Received 01/06/17

# Iterated convergence on Banach space valued functions of abstract $g$-fractional calculus

George A. Anastassiou[1] and Ioannis K. Argyros[2]
[1]Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu
[2]Department of Mathematical Sciences
Cameron University
Lawton, OK 73505, USA
iargyros@cameron.edu

**Abstract**

The novelty of this paper is the design of suitable iterative methods for generating a sequence approximating solutions of equations on Banach spaces. Applications of the semi-local convergence are suggested including Banach space valued functions of fractional calculus, where all integrals are of Bochner-type.

**2010 AMS Subject Classification Codes:** 65G99, 65H10, 26A33, 46B25, 47J25, 47J05.

**Key Words and Phrases:** Iterative method, Banach space, semi-local convergence, $g$-Fractional Calculus, Bochner-type integral.

## 1 Introduction

Let $B_1, B_2$ stand for Banach space and let $\Omega$ stand for an open subset of $B_1$. Let also $U(z,\xi) := \{u \in B_1 : \|u - z\| < \xi\}$ and let $\overline{U}(z,\xi)$ stand for the closure of $U(z,\xi)$.

Many problems in Computational Sciences, Engineering, Mathematical Chemistry, Mathematical Physics, Mathematical Economics and other disciplines can written like

$$F(x) = 0 \tag{1.1}$$

using Mathematical Modeling [1]-[17], where $F : \Omega \to B_2$ is a continuous operator. The solution $x^*$ of equation (1.1) is needed in closed form. However, this is achieved only in special cases. That explains why most solution methods for such equations are usually iterative. There is a plethora of iterative methods for solving equation (1.1), more the [2, 6, 7, 9 - 13, 15, 16].

**Newton's method** [6, 7, 11, 15, 16]:

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n). \tag{1.2}$$

Secant method:

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \tag{1.3}$$

where $[\cdot, \cdot; F]$ denotes a divided difference of order one on $\Omega \times \Omega$ [7, 15, 16].

Newton-like method:

$$x_{n+1} = x_n - E_n^{-1} F(x_n), \tag{1.4}$$

where $E_n = E(F)(x_n)$ and $E : \Omega \to \mathcal{L}(B_1, B_2)$ the space of bounded linear operators from $B_1$ into $B_2$. Other methods can be found in [7], [11], [15], [16] and the references therein.

In the present study we consider the new method defined for each $n = 0, 1, 2, ...$ by

$$x_{n+1} = G(x_n)$$
$$G(x_{n+1}) = G(x_n) - A_n^{-1} F(x_n), \tag{1.5}$$

where $x_0 \in \Omega$ is an initial point, $G : B_3 \to \Omega$ ($B_3$ a Banach space), $A_n = A(F)(x_{n+1}, x_n) = A(x_{n+1}, x_n)$ and $A : \Omega \times \Omega \to \mathcal{L}(B_1, B_2)$. Method (1.5) generates a sequence which we shall show converges to $x^*$ under some Lipschitz-type conditions (to be precised in Section 2). Although method (1.5) (and Section 2) is of independent interest, it is nevertheless designed especially to be used in $g$-Abstract Fractional Calculus (to be precised in Section 3). As far as we know such iterative methods have not yet appeared in connection to solve equations in Abstract Fractional Calculus.

In this paper we present the semi-local convergence of method (1.5) in Section 2. Some applications to Abstract $g$-Fractional Calculus are suggested in Section 3 on a certain Banach space valued functions, where all the integrals are of Bochner-type [8], [14].

## 2 Semi-local Convergence analysis

We present the semi-local convergence analysis of method (1.5) using conditions $(M)$:

$(m_1)$ $F : \Omega \subset B_1 \to B_2$ is continuous, $G : B_3 \to \Omega$ is continuous and $A(x, y) \in \mathcal{L}(B_1, B_2)$ for each $(x, y) \in \Omega \times \Omega$.

$(m_2)$ There exist $\beta > 0$ and $\Omega_0 \subset B_1$ such that $A(x,y)^{-1} \in \mathcal{L}(B_2, B_1)$ for each $(x,y) \in \Omega_0 \times \Omega_0$ and

$$\left\| A(x,y)^{-1} \right\| \leq \beta^{-1}.$$

Set $\Omega_1 = \Omega \cap \Omega_0$.

$(m_3)$ There exists a continuous and nondecreasing function $\psi : [0,+\infty)^3 \rightarrow [0,+\infty)$ such that for each $x, y \in \Omega_1$

$$\| F(x) - F(y) - A(x,y)(G(x) - G(y)) \| \leq$$

$$\beta \psi (\|x-y\|, \|x-x_0\|, \|y-x_0\|) \|G(x) - G(y)\|.$$

$(m_4)$ There exists a continuous and nondecreasing function $\psi_0 : [0,+\infty) \rightarrow [0,+\infty)$ such that for each $x \in \Omega_1$

$$\|G(x) - G(x_0)\| \leq \psi_0 (\|x - x_0\|) \|x - x_0\|.$$

$(m_5)$ For $x_0 \in \Omega_0$ and $x_1 = G(x_0) \in \Omega_0$ there exists $\eta \geq 0$ such that

$$\left\| A(x_1, x_0)^{-1} F(x_0) \right\| \leq \eta.$$

$(m_6)$ There exists $s > 0$ such that

$$\psi(\eta, s, s) < 1,$$

$$\psi_0(s) < 1$$

and

$$\|G(x_0) - x_0\| \leq s \leq \frac{\eta}{1 - q_0},$$

where $q_0 = \psi(\eta, s, s)$.

