0 - t)^r}{h}\right). \quad (174)$$

We distinguish the following cases.

(1) Let $x \geq t > s_0$; then there exist small enough $h > 0$ such that $t \geq s_0 \pm h$.

That is,

$$\chi_{[0,s_0\pm h]}(t) = \chi_{[0,s_0]}(t) = 0.$$

Hence, there exists

$$\frac{\partial g(s_0, t)}{\partial s} = 0, \text{ all } t : s_0 < t \leq x. \quad (175)$$

(2) Let $0 \leq t < s_0$; then there exist small enough $h > 0$ such that $t < s_0 \pm h$.

That is

$$\chi_{[0,s_0\pm h]}(t) = \chi_{[0,s_0]}(t) = 1.$$

In that case

$$\frac{\partial g(s_0, t)}{\partial s} = f(t)\left(\lim_{h\to 0} \frac{(s_0 + h - t)^r - (s_0 - t)^r}{h}\right) = r(s_0 - t)^{r-1} f(t), \quad (176)$$

exists for almost all $t : 0 \leq t < s_0$.

(3) Let $t = s_0$. Then we see that

$$\frac{\partial g_+(s_0, s_0)}{\partial s} = f(s_0)\left(\lim_{h\to 0+} \frac{h^r}{h}\right) = f(s_0)\left(\lim_{h\to 0+} h^{r-1}\right). \quad (177)$$

The last limit does not exist if $0 < r < 1$, equals $f(s_0)$ if $r = 1$ and may not exist, and equals 0 if $r > 1$.

Notice also that

$$\frac{\partial g_-(s_0, s_0)}{\partial s} = f(s_0)\left(\lim_{h\to 0-} \frac{\chi_{[0,s_0+h]}(s_0)\,h^r}{h}\right) = f(s_0)\left(\lim_{h\to 0-} \chi_{[0,s_0+h]} h^{r-1}\right) = 0,$$

by $\chi_{[0,s_0+h]}(s_0) = 0$, $h < 0$.

That is,

$$\frac{\partial g_{-}(s_0, s_0)}{\partial s} = 0. \tag{178}$$

In general as a conclusion we get that $\frac{\partial g(s_0,t)}{\partial s}$ exists for almost all $t \in [0, x]$.

Next we define the difference quotient at s_0,

$$D_{s_0}^{g}(h,t) := f(t)\left(\frac{\chi_{[0,s_0+h]}(t)(s_0+h-t)^r - \chi_{[0,s_0]}(t)(s_0-t)^r}{h}\right), \tag{179}$$

for $h \neq 0$, and $D_{x_0}^{g}(0,t) := 0$.

Again we distinguish the following cases.

(1) Let $x \geq t > s_0$; then there exist small enough $h > 0$ such that $t > s_0 \pm h$.

Clearly then $D_{x_0}^{g}(h,t) = 0$.

(2) Let $0 \leq t < s_0$; then there exist small enough $h > 0$ such that $t < s_0 \pm h$. Thus

$$D_{s_0}^{g}(\pm h,t) = f(t)\left(\frac{(s_0 \pm h - t)^r - (s_0-t)^r}{\pm h}\right). \tag{180}$$

Call $\rho := s_0 - t > 0$; clearly $\rho \leq x$. Define

$$\varphi(t) := \frac{(\rho+h)^r - \rho^r}{h} = \frac{(s_0+h-t)^r - (s_0-t)^r}{h} \tag{181}$$

for h close to zero, $r > 0$.

That is, $D_{s_0}^{g}(h,t) = f(t)\varphi(t)$. If $r = 1$, then $\varphi(h) = 1$ and

$$D_{s_0}^{g}(h,t) = f(t). \tag{182}$$

We now treat the following subcases.

(2 (i)) If $r > 1$, then $\gamma(\rho) := \rho^r$, $0 \leq \rho \leq x$, is convex and increasing. If $h > 0$, then by the mean value theorem we get that

$$\varphi(h) < rx^{r-1}.$$

That is,

$$\left\| D_{s_0}^{g}(h,t) \right\| \leq rx^{r-1} \left\| f(t) \right\|. \tag{183}$$

If $h < 0$, then, similarly, again we get

$$\varphi(h) = \frac{\rho^r - (\rho+h)^r}{-h} < rx^{r-1}.$$

That is, for small $|h|$ we have

$$\left\| D_{s_0}^{g}(h,t) \right\| \leq rx^{r-1} \left\| f(t) \right\|, \quad r \geq 1. \tag{184}$$

(2 (ii)) If $0 < r < 1$, then $\gamma(\rho) := \rho^r$, $0 \leq \rho \leq x$ is concave and increasing. Let $h > 0$; then $\varphi(h) < \rho^{r-1} = (s_0-t)^{r-1}$ and for $h < 0$, again $\varphi(h) \leq \rho^{r-1} = (s_0-t)^{r-1}$.

That is

$$\left\| D_{s_0}^{g}(h,t) \right\| \leq (s_0 - t)^{r-1} \left\| f(t) \right\|, \tag{185}$$

for small $|h|$.

(3) Case of $t = s_0$; then

$$D_{s_0}^{g}(h,s_0) = f(s_0)\, h^{r-1}, \text{ for } h > 0, \tag{186}$$

and

$$D_{s_0}^{g}(h,s_0) = 0, \text{ for } h < 0. \tag{187}$$

So, if $r \geq 1$ we obtain

$$\left\| D_{s_0}^{g}(h,s_0) \right\| \leq \left\| f(s_0) \right\| x^{r-1}, \tag{188}$$

for small $|h|$.

If $0 < r < 1$, then for small $h > 0$ the function $D_{s_0}^{g}(h,s_0)$ may be unbounded.

In conclusion we get:

(I) For $r \geq 1$, that

$$\left\| D_{s_0}^{g}(h,t) \right\| \leq r x^{r-1} \left\| f \right\|_{\infty} < +\infty. \tag{189}$$

for almost all $t \in [0,x]$.

Hence

$$\left\| \frac{\partial g(s_0,t)}{\partial s} \right\| \leq r x^{r-1} \left\| f \right\|_{\infty}, \tag{190}$$

for almost all $t \in [0,x]$.

(II) For $0 < r < 1$, that

$$\left\| D_{s_0}^{g}(h,t) \right\| \leq \lambda(t), \text{ for almost all } t \in [0,x], \tag{191}$$

where

$$\lambda(t) := \begin{cases} (s_0 - t)^{r-1} \left\| f(t) \right\|, & 0 \leq t < s_0, \\ 0, \quad \text{for } s_0 \leq t \leq x. \end{cases} \tag{192}$$

Hence it holds

$$\left\| \frac{\partial g(s_0,t)}{\partial s} \right\| \leq \lambda(t), \quad \text{for almost all } t \in [0,x]. \tag{193}$$

Clearly λ is integrable on $[0,x]$. Then by Theorem 90, p. 39, [8], we get that $\frac{\partial g(s_0,\cdot)}{\partial s}$ defines a Bochner integrable function, and there exists

$$F'(s_0) = \int_0^x \frac{\partial g(s_0,t)}{\partial s} dt = r \int_0^{s_0} (s_0 - t)^{r-1} f(t)\, dt + \int_{s_0}^x 0 dt$$

$$= r \int_0^{s_0} (s_0 - t)^{r-1} f(t)\, dt. \tag{194}$$

That proves the claim. ■

We need

Theorem 35 *Let $x \leq s \leq 0$ and $f \in L_{\infty}([x,0], X)$, $r > 0$, $(X, \|\cdot\|)$ is a Banach space. Define*

$$G(s) = \int_s^0 (t-s)^r f(t)\, dt,$$

the last integral is of Bochner type.

Then there exists

$$G'(s) = -r \int_s^0 (t-s)^{r-1} f(t)\, dt, \tag{195}$$

all $s \in [x,0]$.

Proof. Fix $s_0 \in [x,0]$ and notice that

$$G(s_0) = \int_{s_0}^0 (t-s_0)^r f(t)\, dt = \int_x^0 \chi_{[s_0,0]}(t)(t-s_0)^r f(t)\, dt,$$

where χ is the indicator function.

We call

$$g(s,t) := \chi_{[s,0]}(t)(t-s)^r f(t),$$

which is a Bochner integrable function for every $s \in [x,0]$. That is, $g(s_0,t) = \chi_{[s_0,0]}(t)(t-s_0)^r f(t)$, all $t \in [x,0]$, and $G(s_0) = \int_x^0 g(s_0,t)\, dt$.

We would like to study if there exists

$$\frac{\partial g(s_0,t)}{\partial s} = f(t)\left(\lim_{h\to 0} \frac{\chi(t)_{[s_0+h,0]}(t-s_0-h)^r - \chi(t)_{[s_0,0]}(t-s_0)^r}{h}\right). \tag{196}$$

We distinguish the following cases.

(1) Let $x \leq t < s_0$; then there exist small enough $h > 0$ such that $t < s_0 \pm h$. That is,

$$\chi_{[s_0\pm h,0]}(t) = \chi_{[s_0,0]}(t) = 0.$$

Hence, there exists

$$\frac{\partial g(s_0,t)}{\partial s} = 0, \text{ all } t : x \leq t < s_0. \tag{197}$$

(2) Let $s_0 < t \leq 0$; then there exist small enough $h > 0$ such that $t > s_0 \pm h$. That is

$$\chi_{[s_0\pm h,0]}(t) = \chi_{[s_0,0]}(t) = 1.$$

In that case

$$\frac{\partial g(s_0,t)}{\partial s} = f(t)\left(\lim_{h\to 0} \frac{(t-(s_0+h))^r - (t-s_0)^r}{h}\right) = -r(t-s_0)^{r-1} f(t), \tag{198}$$

exists for almost all $t : s_0 < t \leq 0$.

