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ON SOME OSTROWSKI TYPE INEQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. In this paper, some inequalities of Ostrowski type for the generalized Riemann-Liouville fractional integrals of absolutely continuous functions in terms of the Lebesgue p-norm of the derivatives are obtained. Some examples for the Hadamard and Harmonic fractional integrals are also given.

1. Introduction

In 1938, A. Ostrowski [21], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x\in[a,b]$.

Theorem 1 (Ostrowski, 1938 [21]). *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty(b-a), \tag{1.1}$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

The following result, which is an improvement on Ostrowski's inequality, holds.

Theorem 2 (Dragomir, 2002 [11]). *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ whose derivative $f'\in L_\infty[a,b]$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.2}$$
$$\le\frac{1}{2(b-a)}\left[\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2\right]$$

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$$\leq \begin{cases} \|f'\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a); \\ \\ \frac{1}{2}\left[\|f'\|^p_{[a,x],\infty}+\|f'\|^p_{[x,b],\infty}\right]^{\frac{1}{p}}\left[\left(\frac{x-a}{b-a}\right)^{2q}+\left(\frac{b-x}{b-a}\right)^{2q}\right]^{\frac{1}{q}}(b-a), \\ \\ \textit{where } \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{2}\left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,b],\infty}\right]\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^2(b-a) \end{cases}$$

for all $x\in[a,b]$, *where* $\|\cdot\|_{[m,n],\infty}$ *denotes the usual* ∞*-norm on* $L_\infty[m,n]$, *i.e., we recall that*

$$\|g\|_{[m,n],\infty}=\operatorname*{essup}_{t\in[m,n]}|g(t)|<\infty.$$

Corollary 1. *With the assumptions of Theorem 2 we have the mid-point inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{8}\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right](b-a) \tag{1.3}$$
$$\leq\frac{1}{4}\|f'\|_{[a,b],\infty}(b-a).$$

For other Ostrowski type inequalities, see [6]-[16] and the references therein.

In order to extend these results for fractional integrals, we need the following definitions.

Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J^\alpha_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt$$

for $a<x\leq b$ and

$$J^\alpha_{b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt$$

for $a\leq x<b$, where Γ is the *Gamma function*. For $\alpha=0$, they are defined as $J^0_{a+}f(x)=J^0_{b-}f(x)=f(x)$ for $x\in(a,b)$.

For several inequalities for Riemann-Liouville fractional integrals see [1]-[5], [14]-[29] and the references therein.

Let (a,b) with $-\infty\leq a<b\leq\infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\operatorname{Re}(\alpha)>0$. Also, let g be a *strictly increasing function* on (a,b), having a continuous derivative g' on (a,b). Following [19, p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ by

$$I^\alpha_{a+,g}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{g'(t)f(t)\,dt}{[g(x)-g(t)]^{1-\alpha}},\ a<x\leq b \tag{1.4}$$

and

$$I^\alpha_{b-,g}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{g'(t)f(t)\,dt}{[g(t)-g(x)]^{1-\alpha}},\ a\leq x<b. \tag{1.5}$$

For $g(t) = t$ we have the classical *Riemann-Liouville fractional integrals* introduced above while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [19, p. 111]

$$H_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t)\,dt}{t},\ 0 \le a < x \le b \tag{1.6}$$

and

$$H_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \frac{f(t)\,dt}{t},\ 0 \le a < x < b. \tag{1.7}$$

One can consider the function $g(t) = -t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha} t^{\alpha+1}},\ 0 \le a < x \le b \tag{1.8}$$

and

$$R_{b-}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha} t^{\alpha+1}},\ 0 \le a < x < b. \tag{1.9}$$

Recall also the concept of *generalized mean* generated by a function. If g is a function which maps an interval I of the real line to the real numbers and is both continuous and injective then we can define the *g-mean of two numbers $a, b \in I$* by

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a,b) = A(a,b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0,\infty)$ and $g(t) = \ln t$, then $M_g(a,b) = G(a,b) := \sqrt{ab}$, the *geometric mean*. If $I = (0,\infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a,b) = H(a,b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0,\infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a,b) = M_p(a,b) := \left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*.