$(m_7)$ $\overline{U}(x_0, s) \subset \Omega$.

Next, we present the semi-local convergence analysis for method (1.5) using the conditions $(M)$ and the preceding notation.

**Theorem 2.1** *Assume that the conditions $(M)$ hold. Then, sequence $\{x_n\}$ generated by method (1.5) starting at $x_0 \in \Omega$ is well defined in $U(x_0, s)$, remains in $U(x_0, s)$ for each $n = 0, 1, 2, \ldots$ and converges to a solution $x^* \in \overline{U}(x_0, s)$ of equation $F(x) = 0$. The limit point $x^*$ is the unique solution of equation $F(x) = 0$ in $\overline{U}(x_0, s)$.*

**Proof.** By the definition of $s$ and $(m_5)$, we have $x_1 \in U(x_0, s)$. The proof is based on mathematical induction on $k$. Suppose that $\|x_k - x_{k-1}\| \leq q_0^{k-1} \eta$ and $\|x_k - x_0\| \leq s$.

We get by (1.5), $(m_2)-(m_5)$ in turn that

$$\|G(x_{k+1})-G(x_k)\| = \left\|A_k^{-1}F(x_k)\right\| =$$
$$\left\|A_k^{-1}\left(F(x_k)-F(x_{k-1})-A_{k-1}\left(G(x_k)-G(x_{k-1})\right)\right)\right\|$$
$$\leq \left\|A_k^{-1}\right\| \|F(x_k)-F(x_{k-1})-A_{k-1}\left(G(x_k)-G(x_{k-1})\right)\| \leq$$
$$\beta^{-1}\beta\psi\left(\|x_k-x_{k-1}\|,\|x_{k-1}-x_0\|,\|y_k-x_0\|\right)\|G(x_k)-G(x_{k-1})\| \leq$$
$$\psi(\eta,s,s)\|G(x_k)-G(x_{k-1})\| = q_0\|G(x_k)-G(x_{k-1})\| \leq q_0^k\|x_1-x_0\| \leq q_0^k\eta \tag{2.1}$$

and by $(m_6)$

$$\|x_{k+1}-x_0\| = \|G(x_k)-x_0\| \leq \|G(x_k)-G(x_0)\| + \|G(x_0)-x_0\|$$
$$\leq \psi_0(\|x_k-x_0\|)\|x_k-x_0\| + \|G(x_0)-x_0\|$$
$$\leq \psi_0(s)s + \|G(x_0)-x_0\| \leq s.$$

The induction is completed. Moreover, we have by (2.1) that for $m=0,1,2,...$

$$\|x_{k+m}-x_k\| \leq \frac{1-q_0^m}{1-q_0}q_0^k\eta.$$

It follows from the preceding inequation that sequence $\{G(x_k)\}$ is complete in a Banach space $B_1$ and as such it converges to some $x^* \in \overline{U}(x_0,s)$ (since $\overline{U}(x_0,s)$ is a closed ball). By letting $k\to+\infty$ in (2.1) we get $F(x^*)=0$. We also get by (1.5) that $G(x^*)=x^*$. To show the uniqueness part, let $x^{**}\in U(x_0,s)$ be a solution of equation $F(x)=0$ and $G(x^{**})=x^{**}$. By using (1.5), we obtain in turn that

$$\|x^{**}-G(x_{k+1})\| = \left\|x^{**}-G(x_k)+A_k^{-1}F(x_k)-A_k^{-1}F(x^{**})\right\| \leq$$
$$\left\|A_k^{-1}\right\| \|F(x^{**})-F(x_k)-A_k\left(G(x^{**})-G(x_k)\right)\| \leq$$
$$\beta^{-1}\beta\psi_0\left(\|x^{**}-x_k\|,\|x_{k+1}-x_0\|,\|x_k-x_0\|\right)\|G(x^{**})-G(x_k)\| \leq$$
$$q_0\|G(x^{**})-G(x_k)\| \leq q_0^{k+1}\|x^{**}-x_0\|,$$

so $\lim_{k\to+\infty} x_k = x^{**}$. We have shown that $\lim_{k\to+\infty} x_k = x^*$, so $x^*=x^{**}$. ■

**Remark 2.2** *(1) Condition $(m_2)$ can become part of condition $(m_3)$ by considering*

$(m_3)'$ *There exists a continuous and nondecreasing function $\varphi:[0,+\infty)^3\to[0,+\infty)$ such that for each $x,y\in\Omega_1$*

$$\left\|A(x,y)^{-1}\left[F(x)-F(y)-A(x,y)\left(G(x)-G(y)\right)\right]\right\| \leq$$

$$\varphi\left(\|x-y\|,\|x-x_0\|,\|y-x_0\|\right)\|G(x)-G(y)\|.$$

*Notice that*

$$\varphi(u_1,u_2,u_3)\le\psi(u_1,u_2,u_3)$$

*for each* $u_1\ge 0$, $u_2\ge 0$ *and* $u_3\ge 0$. *Similarly, a function* $\varphi_1$ *can replace* $\psi_1$ *for the uniqueness of the solution part. These replacements are of Mysovskii-type [6], [11], [15] and influence the weaking of the convergence criterion in* $(m_6)$, *error bounds and the precision of* $s$.