(3) Let $t = s_0$. Then we see that

$$\frac{\partial g_+ (s_0, s_0)}{\partial s} = f(s_0) \left(\lim_{h \to 0+} \frac{0 \cdot (-h)^r - 1 \cdot 0^r}{h} \right) = 0. \tag{199}$$

Also we get

$$\frac{\partial g_- (s_0, s_0)}{\partial s} = f(s_0) \left(\lim_{h \to 0-} \frac{1 \cdot (-h)^r - 1 \cdot 0^r}{h} \right) =$$

$$f(s_0) \left(\lim_{h \to 0-} \frac{(-h)^r}{h} \right) = -f(s_0) \left(\lim_{h \to 0-} \frac{(-h)^r}{-h} \right) =$$

$$-f(s_0) \left(\lim_{h \to 0-} (-h)^{r-1} \right) = -f(s_0) \left(\lim_{h \to 0+} h^{r-1} \right). \tag{200}$$

The last limit does not exist if $0 < r < 1$; equals $-f(s_0)$ if $r = 1$ and may not exist; and equals 0 if $r > 1$.

In general as a conclusion we get that $\frac{\partial g(s_0,t)}{\partial s}$ exists for almost all $t \in [x, 0]$.

Next we define the difference quotient at s_0,

$$D_{s_0}^g (h, t) := f(t) \left(\frac{\chi_{[s_0+h,0]} (t) (t - s_0 - h)^r - \chi_{[s_0,0]} (t) (t - s_0)^r}{h} \right), \tag{201}$$

for $h \neq 0$, and $D_{s_0}^g (0, t) := 0$.

Again we distinguish the following cases.

(1) Let $x \leq t < s_0$; then there exist small enough $h > 0$ such that $t < s_0 \pm h$. Clearly then $D_{s_0}^g (h, t) = 0$.

(2) Let $s_0 < t \leq 0$; then there exist small enough $h > 0$ such that $t > s_0 \pm h$. In that case

$$D_{s_0}^g (\pm h, t) = f(t) \left(\frac{(t - (s_0 \pm h))^r - (t - s_0)^r}{\pm h} \right). \tag{202}$$

Call $\rho := t - s_0 > 0$; clearly $0 < \rho \leq |x|$.

Define

$$\varphi(t) := \frac{(\rho - h)^r - \rho^r}{h} = \frac{(t - s_0 - h)^r - (t - s_0)^r}{h}$$

for h close to zero, $r > 0$.

That is,

$$D_{s_0}^g (h, t) = f(t) \varphi(t).$$

If $r = 1$, then $\varphi(h) = -1$ and

$$D_{s_0}^g (h, t) = -f(t).$$

We now treat the following subcases.

(2 (i)) If $r > 1$ and $|h|$ small, then by the mean value theorem we get

$$\left\| D_{s_0}^{g}(h,t) \right\| = \| f(t) \| \, \| \varphi(t) \| \le \| f(t) \| \, r 2^{r-1} |x|^{r-1} . \tag{203}$$

That is, for $r \ge 1$ and small $|h|$ we obtain

$$\left\| D_{s_0}^{g}(h,t) \right\| \le r 2^{r-1} |x|^{r-1} \| f(t) \| . \tag{204}$$

(2 (ii)) If $0 < r < 1$ and $|h|$ small we get the following:

The function $\gamma(\rho) := \rho^r$, $0 \le \rho \le |x|$ is concave and increasing. Let $h > 0$; then

$$|\varphi(h)| = \frac{\rho^r - (\rho - h)^r}{\rho - (\rho - h)} < \rho^{r-1} = (r - s_0)^{r-1} , \tag{205}$$

and for $h < 0$, again

$$|\varphi(h)| = \frac{(\rho - h)^r - \rho^r}{(\rho - h) - \rho} < \rho^{r-1} = (t - s_0)^{r-1} . \tag{206}$$

Therefore we obtain

$$\left\| D_{s_0}^{g}(h,t) \right\| \le \| f(t) \| (t - s_0)^{r-1} , \tag{207}$$

for $0 < r < 1$ and $|h|$ small.

(3) Case of $t = s_0$; then

$$D_{s_0}^{g}(h, s_0) = -f(s_0) \, h^{r-1}, \text{ for } h < 0, \tag{208}$$

and

$$D_{s_0}^{g}(h, s_0) = 0, \text{ for } h > 0.$$

So, if $r \ge 1$ we obtain

$$\left\| D_{s_0}^{g}(h, s_0) \right\| \le \| f(s_0) \| \, |x|^{r-1} , \tag{209}$$

for small $|h|$.

If $0 < r < 1$, then for small $|h|$ with $h < 0$, the function $D_{s_0}^{g}(h, s_0)$ may be unbounded.

In conclusion we get:

(I) For $r \ge 1$, that

$$\left\| D_{s_0}^{g}(h,t) \right\| \le r 2^{r-1} |x|^{r-1} \| f \|_{\infty} < +\infty, \tag{210}$$

for almost all $t \in [x, 0]$.

Hence

$$\left\| \frac{\partial g(s_0, t)}{\partial s} \right\| \le r 2^{r-1} |x|^{r-1} \| f \|_{\infty} , \tag{211}$$

for almost all $t \in [x, 0]$.

(II) For $0 < r < 1$, that

$$\left\| D_{s_0}^{g}(h,t) \right\| \leq \lambda(t), \text{ for almost all } t \in [x,0],$$

where

$$\lambda(t) := \begin{cases} 0, & \text{for } x \leq t \leq s_0, \\ \|f(t)\| (t - s_0)^{r-1}, & s_0 < t \leq 0. \end{cases} \tag{212}$$

Hence it holds

$$\left\| \frac{\partial g(s_0, t)}{\partial s} \right\| \leq \lambda(t), \quad \text{for almost all } t \in [x,0]. \tag{213}$$

Clearly λ is integrable on $[x,0]$.

Then by Theorem 90, p. 39, [8], we get that $\frac{\partial g(s_0,\cdot)}{\partial s}$ defines a Bochner integrable function, and there exists

$$G'(s_0) = \int_x^0 \frac{\partial g(s_0,t)}{\partial s} dt = -r \int_{s_0}^0 (t - s_0)^{r-1} f(t)\, dt + \int_x^{s_0} 0dt \tag{214}$$

$$= -r \int_{s_0}^0 (t - s_0)^{r-1} f(t)\, dt.$$

That proves the claim. ■

We mention

Definition 36 *Let $U \subseteq \mathbb{R}$ be an interval, and X be a Banach space, we denote by $L_1(U, X)$ the Bochner integrable functions from U into X.*

We need

Definition 37 *Let $n \in \mathbb{R}_+$, and $[a,b] \subset \mathbb{R}$, X a Banach space, and $L_1([a,b], X)$. The Bochner integral operator*

$$(J_a^n f)(x) := \frac{1}{\Gamma(n)} \int_a^x (x - t)^{n-1} f(t)\, dt, \tag{215}$$

for $a \leq x \leq b$, is called the Riemann-Liouville fractional Bochner integral operator of order n, where Γ is the gamma function.

For $n = 0$, we set $J_a^0 := I$, the identity operator.

We give

Theorem 38 *([4]) Let $m, n \in \mathbb{R}_+$ and $f \in L_1([a,b], X)$. Then*

$$J_a^m J_a^n f = J_a^{m+n} f = J_a^n J_a^m f, \tag{216}$$

holds almost everywhere on $[a,b]$.

If $f \in C([a,b], X)$ or $m + n \geq 1$, then identity in (216) is valid everywhere on $[a,b]$.

We make

Definition 39 *Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\nu > 0$; $n := \lceil \nu \rceil \in \mathbb{N}$, $\lceil \cdot \rceil$ is the ceiling of the number, $f : [a,b] \to X$. We assume that $f^{(n)} \in L_1([a,b], X)$. We call the Caputo-Bochner left fractional derivative of order ν:*

$$\left(D_{*a}^{\nu} f\right)(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \quad \forall\, x \in [a,b]. \tag{217}$$

If $\nu \in \mathbb{N}$, we set $D_{*a}^{\nu} f := f^{(\nu)}$ the ordinary X-valued derivative, and also set $D_{*a}^{0} f := f$.

We need

Definition 40 *Let $\alpha > 0$, $[a,b] \subset \mathbb{R}$, X is a Banach space, and $f \in L_1([a,b], X)$. The Bochner integral operator*

$$\left(I_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} f(z)\, dz, \tag{218}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function, is called the Riemann-Liouville right fractional Bochner integral operator of order α.

For $\alpha = 0$, we set $I_{b-}^{0} := I$ (the identity operator).

We mention

Theorem 41 *([2]) Let $\alpha, \beta \geq 0$, $f \in L_1([a,b], X)$. Then*

$$I_{b-}^{\alpha} I_{b-}^{\beta} f = I_{b-}^{\alpha+\beta} f = I_{b-}^{\beta} I_{b-}^{\alpha} f, \tag{219}$$

valid almost everywhere on $[a,b]$.

If additionally $f \in C([a,b], X)$ or $\alpha + \beta \geq 1$, then we have identity true on all of $[a,b]$.

We need

Definition 42 *Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m := \lceil \alpha \rceil$, ($\lceil \cdot \rceil$ the ceiling of the number). We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^{\alpha} f\right)(x) := (-1)^m I_{b-}^{m-\alpha} f^{(m)}(x), \tag{220}$$

i.e.