2. Main Results

We have the following representation,

Lemma 1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then for any $x \in (a,b)$ we have*

$$\begin{aligned} &I_{x-,g}^{\alpha} f(a) + I_{x+,g}^{\alpha} f(b) \\ &= \frac{1}{\Gamma(\alpha+1)} \left[(g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha}\right] f(x) \\ &+ \frac{1}{\Gamma(\alpha+1)} \left[\int_x^b (g(b)-g(t))^{\alpha} f'(t)\,dt - \int_a^x (g(t)-g(a))^{\alpha} f'(t)\,dt\right]. \end{aligned} \tag{2.1}$$

In particular, we have

$$I^{\alpha}_{M_g(a,b)-,g}f(a)+I^{\alpha}_{M_g(a,b)+,g}f(b) = \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}f(M_g(a,b)) + \frac{1}{\Gamma(\alpha+1)}\int_{M_g(a,b)}^{b}(g(b)-g(t))^{\alpha}f'(t)\,dt - \frac{1}{\Gamma(\alpha+1)}\int_{a}^{M_g(a,b)}(g(t)-g(a))^{\alpha}f'(t)\,dt. \tag{2.2}$$

Proof. By the definition of generalized Riemann-Liouville fractional integrals, we have

$$I^{\alpha}_{x+,g}f(b)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(g(b)-g(t))^{\alpha-1}g'(t)f(t)\,dt$$

for $a\le x<b$ and

$$I^{\alpha}_{x-,g}f(a)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(g(t)-g(a))^{\alpha-1}g'(t)f(t)\,dt$$

for $a<x\le b$.

Since $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function $[a,b]$, then the Lebesgue integrals

$$\int_{a}^{x}(g(t)-g(a))^{\alpha}f'(t)\,dt \text{ and } \int_{x}^{b}(g(b)-g(t))^{\alpha}f'(t)\,dt$$

exist and integrating by parts, we have

$$\frac{1}{\Gamma(\alpha+1)}\int_{a}^{x}(g(t)-g(a))^{\alpha}f'(t)\,dt = \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}f(x)-\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(g(t)-g(a))^{\alpha-1}g'(t)f(t)\,dt = \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}f(x)-I^{\alpha}_{x-,g}f(a) \tag{2.3}$$

for $a<x\le b$ and

$$\frac{1}{\Gamma(\alpha+1)}\int_{x}^{b}(g(b)-g(t))^{\alpha}f'(t)\,dt = \frac{1}{\Gamma(\alpha)}\int_{x}^{b}(g(b)-g(t))^{\alpha-1}g'(t)f(t)\,dt-\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha}f(x) = I^{\alpha}_{x+,g}f(b)-\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha}f(x) \tag{2.4}$$

for $a\le x<b$.

From (2.3) we get

$$I^{\alpha}_{x-,g}f(a)=\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}f(x) - \frac{1}{\Gamma(\alpha+1)}\int_{a}^{x}(g(t)-g(a))^{\alpha}f'(t)\,dt$$

for $a < x \leq b$ and from (2.4)

$$I_{x+,g}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha+1)} (g(b) - g(x))^{\alpha} f(x) + \frac{1}{\Gamma(\alpha+1)} \int_{x}^{b} (g(b) - g(t))^{\alpha} f'(t) \, dt,$$

for $a \leq x < b$, which by addition produce (2.1). □

We use the *Lebesgue p-norms* defined as

$$\|h\|_{[c,d],\infty} := \operatorname*{essup}_{t\in[c,d]} |h(t)| < \infty \text{ provided } h \in L_{\infty}[c,d]$$

and

$$\|h\|_{[c,d],p} := \left(\int_{c}^{d} |h(t)|^{p} \, dt \right)^{1/p} < \infty \text{ provided } h \in L_{p}[c,d], \ p \geq 1.$$

The following inequalities hold:

Theorem 3. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b).*

(i) If $\frac{f'}{g'} \in L_{\infty}[a,b]$, then for any $x \in (a,b)$ we have

$$\left| I_{x-,g}^{\alpha} f(a) + I_{x+,g}^{\alpha} f(b) - \frac{(g(x) - g(a))^{\alpha} + (g(b) - g(x))^{\alpha}}{\Gamma(\alpha+1)} f(x) \right| \tag{2.5}$$
$$\leq \frac{1}{\Gamma(\alpha+2)} \left[\left\| \frac{f'}{g'} \right\|_{[x,a],\infty} (g(x) - g(a))^{\alpha+1} + \left\| \frac{f'}{g'} \right\|_{[b,x],\infty} (g(b) - g(x))^{\alpha+1} \right]$$
$$\leq \frac{1}{\Gamma(\alpha+2)} \left\| \frac{f'}{g'} \right\|_{[a,b],\infty} \left[(g(x) - g(a))^{\alpha+1} + (g(b) - g(x))^{\alpha+1} \right].$$