*(2) Suppose that there exist* $\beta>0$, $\beta_1>0$ *and* $L\in\mathcal{L}(B_1,B_2)$ *with* $L^{-1}\in\mathcal{L}(B_2,B_1)$ *such that*

$$\left\|L^{-1}\right\|\le\beta^{-1}$$

$$\|A(x,y)-L\|\le\beta_1$$

*and*

$$\beta_2:=\beta^{-1}\beta_1<1.$$

*Then, it follows from the Banach lemma on invertible operators [11], and*

$$\left\|L^{-1}\right\|\|A(x,y)-L\|\le\beta^{-1}\beta_1=\beta_2<1$$

*that* $A(x,y)^{-1}\in\mathcal{L}(B_2,B_1)$. *Let* $\beta=\frac{\beta^{-1}}{1-\beta_2}$. *Then, under these replacements, condition* $(m_2)$ *is implied, therefore it can be dropped from the conditions* $(M)$.

**Remark 2.3** *Section 2 has an interest independent of Section 3. It is worth noticing that the results especially of Theorem 2.1 can apply in Abstract g-Fractional Calculus as illustrated in Section 3. By specializing function* $\psi$, *we can apply the results of say Theorem 2.1 in the examples suggested in Section 3. In particular for (3.33), we choose for* $u_1\ge 0$, $u_2\ge 0$, $u_3\ge 0$

$$\psi(u_1,u_2,u_3)=\frac{\lambda\mu_1^{\alpha}}{\beta\Gamma(\alpha)(\alpha+1)},$$

*if* $|g(x)-g(y)|\le\mu_1$ *for each* $x,y\in[a,b]$;

$$\psi(u_1,u_2,u_3)=\frac{\lambda\mu_2^{\alpha}}{\beta\Gamma(\alpha)(\alpha+1)},$$

*if* $|g(x)-g(y)|\le\xi_2\|x-y\|$ *for each* $x,y\in[a,b]$ *and* $\mu_2=\xi_2|b-a|$;

$$\psi(u_1,u_2,u_3)=\frac{\lambda\mu_3^{\alpha}}{\beta\Gamma(\alpha)(\alpha+1)},$$

*if* $|g(x)|\le\xi_3$ *for each* $x,y\in[a,b]$ *and* $\mu_3=2\xi_3$, *where* $\alpha,\lambda$ *and* $\Gamma$ *are given in Section 3.*

*Other choices of function* $\psi$ *are also possible.*

*Notice that with these choices of function* $\psi$ *and* $f=F$ *and* $g=G$, *crucial condition* $(m_3)$ *is satisfied, which justifies our definition of method (1.5). We can provide similar choices for the other examples of Section 3.*

## 3 Applications to $X$-valued $g$-Fractional Calculus

Here we deal with Banach space $(X, \|\cdot\|)$ valued functions $f$ of real domain $[a,b]$. All integrals here are of Bochner-type, see [14]. The derivatives of $f$ are defined similarly to numerical ones, see [17], pp. 83-86 and p. 93.

Here both backgrounds needed come from [5].

I) We need

**Definition 3.1** *([5]) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$.*

*We define the left generalized $g$-fractional derivative $X$-valued of $f$ of order $\alpha$ as follows:*

$$\left(D_{a+;g}^{\alpha} f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x) - g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{3.1}$$

*$\forall\, x \in [a,b]$. The last integral is of Bochner type.*

*If $\alpha \notin \mathbb{N}$, by [5], we have that $\left(D_{a+;g}^{\alpha} f\right) \in C([a,b], X)$.*

*We see that*

$$\left(J_{a;g}^{n-\alpha}\left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = \left(D_{a+;g}^{\alpha} f\right)(x), \quad \forall\, x \in [a,b]. \tag{3.2}$$

*We set*

$$D_{a+;g}^{n} f(x) := \left(\left(f \circ g^{-1}\right)^{n} \circ g\right)(x) \in C([a,b], X), \quad n \in \mathbb{N}, \tag{3.3}$$

$$D_{a+;g}^{0} f(x) = f(x), \quad \forall\, x \in [a,b].$$

*When $g = id$, then*

$$D_{a+;g}^{\alpha} f = D_{a+;id}^{\alpha} f = D_{*a}^{\alpha} f,$$

*the usual left $X$-valued Caputo fractional derivative, see [4].*

We need the $X$-valued left general fractional Taylor's formula.

**Theorem 3.2** *([5]) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, $a \leq x \leq b$. Then*

$$f(x) = f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) +$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D_{a+;g}^{\alpha} f\right)(t)\, dt =$$

$$f(a)+\sum_{i=1}^{n-1}\frac{(g(x)-g(a))^i}{i!}\left(f\circ g^{-1}\right)^{(i)}(g(a))+ \tag{3.4}$$

$$\frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D_{a+;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz.$$

*The remainder of (3.4) is a continuous function in* $x\in[a,b]$.