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \quad \forall\, x \in [a,b]. \tag{221}$$

We observe that $D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $D_{b-}^{0} f(x) = f(x)$.

We consider ($\alpha > 0$)

$$D_{*a}^{n\alpha} := D_{*a}^{\alpha} D_{*a}^{\alpha} ... D_{*a}^{\alpha} \quad (n\text{-times}),\ n \in \mathbb{N}. \tag{222}$$

Also denote by

$$J_a^{n\alpha} := J_a^{\alpha} J_a^{\alpha} ... J_a^{\alpha} \quad (n\text{-times}). \tag{223}$$

Similarly we consider

$$D_{b-}^{n\alpha} := D_{b-}^{\alpha} D_{b-}^{\alpha} ... D_{b-}^{\alpha} \quad (n\text{-times}), \tag{224}$$

and

$$I_{b-}^{n\alpha} := I_{b-}^{\alpha} I_{b-}^{\alpha} ... I_{b-}^{\alpha} \quad (n\text{-times}). \tag{225}$$

Next we apply Theorems 30, 33, when g =identity map.

We have the following left modified X-valued Taylor's formula.

Theorem 43 *Let $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$. For $k = 1, ..., n$, we assume that $D_{*a}^{k\alpha} f \in C^1([a,b], X)$ and $D_{*a}^{(n+1)\alpha} f \in C([a,b], X)$.*

Then

$$f(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{*a}^{i\alpha} f\right)(a) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (x-t)^{(n+1)\alpha-1} \left(D_{*a}^{(n+1)\alpha} f\right)(t)\, dt, \quad \forall\ x \in [a,b]. \tag{226}$$

We have also the following right modified X-valued Taylor's formula.

Theorem 44 *Let $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$. For $k = 1, ..., n$, we assume that $D_{b-}^{k\alpha} f \in C^1([a,b], X)$ and $D_{b-}^{(n+1)\alpha} f \in C([a,b], X)$.*

Then

$$f(x) = \sum_{i=0}^{n} \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D_{b-}^{i\alpha} f\right)(b) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^b (t-x)^{(n+1)\alpha-1} \left(D_{b-}^{(n+1)\alpha} f\right)(t)\, dt, \quad \forall\ x \in [a,b]. \tag{227}$$

We give

Theorem 45 *Let $0 < \alpha < 1$, $f \in C^m([a,b], X)$, where $(X, \|\cdot\|)$ is a Banach space, $m \in \mathbb{N}$. Assume that $D_{*a}^{k\alpha} f \in C^1([a,b], X)$, $k = 1, ..., n$, $n \in \mathbb{N}$ and $D_{*a}^{(n+1)\alpha} f \in C([a,b], X)$. Suppose that $\left(D_{*a}^{i\alpha} f\right)(a) = 0$, for $i = 0, 2, 3, ..., n$. Let $\gamma > 0$ with $\lceil \gamma \rceil = m < n+1$, such that $m < (n+1)\alpha$, equivalently $\alpha > \frac{m}{n+1}$. Then*

$$\left(D_{*a}^{\gamma} f\right)(x) = \frac{1}{\Gamma((n+1)\alpha - \gamma)} \int_a^x (x-t)^{(n+1)\alpha-\gamma-1} \left(D_{*a}^{(n+1)\alpha} f\right)(t)\, dt, \tag{228}$$

$\forall\ x \in [a,b]$. Furthermore it holds $\left(D_{*a}^{\gamma} f\right) \in C([a,b], X)$.

Proof. Here we have that

$$\left(D_{*a}^{\alpha}f\right)(x)=\frac{1}{\Gamma(1-\alpha)}\int_a^x(x-t)^{-\alpha}f'(t)\,dt,\quad\forall\,x\in[a,b]. \tag{229}$$

We observe that

$$\left\|\left(D_{*a}^{\alpha}f\right)(x)\right\|\leq\frac{1}{\Gamma(1-\alpha)}\int_a^x(x-t)^{-\alpha}\left\|f'(t)\right\|dt \tag{230}$$

$$\leq\frac{1}{\Gamma(1-\alpha)}\left(\int_a^x(x-t)^{-\alpha}\,dt\right)\left\|f'\right\|_{L_\infty([a,b],X)}$$

$$=\frac{(x-a)^{1-\alpha}}{\Gamma(1-\alpha)(1-\alpha)}\left\|f'\right\|_{L_\infty([a,b],X)}=\frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)}\left\|f'\right\|_{L_\infty([a,b],X)}<+\infty. \tag{231}$$

Hence

$$\left\|\left(D_{*a}^{\alpha}f\right)(a)\right\|=0,$$

that is

$$\left(D_{*a}^{\alpha}f\right)(a)=0. \tag{232}$$

The left Caputo fractional derivative of order γ is given by

$$D_{*a}^{\gamma}f=\frac{1}{\Gamma(m-\gamma)}\int_a^x(x-t)^{m-\gamma-1}f^{(m)}(t)\,dt=\left(J_a^{m-\gamma}f^{(m)}\right)(x),\ \ \forall x\in[a,b], \tag{233}$$

which exists everywhere over $[a,b]$.

We set $D_{*a}^{m}f=f^{(m)}$, $m\in\mathbb{N}$.

By Theorem 43 we obtain

$$f(x)=\frac{1}{\Gamma((n+1)\alpha)}\int_a^x(x-t)^{(n+1)\alpha-1}\left(D_{*a}^{(n+1)\alpha}f\right)(t)\,dt,\quad\forall\,x\in[a,b]. \tag{234}$$

By Theorem 34, when $(n+1)\alpha-1>0$, equivalently when $\alpha>\frac{1}{n+1}$, we get that there exists

$$f'(x)=\frac{((n+1)\alpha-1)}{\Gamma((n+1)\alpha)}\int_a^x(x-t)^{(n+1)\alpha-2}\left(D_{*a}^{(n+1)\alpha}f\right)(t)\,dt,\quad\forall\,x\in[a,b]. \tag{235}$$

If $(n+1)\alpha-2>0$, equivalently, if $\alpha>\frac{2}{n+1}$, we get that there exists

$$f''(x)=\frac{((n+1)\alpha-1)((n+1)\alpha-2)}{\Gamma((n+1)\alpha)}\int_a^x(x-t)^{(n+1)\alpha-3}\left(D_{*a}^{(n+1)\alpha}f\right)(t)\,dt, \tag{236}$$

$\forall\,x\in[a,b]$.

In general, if $(n+1)\alpha - m > 0$, equivalently, if $\alpha > \frac{m}{n+1}$, we get that there exists

$$f^{(m)}(x) = \frac{\prod_{j=1}^{m}((n+1)\alpha - j)}{\Gamma((n+1)\alpha)} \int_a^x (x-t)^{(n+1)\alpha - m - 1} \left(D_{*a}^{(n+1)\alpha} f\right)(t)\, dt, \tag{237}$$

$\forall\, x \in [a,b]$.

By Theorem 10, we get that $f^{(m)} \in C([a,b], X)$.

By (215), we derive

$$\begin{aligned} f^{(m)}(x) &= \frac{\prod_{j=1}^{m}((n+1)\alpha - j)\,\Gamma((n+1)\alpha - m)}{\Gamma((n+1)\alpha)} \left(J_a^{((n+1)\alpha - m)}\left(D_{*a}^{(n+1)\alpha} f\right)\right)(x) \\ &= \left(J_a^{((n+1)\alpha - m)}\left(D_{*a}^{(n+1)\alpha} f\right)\right)(x). \end{aligned} \tag{238}$$

We have proved that

$$f^{(m)}(x) = \left(J_a^{((n+1)\alpha - m)}\left(D_{*a}^{(n+1)\alpha} f\right)\right)(x), \quad \forall\, x \in [a,b]. \tag{239}$$

We have that (case of $\gamma < m$)

$$\begin{aligned} \left(D_{*a}^{\gamma} f\right)(x) &= \left(J_a^{m-\gamma} f^{(m)}\right)(x) = \left(J_a^{m-\gamma} J_a^{((n+1)\alpha - m)}\left(D_{*a}^{(n+1)\alpha} f\right)\right)(x) \\ &= \left(J_a^{(n+1)\alpha - \gamma}\left(D_{*a}^{(n+1)\alpha} f\right)\right)(x). \end{aligned} \tag{240}$$

That is

$$\left(D_{*a}^{\gamma} f\right)(x) = \left(J_a^{(n+1)\alpha - \gamma}\left(D_{*a}^{(n+1)\alpha} f\right)\right)(x), \quad \forall\, x \in [a,b]. \tag{241}$$

I.e. we have found the representation formula:

$$\left(D_{*a}^{\gamma} f\right)(x) = \frac{1}{\Gamma((n+1)\alpha - \gamma)} \int_a^x (x-t)^{(n+1)\alpha - \gamma - 1} \left(D_{*a}^{(n+1)\alpha} f\right)(t)\, dt, \tag{242}$$

$\forall\, x \in [a,b]$.

The last formula (242) is true under the assumption $(n+1)\alpha > m$, and since $m \geq \gamma$, it implies $(n+1)\alpha > \gamma$ and $(n+1)\alpha - \gamma > 0$. Furthermore, by Theorem 10, we get that $\left(D_{*a}^{\gamma} f\right) \in C([a,b], X)$.