(ii) If $\frac{f'}{(g')^{\frac{1}{q}}} \in L_{p}[a,b]$, where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in (a,b)$ we have

$$\left| I_{x-,g}^{\alpha} f(a) + I_{x+,g}^{\alpha} f(b) - \frac{(g(x) - g(a))^{\alpha} + (g(b) - g(x))^{\alpha}}{\Gamma(\alpha+1)} f(x) \right| \tag{2.6}$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}}$$
$$\times \left[\left\| \frac{f'}{(g')^{\frac{1}{q}}} \right\|_{[a,x],p} (g(x) - g(a))^{\alpha+\frac{1}{q}} + \left\| \frac{f'}{(g')^{\frac{1}{q}}} \right\|_{[x,b],p} (g(b) - g(x))^{\alpha+\frac{1}{q}} \right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}} \left\| \frac{f'}{(g')^{\frac{1}{q}}} \right\|_{[a,b],p} \left[(g(x) - g(a))^{q\alpha+1} + (g(b) - g(x))^{q\alpha+1} \right]^{1/q}.$$

Proof. (i) Using the identity (2.1) and the properties of modulus, we have

$$\left|I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)-\frac{(g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}}{\Gamma(\alpha+1)}f(x)\right| \tag{2.7}$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\left[\left|\int_x^b(g(b)-g(t))^{\alpha}f'(t)\,dt\right|+\left|\int_a^x(g(t)-g(a))^{\alpha}f'(t)\,dt\right|\right]$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\left[\int_x^b(g(b)-g(t))^{\alpha}|f'(t)|\,dt+\int_a^x(g(t)-g(a))^{\alpha}|f'(t)|\,dt\right]$$
$$=\frac{1}{\Gamma(\alpha+1)}\int_x^b(g(b)-g(t))^{\alpha}\left|\frac{f'(t)}{g'(t)}\right|g'(t)\,dt$$
$$+\frac{1}{\Gamma(\alpha+1)}\int_a^x(g(t)-g(a))^{\alpha}\left|\frac{f'(t)}{g'(t)}\right|g'(t)\,dt$$
$$=:\frac{1}{\Gamma(\alpha+1)}D(x)$$

for any $x\in(a,b)$.

By the properties of integral

$$\int_x^b(g(b)-g(t))^{\alpha}\left|\frac{f'(t)}{g'(t)}\right|g'(t)\,dt\leq\left\|\frac{f'}{g'}\right\|_{[b,x],\infty}\frac{(g(b)-g(x))^{\alpha+1}}{\alpha+1}$$

and

$$\int_a^x(g(t)-g(a))^{\alpha}\left|\frac{f'(t)}{g'(t)}\right|g'(t)\,dt\leq\left\|\frac{f'}{g'}\right\|_{[x,a],\infty}\frac{(g(x)-g(a))^{\alpha+1}}{\alpha+1}$$

for any $x\in(a,b)$.

Therefore

$$\begin{aligned}D(x)&\leq\left\|\frac{f'}{g'}\right\|_{[x,a],\infty}\frac{(g(x)-g(a))^{\alpha+1}}{\alpha+1}+\left\|\frac{f'}{g'}\right\|_{[b,x],\infty}\frac{(g(b)-g(x))^{\alpha+1}}{\alpha+1}\\&\leq\max\left\{\left\|\frac{f'}{g'}\right\|_{[a,x],\infty},\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}\right\}\left[\frac{(g(x)-g(a))^{\alpha+1}+(g(b)-g(x))^{\alpha+1}}{\alpha+1}\right]\\&=\frac{1}{\alpha+1}\left\|\frac{f'}{g'}\right\|_{[a,b],\infty}\left[(g(x)-g(a))^{\alpha+1}+(g(b)-g(x))^{\alpha+1}\right]\end{aligned}$$

for any $x\in(a,b)$ and the inequality (2.5) is thus proved.