Here we are going to operate more generally. We consider $f\in C^n([a,b],X)$. We define the following $X$-valued left $g$-fractional derivative of $f$ of order $\alpha$ as follows:

$$\left(D_{y+;g}^{\alpha}f\right)(x):=\frac{1}{\Gamma(n-\alpha)}\int_y^x(g(x)-g(t))^{n-\alpha-1}g'(t)\left(f\circ g^{-1}\right)^{(n)}(g(t))\,dt, \tag{3.5}$$

for any $a\leq y\leq x\leq b$;

$$D_{y+;g}^{n}f(x)=\left(\left(f\circ g^{-1}\right)^{(n)}\circ g\right)(x),\ \ \forall\ x,y\in[a,b], \tag{3.6}$$

and

$$D_{y+;g}^{0}f(x)=f(x),\ \ \forall\ x\in[a,b]. \tag{3.7}$$

For $\alpha>0$, $\alpha\notin\mathbb{N}$, by convention we set that

$$\left(D_{y+;g}^{\alpha}f\right)(x)=0,\ \ \text{for } x<y,\ \ \forall\ x,y\in[a,b]. \tag{3.8}$$

Similarly, we define

$$\left(D_{x+;g}^{\alpha}f\right)(y):=\frac{1}{\Gamma(n-\alpha)}\int_x^y(g(y)-g(t))^{n-\alpha-1}g'(t)\left(f\circ g^{-1}\right)^{(n)}(g(t))\,dt, \tag{3.9}$$

for any $a\leq x\leq y\leq b$;

$$D_{x+;g}^{n}f(y)=\left(\left(f\circ g^{-1}\right)^{(n)}\circ g\right)(y),\ \ \forall\ x,y\in[a,b], \tag{3.10}$$

and

$$D_{x+;g}^{0}f(y)=f(y),\ \ \forall\ y\in[a,b]. \tag{3.11}$$

For $\alpha>0$, $\alpha\notin\mathbb{N}$, by convention we set that

$$\left(D_{x+;g}^{\alpha}f\right)(y)=0,\ \ \text{for } y<x,\ \ \forall\ x,y\in[a,b]. \tag{3.12}$$

We get that (see [8])

$$\left\|\left(D_{a+;g}^{\alpha}f\right)(x)\right\|\leq\frac{1}{\Gamma(n-\alpha)}\int_a^x(g(x)-g(t))^{n-\alpha-1}g'(t)\left\|\left(f\circ g^{-1}\right)^{(n)}(g(t))\right\|dt \tag{3.13}$$

$$\leq \frac{\left\| \left(f \circ g^{-1}\right)^{(n)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(n-\alpha)} \int_a^x (g(x) - g(t))^{n-\alpha-1} g'(t)\, dt =$$

$$\frac{\left\| \left(f \circ g^{-1}\right)^{(n)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)} (g(x) - g(a))^{n-\alpha} \leq$$

$$\frac{\left\| \left(f \circ g^{-1}\right)^{(n)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)} (g(b) - g(a))^{n-\alpha}, \quad \forall\, x \in [a,b]. \tag{3.14}$$

That is

$$\left(D^{\alpha}_{a+;g} f\right)(a) = 0, \tag{3.15}$$

and

$$\left(D^{\alpha}_{y+;g} f\right)(y) = \left(D^{\alpha}_{x+;g} f\right)(x) = 0, \quad \forall\, x, y \in [a,b]. \tag{3.16}$$

Thus when $\alpha > 0$, $\alpha \notin \mathbb{N}$, both $D^{\alpha}_{y+;g} f$, $D^{\alpha}_{x+;g} f \in C([a,b], X)$, (see [5]).

Hence by Theorem 3.2 we obtain

$$f(x) - f(y) = \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(y))}{k!} (g(x) - g(y))^k +$$

$$\frac{1}{\Gamma(\alpha)} \int_y^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D^{\alpha}_{y+;g} f\right)(t)\, dt, \quad \forall\, x \in [y,b], \tag{3.17}$$

and

$$f(y) - f(x) = \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(x))}{k!} (g(y) - g(x))^k +$$

$$\frac{1}{\Gamma(\alpha)} \int_x^y (g(y) - g(t))^{\alpha-1} g'(t) \left(D^{\alpha}_{x+;g} f\right)(t)\, dt, \quad \forall\, y \in [x,b], \tag{3.18}$$

We define also the following $X$-valued linear operator

$$(A_1(f))(x,y) :=$$

$$\begin{cases} \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(y))}{k!} (g(x) - g(y))^{k-1} + \left(D^{\alpha}_{y+;g} f\right)(x) \frac{(g(x)-g(y))^{\alpha-1}}{\Gamma(\alpha+1)}, & \text{for } x > y, \\ \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x))}{k!} (g(y) - g(x))^{k-1} + \left(D^{\alpha}_{x+;g} f\right)(y) \frac{(g(y)-g(x))^{\alpha-1}}{\Gamma(\alpha+1)}, & \text{for } x < y, \\ f^{(n)}(x), \quad \text{when } x = y, \end{cases} \tag{3.19}$$

$\forall\, x, y \in [a,b]$; $\alpha > 0$, $n = \lceil \alpha \rceil$.