The theorem is proved. ■

We continue with

Theorem 46 *Under the assumptions of Theorem 45, and when* $\frac{\gamma+m}{n+1} < \alpha < 1$, *we get that*

$$\left(D_{*a}^{2\gamma} f\right)(x) = \frac{1}{\Gamma((n+1)\alpha - 2\gamma)} \int_a^x (x-t)^{(n+1)\alpha - 2\gamma - 1} \left(D_{*a}^{(n+1)\alpha} f\right)(t)\, dt, \tag{243}$$

$\forall\ x \in [a,b]$, *and* $\left(D_{*a}^{2\gamma} f\right) \in C([a,b], X)$.

Proof. Call $\lambda := (n+1)\alpha - \gamma - 1$, i.e. $\lambda + 1 = (n+1)\alpha - \gamma$, and call $\delta := (n+1)\alpha$. Then we can write

$$\left(D_{*a}^{\gamma} f\right)(x) = \frac{1}{\Gamma(\lambda+1)} \int_a^x (x-t)^{\lambda} \left(D_{*a}^{\delta} f\right)(t)\, dt, \quad \forall\ x \in [a,b]. \tag{244}$$

If $\lambda > 0$, then

$$\left(D_{*a}^{\gamma} f\right)'(x) = \frac{\lambda}{\Gamma(\lambda+1)} \int_a^x (x-t)^{\lambda-1} \left(D_{*a}^{\delta} f\right)(t)\, dt, \quad \forall\ x \in [a,b]. \tag{245}$$

If $\lambda - 1 > 0$, then

$$\left(D_{*a}^{\gamma} f\right)''(x) = \frac{\lambda(\lambda-1)}{\Gamma(\lambda+1)} \int_a^x (x-t)^{\lambda-2} \left(D_{*a}^{\delta} f\right)(t)\, dt, \quad \forall\ x \in [a,b]. \tag{246}$$

If $\lambda - 2 > 0$, then

$$\left(D_{*a}^{\gamma} f\right)^{(3)}(x) = \frac{\lambda(\lambda-1)(\lambda-2)}{\Gamma(\lambda+1)} \int_a^x (x-t)^{\lambda-3} \left(D_{*a}^{\delta} f\right)(t)\, dt, \quad \forall\ x \in [a,b]. \tag{247}$$

etc.

In general, if $\lambda - m + 1 > 0$, then

$$\left(D_{*a}^{\gamma} f\right)^{(m)}(x) = \frac{\lambda(\lambda-1)(\lambda-2)\dots(\lambda-m+1)}{\Gamma(\lambda+1)} \int_a^x (x-t)^{(\lambda-m+1)-1} \left(D_{*a}^{\delta} f\right)(t)\, dt \tag{248}$$

$$= \frac{\lambda(\lambda-1)(\lambda-2)\dots(\lambda-m+1)\,\Gamma(\lambda-m+1)}{\Gamma(\lambda+1)} \left(J_a^{(\lambda-m+1)} \left(D_{*a}^{\delta} f\right)\right)(x)$$

$$= \left(J_a^{(\lambda-m+1)} \left(D_{*a}^{\delta} f\right)\right)(x), \quad \forall\ x \in [a,b]. \tag{249}$$

That is, if $\lambda - m + 1 > 0$, then

$$\left(D_{*a}^{\gamma} f\right)^{(m)}(x) = \left(J_a^{(\lambda-m+1)} \left(D_{*a}^{\delta} f\right)\right)(x), \quad \forall\ x \in [a,b]. \tag{250}$$

We notice that

$$\left(D_{*a}^{2\gamma} f\right)(x) = \left(D_{*a}^{\gamma}\left(D_{*a}^{\gamma} f\right)\right)(x) = \left(J_a^{m-\gamma}\left(D_{*a}^{\gamma} f\right)^{(m)}\right)(x) = \tag{251}$$

$$\left(J_a^{m-\gamma} J_a^{\lambda-m+1}\left(D_{*a}^{\delta} f\right)\right)(x) = \left(J_a^{\lambda-\gamma+1}\left(D_{*a}^{\delta} f\right)\right)(x) =$$

$$\left(J_a^{(n+1)\alpha-\gamma-1-\gamma+1}\left(D_{*a}^{\delta} f\right)\right)(x) = \left(J_a^{(n+1)\alpha-2\gamma}\left(D_{*a}^{\delta} f\right)\right)(x). \tag{252}$$

That is

$$\left(D_{*a}^{2\gamma} f\right)(x) = \left(J_a^{(n+1)\alpha-2\gamma}\left(D_{*a}^{(n+1)\alpha} f\right)\right)(x), \quad \forall\ x \in [a,b], \tag{253}$$

under the condition $\frac{\gamma+m}{n+1} < \alpha < 1$.

The theorem is proved. ∎

We give

Theorem 47 *Under the assumptions of Theorem 45, and when* $\frac{m+2\gamma}{n+1} < \alpha < 1$, *we obtain that*

$$\left(D_{*a}^{3\gamma} f\right)(x) = \frac{1}{\Gamma\left((n+1)\alpha - 3\gamma\right)} \int_a^x (x-t)^{(n+1)\alpha - 3\gamma - 1} \left(D_{*a}^{(n+1)\alpha} f\right)(t)\, dt, \tag{254}$$

$\forall\ x \in [a,b]$, *and* $\left(D_{*a}^{3\gamma} f\right) \in C\left([a,b], X\right)$.

Proof. Call $\rho := (n+1)\alpha - 2\gamma - 1$, i.e. $\rho + 1 = (n+1)\alpha - 2\gamma$, and call $\delta := (n+1)\alpha$. Then we can write

$$\left(D_{*a}^{2\gamma} f\right)(x) = \frac{1}{\Gamma(\rho+1)} \int_a^x (x-t)^{\rho} \left(D_{*a}^{\delta} f\right)(t)\, dt, \quad \forall\ x \in [a,b]. \tag{255}$$

If $\rho > 0$, then

$$\left(D_{*a}^{2\gamma} f\right)'(x) = \frac{\rho}{\Gamma(\rho+1)} \int_a^x (x-t)^{\rho-1} \left(D_{*a}^{\delta} f\right)(t)\, dt, \quad \forall\ x \in [a,b]. \tag{256}$$

If $\rho - 1 > 0$, then

$$\left(D_{*a}^{2\gamma} f\right)''(x) = \frac{\rho(\rho-1)}{\Gamma(\rho+1)} \int_a^x (x-t)^{\rho-2} \left(D_{*a}^{\delta} f\right)(t)\, dt, \quad \forall\ x \in [a,b]. \tag{257}$$

If $\rho - 2 > 0$, then

$$\left(D_{*a}^{2\gamma} f\right)^{(3)}(x) = \frac{\rho(\rho-1)(\rho-2)}{\Gamma(\rho+1)} \int_a^x (x-t)^{\rho-3} \left(D_{*a}^{\delta} f\right)(t)\, dt, \quad \forall\ x \in [a,b]. \tag{258}$$

etc.

In general, if $\rho - m + 1 > 0$, then

$$\left(D_{*a}^{2\gamma} f\right)^{(m)}(x) = \frac{\rho(\rho-1)(\rho-2)\dots(\rho-m+1)}{\Gamma(\rho+1)} \int_a^x (x-t)^{(\rho-m+1)-1} \left(D_{*a}^{\delta} f\right)(t)\, dt \tag{259}$$

$$= \frac{\rho(\rho-1)(\rho-2)\dots(\rho-m+1)\,\Gamma(\rho-m+1)}{\Gamma(\rho+1)} \left(J_a^{(\rho-m+1)}\left(D_{*a}^{\delta} f\right)\right)(x)$$

$$= \left(J_a^{(\rho-m+1)}\left(D_{*a}^{\delta} f\right)\right)(x), \quad \forall\ x \in [a,b]. \tag{260}$$

That is, if $\rho - m + 1 > 0$, then

$$\left(D_{*a}^{2\gamma} f\right)^{(m)}(x) = \left(J_a^{(\rho-m+1)}\left(D_{*a}^{\delta} f\right)\right)(x), \quad \forall\ x \in [a,b]. \tag{261}$$

We notice that

$$\left(D_{*a}^{3\gamma} f\right)(x) = \left(D_{*a}^{\gamma}\left(D_{*a}^{2\gamma} f\right)\right)(x) = \left(J_a^{m-\gamma}\left(D_{*a}^{2\gamma} f\right)^{(m)}\right)(x) = \tag{262}$$

$$\left(J_a^{m-\gamma} J_a^{\rho-m+1}\left(D_{*a}^{\delta} f\right)\right)(x) = \left(J_a^{\rho-\gamma+1}\left(D_{*a}^{\delta} f\right)\right)(x) =$$

$$\left(J_{a}^{(n+1)\alpha-2\gamma-1-\gamma+1}\left(D_{*a}^{\delta}f\right)\right)(x)=\left(J_{a}^{(n+1)\alpha-3\gamma}\left(D_{*a}^{(n+1)\alpha}f\right)\right)(x). \quad (263)$$

That is, if $\frac{m+2\gamma}{n+1}<\alpha<1$, we get

$$\left(D_{*a}^{3\gamma}f\right)(x)=\left(J_{a}^{(n+1)\alpha-3\gamma}\left(D_{*a}^{(n+1)\alpha}f\right)\right)(x), \quad \forall\, x\in[a,b]. \quad (264)$$

The theorem is proved. ■

In general, we derive the iterated left fractional derivative formula:

Theorem 48 *Under the assumptions of Theorem 45, and when* $\frac{m+(k-1)\gamma}{n+1}<\alpha<1$, $k\in\mathbb{N}$, *we obtain that*

$$\left(D_{*a}^{k\gamma}f\right)(x)=\frac{1}{\Gamma\left((n+1)\alpha-k\gamma\right)}\int_{a}^{x}(x-t)^{(n+1)\alpha-k\gamma-1}\left(D_{*a}^{(n+1)\alpha}f\right)(t)\,dt, \quad (265)$$