We may assume that (see [12], p. 3)

$$\|(A_1(f))(x,x) - (A_1(f))(y,y)\| = \left\| f^{(n)}(x) - f^{(n)}(y) \right\| \tag{3.20}$$

$$\left\| \left( f^{(n)} \circ g^{-1} \right) (g(x)) - \left( f^{(n)} \circ g^{-1} \right) (g(y)) \right\| \leq \Phi \left| g(x) - g(y) \right|, \quad \forall\, x, y \in [a,b];$$

where $\Phi > 0$.

We estimate and have

i) case of $x > y$ :

$$\| f(x) - f(y) - (A_1(f))(x,y)(g(x) - g(y)) \| =$$

$$\left\| \frac{1}{\Gamma(\alpha)} \int_y^x (g(x) - g(t))^{\alpha-1} g'(t) \left( D_{y+;g}^{\alpha} f \right)(t)\, dt - \right.$$

$$\left. \left( D_{y+;g}^{\alpha} f \right)(x) \frac{(g(x) - g(y))^{\alpha}}{\Gamma(\alpha+1)} \right\| \tag{3.21}$$

(by [1]. p. 426, Theorem 11.43)

$$= \frac{1}{\Gamma(\alpha)} \left\| \int_y^x (g(x) - g(t))^{\alpha-1} g'(t) \left( \left( D_{y+;g}^{\alpha} f \right)(t) - \left( D_{y+;g}^{\alpha} f \right)(x) \right) dt \right\|$$

(by [8])

$$\leq \frac{1}{\Gamma(\alpha)} \int_y^x (g(x) - g(t))^{\alpha-1} g'(t) \left\| \left( D_{y+;g}^{\alpha} f \right)(t) - \left( D_{y+;g}^{\alpha} f \right)(x) \right\| dt \tag{3.22}$$

(we assume that

$$\left\| \left( D_{y+;g}^{\alpha} f \right)(t) - \left( D_{y+;g}^{\alpha} f \right)(x) \right\| \leq \lambda_1 \left| g(t) - g(x) \right|, \tag{3.23}$$

$\forall\, t, x, y \in [a,b] : x \geq t \geq y;\ \lambda_1 > 0$)

$$\leq \frac{\lambda_1}{\Gamma(\alpha)} \int_y^x (g(x) - g(t))^{\alpha-1} g'(t) (g(x) - g(t))\, dt =$$

$$\frac{\lambda_1}{\Gamma(\alpha)} \int_y^x (g(x) - g(t))^{\alpha} g'(t)\, dt = \frac{\lambda_1}{\Gamma(\alpha)} \frac{(g(x) - g(y))^{\alpha+1}}{(\alpha+1)}. \tag{3.24}$$

We have proved that

$$\| f(x) - f(y) - (A_1(f))(x,y)(g(x) - g(y)) \| \leq$$

$$\frac{\lambda_1}{\Gamma(\alpha)} \frac{(g(x) - g(y))^{\alpha+1}}{(\alpha+1)}, \tag{3.25}$$

$\forall\, x, y \in [a,b] : x > y$.

ii) case of $y > x$ : We have that

$$\| f(x) - f(y) - (A_1(f))(x,y)(g(x) - g(y)) \| = \tag{3.26}$$

$$\| f(y) - f(x) - (A_1(f))(x,y)(g(y) - g(x)) \| =$$

$$\left\| \frac{1}{\Gamma(\alpha)} \int_x^y (g(y) - g(t))^{\alpha-1} g'(t) \left(D_{x+;g}^{\alpha} f\right)(t)\, dt - \right.$$

$$\left. \left(D_{x+;g}^{\alpha} f\right)(y) \frac{(g(y) - g(x))^{\alpha}}{\Gamma(\alpha+1)} \right\| =$$

$$\frac{1}{\Gamma(\alpha)} \left\| \int_x^y (g(y) - g(t))^{\alpha-1} g'(t) \left( \left(D_{x+;g}^{\alpha} f\right)(t) - \left(D_{x+;g}^{\alpha} f\right)(y) \right) dt \right\| \leq \tag{3.27}$$

$$\frac{1}{\Gamma(\alpha)} \int_x^y (g(y) - g(t))^{\alpha-1} g'(t) \left\| \left(D_{x+;g}^{\alpha} f\right)(t) - \left(D_{x+;g}^{\alpha} f\right)(y) \right\| dt \tag{3.28}$$

(we assume here that

$$\left\| \left(D_{x+;g}^{\alpha} f\right)(t) - \left(D_{x+;g}^{\alpha} f\right)(y) \right\| \leq \lambda_2 |g(t) - g(y)|, \tag{3.29}$$

$\forall\ t, y, x \in [a, b] : y \geq t \geq x; \lambda_2 > 0)$

$$\leq \frac{\lambda_2}{\Gamma(\alpha)} \int_x^y (g(y) - g(t))^{\alpha-1} g'(t) (g(y) - g(t))\, dt = \tag{3.30}$$

$$\frac{\lambda_2}{\Gamma(\alpha)} \int_x^y (g(y) - g(t))^{\alpha} g'(t)\, dt = \frac{\lambda_2}{\Gamma(\alpha)} \frac{(g(y) - g(x))^{\alpha+1}}{(\alpha+1)}. \tag{3.31}$$

We have proved that

$$\|f(x) - f(y) - (A_1(f))(x, y)(g(x) - g(y))\| \leq \tag{3.32}$$

$$\frac{\lambda_2}{\Gamma(\alpha)} \frac{(g(y) - g(x))^{\alpha+1}}{(\alpha+1)}, \quad \forall\ x, y \in [a, b] : y > x.$$

**Conclusion 3.3** *Set* $\lambda := \max(\lambda_1, \lambda_2)$. *Then*

$$\|f(x) - f(y) - (A_1(f))(x, y)(g(x) - g(y))\| \leq$$

$$\frac{\lambda}{\Gamma(\alpha)} \frac{|g(x) - g(y)|^{\alpha+1}}{(\alpha+1)}, \quad \forall\ x, y \in [a, b]. \tag{3.33}$$

Notice that (3.33) is trivially true when $x = y$.