$\forall\, x\in[a,b]$, *and* $\left(D_{*a}^{k\gamma}f\right)\in C\left([a,b],X\right).$

We give

Theorem 49 *Let* $0<\alpha<1$, $f\in C^{m}\left([a,b],X\right)$, *where* $(X,\|\cdot\|)$ *is a Banach space,* $m\in\mathbb{N}$. *Assume that* $D_{b-}^{k\alpha}f\in C^{1}\left([a,b],X\right)$, *for* $k=1,...,n$, $n\in\mathbb{N}$ *and* $D_{b-}^{(n+1)\alpha}f\in C\left([a,b],X\right)$. *Suupose that* $\left(D_{b-}^{i\alpha}f\right)(b)=0$, $i=0,2,3,...,n$. *Let* $\gamma>0$ *with* $\lceil\gamma\rceil=m<n+1$, *such that* $m<(n+1)\alpha$, *equivalently,* $\alpha>\frac{m}{n+1}$. *Then*

$$\left(D_{b-}^{\gamma}f\right)(x)=\frac{1}{\Gamma\left((n+1)\alpha-\gamma\right)}\int_{x}^{b}(z-x)^{(n+1)\alpha-\gamma-1}\left(D_{b-}^{(n+1)\alpha}f\right)(z)\,dz, \quad (266)$$

$\forall\, x\in[a,b]$. *Furthermore it holds* $\left(D_{b-}^{\gamma}f\right)\in C\left([a,b],X\right).$

Proof. Here we have that

$$\left(D_{b-}^{\alpha}f\right)(x)=\frac{-1}{\Gamma(1-\alpha)}\int_{x}^{b}(J-x)^{-\alpha}f'(J)\,dJ, \quad \forall x\in[a,b]. \quad (267)$$

We observe that

$$\left\|\left(D_{b-}^{\alpha}f\right)(x)\right\|\leq\frac{1}{\Gamma(1-\alpha)}\int_{x}^{b}(J-x)^{-\alpha}\left\|f'(J)\right\|dJ \quad (268)$$

$$\leq\frac{1}{\Gamma(1-\alpha)}\left(\int_{x}^{b}(J-x)^{-\alpha}dJ\right)\left\|f'\right\|_{L_{\infty}([a,b],X)}$$

$$=\frac{(b-x)^{1-\alpha}}{\Gamma(2-\alpha)}\left\|f'\right\|_{L_{\infty}([a,b],X)}<+\infty.$$

Hence

$$\left\|\left(D_{b-}^{\alpha}f\right)(b)\right\|=0, \quad (269)$$

that is

$$\left(D_{b-}^{\alpha} f\right)(b) = 0. \tag{270}$$

The right Caputo fractional derivative X-valued of order γ is given by

$$\left(D_{b-}^{\gamma} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\gamma)} \int_x^b (z-x)^{m-\gamma-1} f^{(m)}(z)\, dz, \ \ \forall\, x \in [a,b], \tag{271}$$

Notice that

$$\left(D_{b-}^{\gamma} f\right)(x) = (-1)^m I_{b-}^{m-\gamma} f^{(m)}(x), \ \ \forall\, x \in [a,b]. \tag{272}$$

We set $D_{b-}^{0} f = f$, and $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, $\forall\, x \in [a,b]$.

By (227) we obtain

$$f(x) = \frac{1}{\Gamma((n+1)\alpha)} \int_x^b (z-x)^{(n+1)\alpha-1} \left(D_{b-}^{(n+1)\alpha} f\right)(z)\, dz, \ \ \forall\, x \in [a,b]. \tag{273}$$

Call $\delta := (n+1)\alpha$, then we have

$$f(x) = \frac{1}{\Gamma(\delta)} \int_x^b (z-x)^{\delta-1} \left(D_{b-}^{\delta} f\right)(z)\, dz, \ \ \forall\, x \in [a,b]. \tag{274}$$

By Theorem 35, when $\delta - 1 > 0$, we get that there exists

$$f'(x) = \frac{(-1)(\delta-1)}{\Gamma(\delta)} \int_x^b (z-x)^{\delta-2} \left(D_{b-}^{\delta} f\right)(z)\, dz, \ \ \forall\, x \in [a,b]. \tag{275}$$

If $\delta - 2 > 0$, then

$$f''(x) = \frac{(-1)^2(\delta-1)(\delta-2)}{\Gamma(\delta)} \int_x^b (z-x)^{\delta-3} \left(D_{b-}^{\delta} f\right)(z)\, dz, \ \ \forall\, x \in [a,b]. \tag{276}$$

In general, if $\delta - m > 0$, equivalently, if $\alpha > \frac{m}{n+1}$, we get that there exists

$$f^{(m)}(x) = \frac{(-1)^m \prod_{j=1}^{m} (\delta - j)}{\Gamma(\delta)} \int_x^b (z-x)^{\delta-m-1} \left(D_{b-}^{\delta} f\right)(z)\, dz, \ \ \forall\, x \in [a,b]. \tag{277}$$

By Theorem 11, we get $f^{(m)} \in C([a,b], X)$.

By (218), we derive

$$\begin{aligned} f^{(m)}(x) &= \frac{(-1)^m \prod_{j=1}^{m} (\delta - j)\, \Gamma(\delta-m) \left(I_{b-}^{\delta-m}\left(D_{b-}^{\delta} f\right)\right)(x)}{\Gamma(\delta)} \\ &= (-1)^m \left(I_{b-}^{\delta-m}\left(D_{b-}^{\delta} f\right)\right)(x), \ \ \forall\, x \in [a,b]. \end{aligned} \tag{278}$$

We have proved that

$$f^{(m)}(x) = (-1)^m \left(I_{b-}^{\delta-m} \left(D_{b-}^{\delta} f \right) \right)(x), \quad \forall\, x \in [a,b]. \tag{279}$$

We have that (case of $\gamma < m$)

$$\begin{aligned} \left(D_{b-}^{\gamma} f \right)(x) &= (-1)^m \left(I_{b-}^{m-\gamma} f^{(m)} \right)(x) = (-1)^{2m} \left(I_{b-}^{m-\gamma} \left(I_{b-}^{\delta-m} \left(D_{b-}^{\delta} f \right) \right) \right)(x) \\ &= \left(I_{b-}^{\delta-\gamma} \left(D_{b-}^{\delta} f \right) \right)(x), \quad \forall\, x \in [a,b]. \end{aligned} \tag{280}$$

That is

$$\left(D_{b-}^{\gamma} f \right)(x) = \left(I_{b-}^{(n+1)\alpha-\gamma} \left(D_{b-}^{(n+1)\alpha} f \right) \right)(x), \quad \forall\, x \in [a,b]. \tag{281}$$

I.e. we have found the representation formula:

$$\left(D_{b-}^{\gamma} f \right)(x) = \frac{1}{\Gamma((n+1)\alpha-\gamma)} \int_x^b (z-x)^{(n+1)\alpha-\gamma-1} \left(D_{b-}^{(n+1)\alpha} f \right)(z)\, dz, \tag{282}$$

$\forall\, x \in [a,b]$.

The last formula (282) is true under the assumption $(n+1)\alpha > m$, and since $m \geq \gamma$, it implies $(n+1)\alpha > \gamma$ and $(n+1)\alpha - \gamma > 0$. Furthermore, by Theorem 11, we get that $\left(D_{b-}^{\gamma} f \right) \in C([a,b], X)$.

The theorem is proved. ■

We continue with

Theorem 50 *Under the assumptions of Theorem 49, and when* $\frac{\gamma+m}{n+1} < \alpha < 1$, *we get that*

$$\left(D_{b-}^{2\gamma} f \right)(x) = \frac{1}{\Gamma((n+1)\alpha-2\gamma)} \int_x^b (z-x)^{(n+1)\alpha-2\gamma-1} \left(D_{b-}^{(n+1)\alpha} f \right)(z)\, dz, \tag{283}$$

$\forall\, x \in [a,b]$. *Furthermore it holds* $\left(D_{b-}^{2\gamma} f \right) \in C([a,b], X)$.

Proof. Call $\lambda := (n+1)\alpha - \gamma - 1$, i.e. $\lambda + 1 = (n+1)\alpha - \gamma$, and call $\delta := (n+1)\alpha$. Then we can write

$$\left(D_{b-}^{\gamma} f \right)(x) = \frac{1}{\Gamma(\lambda+1)} \int_x^b (z-x)^{\lambda} \left(D_{b-}^{\delta} f \right)(z)\, dz, \quad \forall\, x \in [a,b]. \tag{284}$$

If $\lambda > 0$, then

$$\left(D_{b-}^{\gamma} f \right)'(x) = \frac{(-1)\lambda}{\Gamma(\lambda+1)} \int_x^b (z-x)^{\lambda-1} \left(D_{b-}^{\delta} f \right)(z)\, dz, \quad \forall\, x \in [a,b]. \tag{285}$$

If $\lambda - 1 > 0$, then

$$\left(D_{b-}^{\gamma} f\right)'' (x) = \frac{(-1)^2 \lambda (\lambda - 1)}{\Gamma (\lambda + 1)} \int_x^b (z - x)^{\lambda - 2} \left(D_{b-}^{\delta} f\right) (z) \, dz, \quad \forall \ x \in [a, b] . \tag{286}$$

If $\lambda - 2 > 0$, then

$$\left(D_{b-}^{\gamma} f\right)^{(3)} (x) = \frac{(-1)^3 \lambda (\lambda - 1) (\lambda - 2)}{\Gamma (\lambda + 1)} \int_x^b (z - x)^{\lambda - 3} \left(D_{b-}^{\delta} f\right) (z) \, dz, \quad \forall \ x \in [a, b] . \tag{287}$$

etc.