One may assume that

$$\frac{\lambda}{\Gamma(\alpha)} < 1. \tag{3.34}$$

Now based on (3.20) and (3.33), we can apply our numerical methods presented in this article to solve $f(x) = 0$.

II) In the next background again we use [5].

We need

**Definition 3.4** *([5])Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$.*

*We define the right generalized $g$-fractional derivative $X$-valued of $f$ of order $\alpha$ as follows:*

$$\left(D_{b-;g}^{\alpha} f\right)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (g(t) - g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{3.35}$$

*$\forall\, x \in [a,b]$. The last integral is of Bochner type.*

*If $\alpha \notin \mathbb{N}$, by [5], we have that $\left(D_{b-;g}^{\alpha} f\right) \in C([a,b], X)$.*

*We see that*

$$J_{b-;g}^{n-\alpha} \left((-1)^n \left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x) = \left(D_{b-;g}^{\alpha} f\right)(x), \quad a \leq x \leq b. \tag{3.36}$$

*We set*

$$D_{b-;g}^{n} f(x) := (-1)^n \left(\left(f \circ g^{-1}\right)^{n} \circ g\right)(x) \in C([a,b], X), \quad n \in \mathbb{N}, \tag{3.37}$$

$$D_{b-;g}^{0} f(x) := f(x), \quad \forall\, x \in [a,b].$$

*When $g = id$, then*

$$D_{b-;g}^{\alpha} f(x) = D_{b-;id}^{\alpha} f(x) = D_{b-}^{\alpha} f, \tag{3.38}$$

*the usual right $X$-valued Caputo fractional derivative, see [3].*

We also need the Taylor's formula.

**Theorem 3.5** *([5])Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, $a \leq x \leq b$. Then*

$$\begin{aligned} f(x) = f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) + \\ \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha} f\right)(t)\, dt = \\ f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) + \\ \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{\alpha-1} \left(\left(D_{b-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz. \end{aligned} \tag{3.39}$$

*The remainder of (3.39) is a continuous function in $x \in [a,b]$.*

Here we are going to operate more generally. We consider $f \in C^n([a,b],X)$. We define the following $X$-valued right $g$-fractional derivative of $f$ of order $\alpha$ as follows:

$$\left(D_{y-;g}^{\alpha}f\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^y (g(t)-g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{3.40}$$

$\forall\, x \in [a,y]$; where $y \in [a,b]$;

$$\left(D_{y-;g}^{n}f\right)(x) = (-1)^n \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x), \ \ \forall\, x,y \in [a,b], \tag{3.41}$$

$$\left(D_{y-;g}^{0}f\right)(x) = f(x), \ \ \forall\, x \in [a,b]. \tag{3.42}$$

For $\alpha > 0$, $\alpha \notin \mathbb{N}$, by convention we set that

$$\left(D_{y-;g}^{\alpha}f\right)(x) = 0, \ \text{ for } x > y, \ \ \forall\, x,y \in [a,b]. \tag{3.43}$$

Similarly, we define

$$\left(D_{x-;g}^{\alpha}f\right)(y) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_y^x (g(t)-g(y))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{3.44}$$

$\forall\, y \in [a,x]$, where $x \in [a,b]$;

$$\left(D_{x-;g}^{n}f\right)(y) = (-1)^n \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(y), \ \ \forall\, x,y \in [a,b], \tag{3.45}$$

$$\left(D_{x-;g}^{0}f\right)(y) = f(y), \ \ \forall\, y \in [a,b]. \tag{3.46}$$

For $\alpha > 0$, $\alpha \notin \mathbb{N}$, by convention we set that

$$\left(D_{x-;g}^{\alpha}f\right)(y) = 0, \ \text{ for } y > x, \ \ \forall\, x,y \in [a,b]. \tag{3.47}$$

We get that

$$\left\|\left(D_{b-;g}^{\alpha}f\right)(x)\right\| \leq \frac{\left\|\left(f \circ g^{-1}\right)^{(n)} \circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)} (g(b)-g(x))^{n-\alpha} \leq \tag{3.48}$$

$$\frac{\left\|\left(f \circ g^{-1}\right)^{(n)} \circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)} (g(b)-g(a))^{n-\alpha}, \ \ \forall\, x \in [a,b].$$

That is

$$\left(D_{b-;g}^{\alpha}f\right)(b) = 0, \tag{3.49}$$

and

$$\left(D_{y-;g}^{\alpha}f\right)(y) = \left(D_{x-;g}^{\alpha}f\right)(x) = 0, \ \ \forall\, x,y \in [a,b]. \tag{3.50}$$

Thus when $\alpha > 0$, $\alpha \notin \mathbb{N}$, both $D_{y-;g}^{\alpha}f$, $D_{x-;g}^{\alpha}f \in C([a,b],X)$, see [5].