In general, if $\lambda - m + 1 > 0$, then

$$\begin{gathered}
\left(D_{b-}^{\gamma} f\right)^{(m)} (x) = \\
\frac{(-1)^m \lambda (\lambda - 1) (\lambda - 2) ... (\lambda - m + 1)}{\Gamma (\lambda + 1)} \int_x^b (z - x)^{(\lambda - m + 1) - 1} \left(D_{b-}^{\delta} f\right) (z) \, dz \\
= \frac{(-1)^m \lambda (\lambda - 1) (\lambda - 2) ... (\lambda - m + 1) \Gamma (\lambda - m + 1) \left(I_{b-}^{(\lambda - m + 1)} \left(D_{b-}^{\delta} f\right)\right) (x)}{\Gamma (\lambda + 1)} \\
= (-1)^m \left(I_{b-}^{(\lambda - m + 1)} \left(D_{b-}^{\delta} f\right)\right) (x) , \quad \forall \ x \in [a, b] .
\end{gathered} \tag{288}$$

That is, if $\lambda - m + 1 > 0$, then

$$\left(D_{b-}^{\gamma} f\right)^{(m)} (x) = (-1)^m \left(I_{b-}^{(\lambda - m + 1)} \left(D_{b-}^{\delta} f\right)\right) (x) , \quad \forall \ x \in [a, b] . \tag{289}$$

We notice that

$$\begin{gathered}
\left(D_{b-}^{2\gamma} f\right) (x) = \left(D_{b-}^{\gamma} \left(D_{b-}^{\gamma} f\right)\right) (x) = (-1)^m \left(I_{b-}^{m - \gamma} \left(D_{b-}^{\gamma} f\right)^{(m)}\right) (x) = \\
(-1)^{2m} \left(I_{b-}^{m - \gamma} I_{b-}^{\lambda - m + 1} \left(D_{b-}^{\delta} f\right)\right) (x) = \left(I_{b-}^{\lambda - \gamma + 1} \left(D_{b-}^{\delta} f\right)\right) (x) = \\
\left(I_{b-}^{(n+1)\alpha - \gamma - 1 - \gamma + 1} \left(D_{b-}^{\delta} f\right)\right) (x) = \left(I_{b-}^{(n+1)\alpha - 2\gamma} \left(D_{b-}^{\delta} f\right)\right) (x) ,
\end{gathered} \tag{290}$$

$\forall \ x \in [a, b]$.

That is

$$\left(D_{b-}^{2\gamma} f\right) (x) = \left(I_{b-}^{(n+1)\alpha - 2\gamma} \left(D_{b-}^{(n+1)\alpha} f\right)\right) (x) , \quad \forall \ x \in [a, b] , \tag{291}$$

under the condition $\frac{\gamma + m}{n+1} < \alpha < 1$.

The theorem is proved. ■

We give

Theorem 51 *Under the assumptions of Theorem 49, and when* $\frac{m+2\gamma}{n+1} < \alpha < 1$, *we get that*

$$\left(D_{b-}^{3\gamma} f\right)(x) = \frac{1}{\Gamma\left((n+1)\alpha - 3\gamma\right)} \int_x^b (z-x)^{(n+1)\alpha - 3\gamma - 1} \left(D_{b-}^{(n+1)\alpha} f\right)(z)\, dz, \tag{292}$$

$\forall\, x \in [a,b]$, *and* $\left(D_{b-}^{3\gamma} f\right) \in C\left([a,b], X\right)$.

Proof. Call $\rho := (n+1)\alpha - 2\gamma - 1$, i.e. $\rho + 1 = (n+1)\alpha - 2\gamma$, and call again $\delta := (n+1)\alpha$. Then we can write

$$\left(D_{b-}^{2\gamma} f\right)(x) = \frac{1}{\Gamma(\rho+1)} \int_x^b (z-x)^{\rho} \left(D_{b-}^{\delta} f\right)(z)\, dz, \quad \forall\, x \in [a,b]. \tag{293}$$

If $\rho > 0$, then

$$\left(D_{b-}^{2\gamma} f\right)'(x) = \frac{(-1)\,\rho}{\Gamma(\rho+1)} \int_x^b (z-x)^{\rho-1} \left(D_{b-}^{\delta} f\right)(z)\, dz, \quad \forall\, x \in [a,b]. \tag{294}$$

If $\rho - 1 > 0$, then

$$\left(D_{b-}^{2\gamma} f\right)''(x) = \frac{(-1)^2\,\rho(\rho-1)}{\Gamma(\rho+1)} \int_x^b (z-x)^{\rho-2} \left(D_{b-}^{\delta} f\right)(z)\, dz, \quad \forall\, x \in [a,b]. \tag{295}$$

If $\rho - 2 > 0$, then

$$\left(D_{b-}^{2\gamma} f\right)^{(3)}(x) = \frac{(-1)^3\,\rho(\rho-1)(\rho-2)}{\Gamma(\rho+1)} \int_x^b (z-x)^{\rho-3} \left(D_{b-}^{\delta} f\right)(z)\, dz, \quad \forall\, x \in [a,b]. \tag{296}$$

etc.

In general, if $\rho - m + 1 > 0$, then

$$\left(D_{b-}^{2\gamma} f\right)^{(m)}(x) =$$

$$\frac{(-1)^m\,\rho(\rho-1)(\rho-2)\ldots(\rho-m+1)}{\Gamma(\rho+1)} \int_x^b (z-x)^{(\rho-m+1)-1} \left(D_{b-}^{\delta} f\right)(z)\, dz \tag{297}$$

$$= \frac{(-1)^m\,\rho(\rho-1)(\rho-2)\ldots(\rho-m+1)\,\Gamma(\rho-m+1)\left(I_{b-}^{(\rho-m+1)}\left(D_{b-}^{\delta} f\right)\right)(x)}{\Gamma(\rho+1)}$$

$$= (-1)^m \left(I_{b-}^{(\rho-m+1)}\left(D_{b-}^{\delta} f\right)\right)(x), \quad \forall\, x \in [a,b]. \tag{298}$$

That is, if $\rho - m + 1 > 0$, then

$$\left(D_{b-}^{2\gamma} f\right)^{(m)}(x) = (-1)^m \left(I_{b-}^{(\rho-m+1)}\left(D_{b-}^{\delta} f\right)\right)(x), \quad \forall\, x \in [a,b]. \tag{299}$$

We notice that

$$\left(D_{b-}^{3\gamma} f\right)(x) = \left(D_{b-}^{\gamma}\left(D_{b-}^{2\gamma} f\right)\right)(x) = (-1)^m \left(I_{b-}^{m-\gamma}\left(D_{b-}^{2\gamma} f\right)^{(m)}\right)(x) =$$

$$(-1)^{2m}\left(I_{b-}^{m-\gamma} I_{b-}^{\rho-m+1}\left(D_{b-}^{\delta} f\right)\right)(x) = \left(I_{b-}^{\rho-\gamma+1}\left(D_{b-}^{\delta} f\right)\right)(x) = \tag{300}$$

$$\left(I_{b-}^{(n+1)\alpha-2\gamma-1-\gamma+1}\left(D_{b-}^{(n+1)\delta} f\right)\right)(x) = \left(I_{b-}^{(n+1)\alpha-3\gamma}\left(D_{b-}^{(n+1)\alpha} f\right)\right)(x), \quad \forall\, x \in [a,b].$$

That is, if $\frac{m+2\gamma}{n+1} < \alpha < 1$, we get

$$\left(D_{b-}^{3\gamma} f\right)(x) = \left(I_{b-}^{(n+1)\alpha-3\gamma}\left(D_{b-}^{(n+1)\alpha} f\right)\right)(x), \quad \forall\, x \in [a,b]. \tag{301}$$

We have proved the theorem. ■

In general, we derive the iterated right fractional derivative formula:

Theorem 52 *Under the assumptions of Theorem 49, and when* $\frac{m+(k-1)\gamma}{n+1} <$ $\alpha < 1$, $k \in \mathbb{N}$, *we get that:*

$$\left(D_{b-}^{k\gamma} f\right)(x) = \frac{1}{\Gamma((n+1)\alpha - k\gamma)} \int_x^b (z-x)^{(n+1)\alpha-k\gamma-1}\left(D_{b-}^{(n+1)\alpha} f\right)(z)\, dz, \tag{302}$$

$\forall\, x \in [a,b]$, *and* $\left(D_{b-}^{k\gamma} f\right) \in C([a,b], X)$.

Next we give a related generalized fractional Ostrowski type inequality:

Theorem 53 *Let* $g \in C^1([a,b])$ *and strictly increasing, such that* $g^{-1} \in C^1([g(a), g(b)])$, *and* $0 < \alpha < 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$, *where* $(X, \|\cdot\|)$ *is a Banach space. Let* $x_0 \in [a,b]$ *be fixed. Assume that* $F_k^{x_0} := D_{x_0-;g}^{k\alpha} f$, *for* $k = 1, ..., n$, *fulfill* $F_k^{x_0} \in C^1([a,b], X)$ *and* $F_{n+1}^{x_0} \in C([a,x_0], X)$ *and* $\left(D_{x_0-;g}^{i\alpha} f\right)(x_0) = 0$, $i = 1, ..., n$.

Similarly, we assume that $G_k^{x_0} := D_{x_0+;g}^{k\alpha} f$, *for* $k = 1, ..., n$, *fulfill* $G_k^{x_0} \in C^1([x_0,b], X)$ *and* $G_{n+1}^{x_0} \in ([x_0,b], X)$ *and* $\left(D_{x_0+;g}^{i\alpha} f\right)(x_0) = 0$, $i = 1, ..., n$.

Then

$$\left\| \frac{1}{b-a}\int_a^b f(x)\,dx - f(x_0) \right\| \leq \frac{1}{(b-a)\,\Gamma((n+1)\alpha+1)} \cdot$$

$$\left\{ (g(b) - g(x_0))^{(n+1)\alpha}(b-x_0)\left\| D_{x_0+;g}^{(n+1)\alpha} f \right\|_{\infty,[x_0,b]} + \right.$$

$$\left. (g(x_0) - g(a))^{(n+1)\alpha}(x_0-a)\left\| D_{x_0-;g}^{(n+1)\alpha} f \right\|_{\infty,[a,x_0]} \right\}. \tag{303}$$

Proof. By (171), we obtain

$$f(x)-f(x_0)=\frac{1}{\Gamma((n+1)\alpha)}\int_x^{x_0}(g(t)-g(x))^{(n+1)\alpha-1}g'(t)\left(D_{x_0-;g}^{(n+1)\alpha}f\right)(t)\,dt, \tag{304}$$

$\forall\ x\in[a,x_0]$.

Hence it holds

$$\|f(x)-f(x_0)\|\leq$$
$$\frac{1}{\Gamma((n+1)\alpha)}\int_x^{x_0}(g(t)-g(x))^{(n+1)\alpha-1}g'(t)\left\|\left(D_{x_0-;g}^{(n+1)\alpha}f\right)(t)\right\|dt\leq \tag{305}$$
$$\frac{\left\|D_{x_0-;g}^{(n+1)\alpha}f\right\|_{\infty,[a,x_0]}}{\Gamma((n+1)\alpha)}\frac{(g(x_0)-g(x))^{(n+1)\alpha}}{(n+1)\alpha}.$$

We have proved that

$$\|f(x)-f(x_0)\|\leq\frac{(g(x_0)-g(x))^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}\left\|D_{x_0-;g}^{(n+1)\alpha}f\right\|_{\infty,[a,x_0]}, \tag{306}$$

$\forall\ x\in[a,x_0]$.

Also, by (155), we obtain

$$f(x)-f(x_0)=\frac{1}{\Gamma((n+1)\alpha)}\int_{x_0}^{x}(g(x)-g(t))^{(n+1)\alpha-1}g'(t)\left(D_{x_0+;g}^{(n+1)\alpha}f\right)(t)\,dt, \tag{307}$$

$\forall\ x\in[x_0,b]$.

Hence

$$\|f(x)-f(x_0)\|\leq\frac{(g(x)-g(x_0))^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}\left\|D_{x_0+;g}^{(n+1)\alpha}f\right\|_{\infty,[x_0,b]}, \tag{308}$$

$\forall\ x\in[x_0,b]$.

Next we see that

$$\left\|\frac{1}{b-a}\int_a^b f(x)\,dx-f(x_0)\right\|=\frac{1}{b-a}\left\|\int_a^b(f(x)-f(x_0))\,dx\right\|\leq$$
$$\frac{1}{b-a}\int_a^b\|f(x)-f(x_0)\|\,dx= \tag{309}$$
$$\frac{1}{b-a}\left\{\int_a^{x_0}\|f(x)-f(x_0)\|\,dx+\int_{x_0}^{b}\|f(x)-f(x_0)\|\,dx\right\}\leq$$
$$\frac{1}{(b-a)\Gamma((n+1)\alpha+1)}\left\{\left(\int_a^{x_0}(g(x_0)-g(x))^{(n+1)\alpha}dx\right)\left\|D_{x_0-;g}^{(n+1)\alpha}f\right\|_{\infty,[a,x_0]}\right. \tag{310}$$

$$+\left(\int_{x_0}^{b}\left(g\left(x\right)-g\left(x_0\right)\right)^{(n+1)\alpha}dx\right)\left\|D_{x_0+;g}^{(n+1)\alpha}f\right\|_{\infty,[x_0,b]}\Bigg\}\leq$$

$$\frac{1}{(b-a)\,\Gamma\left((n+1)\,\alpha+1\right)}\left\{\left(g\left(x_0\right)-g\left(a\right)\right)^{(n+1)\alpha}\left(x_0-a\right)\left\|D_{x_0-;g}^{(n+1)\alpha}f\right\|_{\infty,[a,x_0]}\right.$$
$$(311)$$
$$\left.+\left(g\left(b\right)-g\left(x_0\right)\right)^{(n+1)\alpha}\left(b-x_0\right)\left\|D_{x_0+;g}^{(n+1)\alpha}f\right\|_{\infty,[x_0,b]}\right\},$$

proving the claim. ■

One can prove many analytic inequalities based on this Banach space setting and our many results presented here. Since this article turns out to be very long we choose to omit this interesting task leaving it to others.

4 Applications

We make

Remark 54 *Some examples for g follow:*

$$\begin{array}{l} g\left(x\right)=x,\quad x\in[a,b]\,,\\ g\left(x\right)=e^{x},\quad x\in[a,b]\subset\mathbb{R}, \end{array}\tag{312}$$

also

$$\begin{array}{l} g\left(x\right)=\sin x,\\ g\left(x\right)=\tan x,\ \text{when } x\in[a,b]:=\left[-\frac{\pi}{2}+\varepsilon,\frac{\pi}{2}-\varepsilon\right],\ \varepsilon>0\ \text{small}, \end{array}\tag{313}$$

and

$$g\left(x\right)=\cos x,\ \text{when } x\in[a,b]:=[\pi+\varepsilon,2\pi-\varepsilon]\,,\ \varepsilon>0\ \text{small}.\tag{314}$$

Above all g's are strictly increasing, $g\in C^{1}\left([a,b]\right)$, and $g^{-1}\in C^{n}\left(\left[g\left(a\right),g\left(b\right)\right]\right)$, for any $n\in\mathbb{N}$.

We give

Theorem 55 *Let $n\in\mathbb{N}$ and $f\in C^{n}\left([a,b]\,,X\right)$, where $[a,b]\subset\mathbb{R}$ and $\left(X,\|\cdot\|\right)$ is a Banach spacxe. Let any $x,y\in[a,b]$. Then*

$$f\left(x\right)=f\left(y\right)+\sum_{i=1}^{n-1}\frac{\left(e^{x}-e^{y}\right)^{i}}{i!}\left(f\circ\ln\right)^{(i)}\left(e^{y}\right)+R_{n}\left(y,x,e^{t}\right),\tag{315}$$

where

$$R_{n}\left(y,x,e^{t}\right)=\frac{1}{(n-1)!}\int_{y}^{x}\left(e^{x}-e^{t}\right)^{n-1}\left(f\circ\ln\right)^{(n)}\left(e^{t}\right)e^{t}dt=\tag{316}$$

$$\frac{1}{(n-1)!}\int_{e^{y}}^{e^{x}}\left(e^{x}-z\right)^{n-1}\left(f\circ\ln\right)^{(n)}\left(z\right)dz.$$

Proof. By Corollary 9, for $g(t) = e^t$. ■

We give

Theorem 56 *Here $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space, $F : [a,b] \to X$. Let $r > 0$ and $F \in L_\infty([a,b], X)$ and*

$$G(s) = \int_a^s \left(e^s - e^t\right)^{r-1} e^t F(t)\, dt, \tag{317}$$

all $s \in [a,b]$. Then $G \in AC([a,b], X)$ for $r \geq 1$ and $G \in C([a,b], X)$ for $r \in (0,1)$.

Proof. By Theorem 10. ■

We present

Theorem 57 *Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n\left(\left[-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon\right], X\right)$, where $\varepsilon > 0$ is small, and $(X, \|\cdot\|)$ is a Banach space, $-\frac{\pi}{2} + \varepsilon \leq x \leq \frac{\pi}{2} - \varepsilon$.*

Then

$$f(x) = f\left(-\frac{\pi}{2} + \varepsilon\right) + \sum_{i=1}^{n-1} \frac{\left(\sin x - \sin\left(-\frac{\pi}{2} + \varepsilon\right)\right)^i}{i!} \left(f \circ \sin^{-1}\right)^{(i)} \left(\sin\left(-\frac{\pi}{2} + \varepsilon\right)\right) +$$

$$\frac{1}{\Gamma(\alpha)} \int_{-\frac{\pi}{2}+\varepsilon}^{x} (\sin x - \sin t)^{\alpha-1} \cos t \left(D^{\alpha}_{\left(-\frac{\pi}{2}+\varepsilon\right)+;\sin} f\right)(t)\, dt = \tag{318}$$

$$f\left(-\frac{\pi}{2} + \varepsilon\right) + \sum_{i=1}^{n-1} \frac{\left(\sin x - \sin\left(-\frac{\pi}{2} + \varepsilon\right)\right)^i}{i!} \left(f \circ \sin^{-1}\right)^{(i)} \left(\sin\left(-\frac{\pi}{2} + \varepsilon\right)\right) +$$

$$\frac{1}{\Gamma(\alpha)} \int_{\sin\left(-\frac{\pi}{2}+\varepsilon\right)}^{\sin x} (\sin x - z)^{\alpha-1} \left(\left(D^{\alpha}_{\left(-\frac{\pi}{2}+\varepsilon\right)+;\sin} f\right) \circ \sin^{-1}\right)(z)\, dz.$$