Hence by Theorem 3.5 we obtain

$$f(x)-f(y)=\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^{k}+$$

$$\frac{1}{\Gamma(\alpha)}\int_{x}^{y}(g(t)-g(x))^{\alpha-1}g'(t)\left(D_{y-;g}^{\alpha}f\right)(t)\,dt,\quad \text{all } a\leq x\leq y\leq b. \tag{3.51}$$

Also, we have

$$f(y)-f(x)=\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(x))}{k!}(g(y)-g(x))^{k}+$$

$$\frac{1}{\Gamma(\alpha)}\int_{y}^{x}(g(t)-g(y))^{\alpha-1}g'(t)\left(D_{x-;g}^{\alpha}f\right)(t)\,dt,\quad \text{all } a\leq y\leq x\leq b. \tag{3.52}$$

We define also the following $X$-valued linear operator

$$(A_2(f))(x,y):=$$

$$\begin{cases}\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^{k-1}-\left(D_{y-;g}^{\alpha}f\right)(x)\frac{(g(y)-g(x))^{\alpha-1}}{\Gamma(\alpha+1)}, & \text{for } x<y,\\ \sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(x))}{k!}(g(y)-g(x))^{k-1}-\left(D_{x-;g}^{\alpha}f\right)(y)\frac{(g(x)-g(y))^{\alpha-1}}{\Gamma(\alpha+1)}, & \text{for } x>y,\\ f^{(n)}(x), \quad \text{when } x=y,\end{cases} \tag{3.53}$$

$\forall\ x,y\in[a,b]$; $\alpha>0$, $n=\lceil\alpha\rceil$.

We may assume that ([12], p. 3)

$$\|(A_2(f))(x,x)-(A_2(f))(y,y)\|=\left\|f^{(n)}(x)-f^{(n)}(y)\right\| \tag{3.54}$$

$$\leq\Phi^{*}|g(x)-g(y)|,\quad\forall\ x,y\in[a,b];$$

where $\Phi^{*}>0$.

We estimate and have

i) case of $x<y$:

$$\|f(x)-f(y)-(A_2(f))(x,y)(g(x)-g(y))\|=$$

$$\left\|\frac{1}{\Gamma(\alpha)}\int_{x}^{y}(g(t)-g(x))^{\alpha-1}g'(t)\left(D_{y-;g}^{\alpha}f\right)(t)\,dt-\right.$$

$$\left.\left(D_{y-;g}^{\alpha}f\right)(x)\frac{(g(y)-g(x))^{\alpha}}{\Gamma(\alpha+1)}\right\| \tag{3.55}$$

(by [1], p. 426, Theorem 11.43)

$$=\frac{1}{\Gamma(\alpha)}\left\|\int_{x}^{y}(g(t)-g(x))^{\alpha-1}g'(t)\left(\left(D_{y-;g}^{\alpha}f\right)(t)-\left(D_{y-;g}^{\alpha}f\right)(x)\right)dt\right\|$$

(by [8])

$$\leq \frac{1}{\Gamma(\alpha)} \int_x^y (g(t) - g(x))^{\alpha-1} g'(t) \left\| \left(D_{y-;g}^{\alpha} f\right)(t) - \left(D_{y-;g}^{\alpha} f\right)(x) \right\| dt \tag{3.56}$$

(we assume that

$$\left\| \left(D_{y-;g}^{\alpha} f\right)(t) - \left(D_{y-;g}^{\alpha} f\right)(x) \right\| \leq \rho_1 |g(t) - g(x)|, \tag{3.57}$$

$\forall\ t, x, y \in [a, b] : y \geq t \geq x;\ \rho_1 > 0)$

$$\leq \frac{\rho_1}{\Gamma(\alpha)} \int_x^y (g(t) - g(x))^{\alpha-1} g'(t) (g(t) - g(x)) dt =$$

$$\frac{\rho_1}{\Gamma(\alpha)} \int_x^y (g(t) - g(x))^{\alpha} g'(t) dt = \frac{\rho_1}{\Gamma(\alpha)} \frac{(g(y) - g(x))^{\alpha+1}}{(\alpha+1)}. \tag{3.58}$$

We have proved that

$$\|f(x) - f(y) - (A_2(f))(x, y)(g(x) - g(y))\| \leq$$

$$\frac{\rho_1}{\Gamma(\alpha)} \frac{(g(y) - g(x))^{\alpha+1}}{(\alpha+1)}, \tag{3.59}$$

$\forall\ x, y \in [a, b] : x < y.$

ii) case of $x > y$ :

$$\|f(x) - f(y) - (A_2(f))(x, y)(g(x) - g(y))\| =$$

$$\|f(y) - f(x) - (A_2(f))(x, y)(g(y) - g(x))\| = \tag{3.60}$$

$$\|f(y) - f(x) + (A_2(f))(x, y)(g(x) - g(y))\| =$$

$$\left\| \frac{1}{\Gamma(\alpha)} \int_y^x (g(t) - g(y))^{\alpha-1} g'(t) \left(D_{x-;g}^{\alpha} f\right)(t) dt - \right.$$

$$\left. \left(D_{x-;g}^{\alpha} f\right)(y) \frac{(g(x) - g(y))^{\alpha}}{\Gamma(\alpha+1)} \right\| =$$

$$\frac{1}{\Gamma(\alpha)} \left\| \int_y^x (g(t) - g(y))^{\alpha-1} g'(t) \left( \left(D_{x-;g}^{\alpha} f\right)(t) - \left(D_{x-;g}^{\alpha} f\right)(y) \right) dt \right\| \leq \tag{3.61}$$

$$\frac{1}{\Gamma(\alpha)} \int_y^x (g(t) - g(y))^{\alpha-1} g'(t) \left\| \left(D_{x-;g}^{\alpha} f\right)(t) - \left(D_{x-;g}^{\alpha} f\right)(y) \right\| dt \tag{3.62}$$