Proof. By Theorem 18. ■

We continue with

Theorem 58 *Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n\left(\left[-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon\right], X\right)$, where $\varepsilon > 0$ is small, and $(X, \|\cdot\|)$ is a Banach space, $-\frac{\pi}{2} + \varepsilon \leq x \leq \frac{\pi}{2} - \varepsilon$.*

Then

$$f(x) = f\left(\frac{\pi}{2} - \varepsilon\right) + \sum_{i=1}^{n-1} \frac{\left(\tan x - \tan\left(\frac{\pi}{2} - \varepsilon\right)\right)^i}{i} \left(f \circ \tan^{-1}\right)^{(i)} \left(\tan\left(\frac{\pi}{2} - \varepsilon\right)\right) +$$

$$\frac{1}{\Gamma(\alpha)} \int_{x}^{\frac{\pi}{2}-\varepsilon} (\tan t - \tan x)^{\alpha-1} \sec^2 t \left(D^{\alpha}_{\left(\frac{\pi}{2}-\varepsilon\right)-;\tan} f\right)(t)\, dt = \tag{319}$$

$$f\left(\frac{\pi}{2} - \varepsilon\right) + \sum_{i=1}^{n-1} \frac{\left(\tan x - \tan\left(\frac{\pi}{2} - \varepsilon\right)\right)^i}{i!} \left(f \circ \tan^{-1}\right)^{(i)} \left(\tan\left(\frac{\pi}{2} - \varepsilon\right)\right) +$$

$$\frac{1}{\Gamma(\alpha)} \int_{\tan x}^{\tan\left(\frac{\pi}{2}-\varepsilon\right)} (z - \tan x)^{\alpha-1} \left(\left(D^{\alpha}_{\left(\frac{\pi}{2}-\varepsilon\right)-;\tan} f\right) \circ \tan^{-1}\right)(z)\, dz.$$

Proof. By Theorem 19. ■

We derive

Theorem 59 *Let $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$. Let $F_k := D^{k\alpha}_{a+;e^t} f$, $k = 1, ..., n$, that fulfill $F_k \in C^1([a,b], X)$ and $F_{n+1} \in C([a,b], X)$. Then*

$$f(x) = \sum_{i=0}^{n} \frac{(e^x - e^a)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D^{i\alpha}_{a+;e^t} f\right)(a) + \tag{320}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x \left(e^x - e^t\right)^{(n+1)\alpha - 1} e^t \left(D^{(n+1)\alpha}_{a+;e^t} f\right)(t)\, dt,$$

$\forall\, x \in [a,b]$.

Proof. By Theorem 30. ■

We further have

Theorem 60 *Let $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1\left(\left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right], X\right)$, $\varepsilon > 0$, small. Suppose that $F_k := D^{k\alpha}_{\left(\frac{\pi}{2}-\varepsilon\right)-;\tan} f$, for $k = 1, ..., n$, fulfill $F_k \in C^1\left(\left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right], X\right)$ and $F_{n+1} \in C\left(\left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right], X\right)$. Then*

$$f(x) = \sum_{i=0}^{n} \frac{\left(\tan\left(\frac{\pi}{2}-\varepsilon\right) - \tan x\right)^{i\alpha}}{\Gamma(i\alpha+1)} \left(D^{i\alpha}_{\left(\frac{\pi}{2}-\varepsilon\right)-;\tan} f\right)\left(\frac{\pi}{2}-\varepsilon\right) + \tag{321}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^{\left(\frac{\pi}{2}-\varepsilon\right)} (\tan t - \tan x)^{(n+1)\alpha-1} \sec^2 t \left(D^{(n+1)\alpha}_{\left(\frac{\pi}{2}-\varepsilon\right)-;\tan} f\right)(t)\, dt,$$

$\forall\, t \in \left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right]$.

Proof. By Theorem 33. ■

We give the following Ostrowski type fractional inequality:

Theorem 61 *Let $0 < \alpha < 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$, where $(X, \|\cdot\|)$ is a Banach space, $x_0 \in [a,b]$. Assume that $F_k^{x_0} := D^{k\sigma}_{x_0-;e^t} f$, for $k = 1, ..., n$, fulfill $F_k^{x_0} \in C^1([a,x_0], X)$ and $F_{n+1}^{x_0} \in C([a,x_0], X)$ and $\left(D^{i\alpha}_{x_0-;e^t} f\right)(x_0) = 0$, $i = 1, ..., n$.*

Similarly, we assume that $G_k^{x_0} := D^{k\alpha}_{x_0+;e^t} f$, for $k = 1, ..., n$, fulfill $G_k^{x_0} \in C^1([x_0,b], X)$ and $G_{n+1}^{x_0} \in C([x_0,b], X)$ and $\left(D^{i\alpha}_{x_0+;e^t} f\right)(x_0) = 0$, $i = 1, ..., n$.

Then

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(x_0) \right\| \leq \frac{1}{(b-a)\Gamma((n+1)\alpha+1)} \cdot \tag{322}$$

$$\left\{ \left(e^b - e^{x_0}\right)^{(n+1)\alpha} (b - x_0) \left\| D^{(n+1)\alpha}_{x_0+;e^t} f \right\|_{\infty,[x_0,b]} + \right.$$

$$\left. (e^{x_0} - e^a)^{(n+1)\alpha} (x_0 - a) \left\| D^{(n+1)\alpha}_{x_0-;e^t} f \right\|_{\infty,[a,x_0]} \right\}.$$

Proof. By Theorem 53 for $g(t) = e^t$. ■

We finish with

Theorem 62 *Let* $0 < \alpha < 1$, $n \in \mathbb{N}$, $f \in C^1([\pi+\varepsilon, 2\pi-\varepsilon], X)$, $\varepsilon > 0$ *small, where* $(X, \|\cdot\|)$ *is a Banach space,* $x_0 \in [\pi+\varepsilon, 2\pi-\varepsilon]$. *Assume that* $F_k^{x_0} := D_{x_0-;\cos}^{k\alpha} f$, *for* $k = 1, ..., n$, *fulfill* $F_k^{x_0} \in C^1([\pi+\varepsilon, x_0], X)$ *and* $F_{n+1}^{x_0} \in C([\pi+\varepsilon, x_0], X)$ *and* $\left(D_{x_0-;\cos}^{i\alpha} f\right)(x_0) = 0$, $i = 1, ..., n$.

Similarly, we assume that $G_k^{x_0} := D_{x_0+;\cos}^{k\alpha} f$, *for* $k = 1, ..., n$, *fulfill* $G_k^{x_0} \in C^1([x_0, 2\pi-\varepsilon], X)$ *and* $G_{n+1}^{x_0} \in C([x_0, 2\pi-\varepsilon], X)$ *and* $\left(D_{x_0+;\cos}^{i\alpha} f\right)(x_0) = 0$, $i = 1, ..., n$.

Then

$$\left\| \frac{1}{\pi - 2\varepsilon} \int_{\pi+\varepsilon}^{2\pi-\varepsilon} f(x)\, dx - f(x_0) \right\| \leq \frac{1}{(\pi - 2\varepsilon)\, \Gamma((n+1)\alpha + 1)} \cdot$$

$$\left\{ (\cos(2\pi-\varepsilon) - \cos x_0)^{(n+1)\alpha} (2\pi - \varepsilon - x_0) \left\| D_{x_0+;\cos}^{(n+1)\alpha} f \right\|_{\infty,[x_0, 2\pi-\varepsilon]} + \right.$$

$$\left. (\cos x_0 - \cos(\pi+\varepsilon))^{(n+1)\alpha} (x_0 - \pi - \varepsilon) \left\| D_{x_0-;\cos}^{(n+1)\alpha} f \right\|_{\infty,[\pi+\varepsilon, x_0]} \right\}. \quad (323)$$

Proof. By Theorem 53 for $g(t) = \cos ine$. ■

References

[1] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis*, Springer, New York, 2006.

[2] G.A. Anastassiou, *Strong Right Fractional Calculus for Banach space valued functions*, 'Revista Proyecciones, Vol. 36, No. 1 (2017), 149-186.

[3] G.A. Anastassiou, *Strong mixed and generalized fractional calculus for Banach space valued functions,* Mat. Vesnik, accepted for publication, 2016.

[4] G.A. Anastassiou, *A strong Fractional Calculus Theory for Banach space valued functions*, Nonlinear Functional Analysis and Applications (Korea), accepted for publication, 2017.

[5] Appendix F, *The Bochner integral and vector-valued L_p-spaces*, https://isem.math.kit.edu/images/f/f7/AppendixF.pdf.

[6] Bochner integral. *Encyclopedia of Mathematics.* URL: http://www.encyclopediaofmath.org/index.php?title=Bochner_integral&oldid=38659.

[7] J.A. Canavati, *The Riemann-Liouville integral*, Nieun Archief Voor Wiskunde, 5 (1)(1987), 53-75.

[8] P. Hajek, M. Johanis, *Smooth Analysis in Banach spaces*, De Gruyter, Berlin, Boston, 2014.

[9] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.

[10] Rong-Qing Jia, *Chapter 3. Absolutely Continuous Functions*, https://www.ualberta.ca/~rjia/Math418/Notes/Chap.3.pdf.

[11] H.L. Royden, *Real Analysis*, 2nd Edition, Macmillan Publishing Co., Inc., New York, 1968.

[12] C. Volintiru, *A proof of the fundamental theorem of Calculus using Hausdorff measures*, Real Analysis Exchange, 26 (1), 2000/2001, pp. 381-390.