(we assume that

$$\left\| \left(D_{x-;g}^{\alpha} f\right)(t) - \left(D_{x-;g}^{\alpha} f\right)(y) \right\| \leq \rho_2 |g(t) - g(y)|, \tag{3.63}$$

$\forall\ t, y, x \in [a, b] : x \geq t \geq y; \rho_2 > 0)$

$$\leq \frac{\rho_2}{\Gamma(\alpha)} \int_y^x (g(t) - g(y))^{\alpha-1} g'(t) (g(t) - g(y)) dt =$$

$$\frac{\rho_2}{\Gamma(\alpha)} \int_y^x (g(t) - g(y))^{\alpha} g'(t)\, dt = \tag{3.64}$$

$$\frac{\rho_2}{\Gamma(\alpha)} \frac{(g(x) - g(y))^{\alpha+1}}{(\alpha+1)}. \tag{3.65}$$

We have proved that

$$\|f(x) - f(y) - (A_2(f))(x,y)(g(x) - g(y))\| \leq$$

$$\frac{\rho_2}{\Gamma(\alpha)} \frac{(g(x) - g(y))^{\alpha+1}}{(\alpha+1)}, \quad \forall\, x, y \in [a,b] : x > y. \tag{3.66}$$

**Conclusion 3.6** *Set* $\rho := \max(\rho_1, \rho_2)$. *Then*

$$\|f(x) - f(y) - (A_2(f))(x,y)(g(x) - g(y))\| \leq$$

$$\frac{\rho}{\Gamma(\alpha)} \frac{|g(x) - g(y)|^{\alpha+1}}{(\alpha+1)}, \quad \forall\, x, y \in [a,b]. \tag{3.67}$$

Notice that (3.67) is trivially true when $x = y$.

One may assume that

$$\frac{\rho}{\Gamma(\alpha)} < 1. \tag{3.68}$$

Now based on (3.54) and (3.67), we can apply our numerical methods presented in this article to solve $f(x) = 0$.

In both fractional applications $\alpha + 1 \geq 2$, iff $\alpha \geq 1$.

Also some examples for $g$ follow:

$$\begin{array}{l} g(x) = e^x, \;\; x \in [a,b] \subset \mathbb{R}, \\ g(x) = \sin x, \\ g(x) = \tan x, \\ \text{where } x \in \left[-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon\right], \text{ where } \varepsilon > 0 \text{ small.} \end{array} \tag{3.69}$$

# References


[1] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis*, Springer, New York, 2006.

[2] S. Amat, S. Busquier, S. Plaza, *Chaotic dynamics of a third-order Newton-type method*, J. Math. Anal. Applic. 366, 1, (2010), 164-174.

[3] G.A. Anastassiou, *Strong Right Fractional Calculus for Banach space valued functions*, Revista Proyecciones, Vol. 36, No. 1 (2017), 149-186.

[4] G.A. Anastassiou, *A strong Fractional Calculus Theory for Banach space valued functions*, Nonlinear Functional Analysis and Applications (Korea), accepted for publication, 2017.

[5] G.A. Anastassiou, *Principles of general fractional analysis for Banach space valued functions*, submitted for publication (2017).

[6] I.K. Argyros, *A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space*, J. Math. Anal. Appl. 298, (2004), 374-397.

[7] I.K. Argyros, A. Magrenan, *Iterative methods and their dynamics with applications*, CRC Press, New York, 2017.

[8] Bochner integral. *Encyclopedia of Mathematics*. URL: http://www.ency clopedia ofmath.org/index.php?title=Bochner_integral& oldid=38659.

[9] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. 37, (1962), 74-79.

[10] J.A. Ezquerro, J.M. Gutierrez, M.A. Hernandez, N. Romero, M.J. Rubio, *The Newton method: From Newton to Kantorovich* (Spanish), Gac. R. Soc. Mat. Esp. 13, (2010), 53-76.

[11] L.V. Kantorovich, G.P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, New York, 1982.

[12] G.E. Ladas and V. Lakshmikantham, *Differential equations in abstract spaces,* Academic Press, New York, London, 1972.

[13] A. Magrenan, *A new tool to study real dynamics: The convergence plane*, Appl. Math. Comput. 248, (2014), 215-224.

[14] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.

[15] F.A. Potra, V. Ptak, *Nondiscrete induction and iterative processes*, Pitman Publ., London, 1984.

[16] P.D. Proinov, *New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems*, J. Complexity, 26, (2010), 3-42.

[17] G.E. Shilov, *Elementary Functional Analysis*, Dover Publications, Inc., New York, 1996.