(ii) By Hölder's weighted integral inequality

$$\left|\int_c^d u(t)\,v(t)\,w(t)\,dt\right|\leq\left(\int_c^d|u(t)|^p w(t)\,dt\right)^{1/p}\left(\int_c^d|v(t)|^q w(t)\,dt\right)^{1/q}$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $w\geq0$ a.e. on $[c,d]$, we also have

$$\begin{aligned}
&\int_a^x (g(t)-g(a))^\alpha \left|\frac{f'(t)}{g'(t)}\right| g'(t)\,dt \\
&\le \left(\int_a^x \left|\frac{f'(t)}{g'(t)}\right|^p g'(t)\,dt\right)^{1/p} \left(\int_a^x (g(t)-g(a))^{q\alpha} g'(t)\,dt\right)^{1/q} \\
&= \left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,x],p} \frac{(g(x)-g(a))^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{1/q}}
\end{aligned}$$

and, similarly

$$\int_x^b (g(b)-g(t))^\alpha \left|\frac{f'(t)}{g'(t)}\right| g'(t)\,dt \le \left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[x,b],p} \frac{(g(b)-g(x))^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{1/q}}.$$

Therefore, by Hölder's discrete inequality

$$mn+uv\le (m^p+u^p)^{1/p}(n^q+v^q)^{1/q}$$

where $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m,\, n,\, u,\, v\ge 0$, we have further

$$\begin{aligned}
C(x) &\le \left[\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,x],p} \frac{(g(x)-g(a))^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{1/q}} + \left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[x,b],p} \frac{(g(b)-g(x))^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{1/q}}\right] \\
&\le \frac{1}{(\alpha q+1)^{1/q}} \left[\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,x],p}^p + \left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[x,b],p}^p\right]^{1/p} \\
&\quad \times \left[\left((g(x)-g(a))^{\alpha+\frac{1}{q}}\right)^q + \left((g(b)-g(x))^{\alpha+\frac{1}{q}}\right)^q\right]^{1/q} \\
&= \frac{1}{(\alpha q+1)^{1/q}} \left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,b],p} \left[(g(x)-g(a))^{q\alpha+1} + (g(b)-g(x))^{q\alpha+1}\right]^{1/q}.
\end{aligned}$$

By making use of (2.7) we get the desired (2.6). □

Corollary 2. *With the assumptions of Theorem 3, we have*

$$\begin{aligned}
(2.8)\quad &\left| I^\alpha_{M_g(a,b)-,g} f(a) + I^\alpha_{M_g(a,b)+,g} f(b) - \frac{(g(b)-g(a))^\alpha}{2^{\alpha-1}\Gamma(\alpha+1)} f(M_g(a,b)) \right| \\
&\le \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)} \left[\left\|\frac{f'}{g'}\right\|_{[a,M_g(a,b)],\infty} + \left\|\frac{f'}{g'}\right\|_{[M_g(a,b),b],\infty}\right] (g(b)-g(a))^{\alpha+1} \\
&\le \frac{1}{2^\alpha\Gamma(\alpha+2)} \left\|\frac{f'}{g'}\right\|_{[a,b],\infty} (g(b)-g(a))^{\alpha+1}
\end{aligned}$$

provided $\frac{f'}{g'} \in L_\infty [a,b]$ *and*

$$\left| I^\alpha_{M_g(a,b)-,g} f(a) + I^\alpha_{M_g(a,b)+,g} f(b) - \frac{(g(b)-g(a))^\alpha}{2^{\alpha-1}\Gamma(\alpha+1)} f(M_g(a,b)) \right| \tag{2.9}$$
$$\leq \frac{(g(b)-g(a))^{\alpha+\frac{1}{q}}}{2^{\alpha+\frac{1}{q}}\Gamma(\alpha+1)(\alpha q+1)^{1/q}} \left[\left\| \frac{f'}{(g')^{\frac{1}{q}}} \right\|_{[a,M_g(a,b)],p} + \left\| \frac{f'}{(g')^{\frac{1}{q}}} \right\|_{[M_g(a,b),b],p} \right]$$
$$\leq \frac{1}{2^\alpha \Gamma(\alpha+1)(\alpha q+1)^{1/q}} \left\| \frac{f'}{(g')^{\frac{1}{q}}} \right\|_{[a,b],p} (g(b)-g(a))^{\alpha+\frac{1}{q}},$$

provided $\frac{f'}{(g')^{\frac{1}{q}}} \in L_p [a,b]$, *where* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

3. Applications

If we take $g(t)=t$, $t\in[a,b]$ in the above inequalities, then we get the following results for the classical Riemann-Liouville fractional integrals of the absolutely continuous function $f:[a,b]\to\mathbb{C}$

$$\left| J^\alpha_{x-} f(a) + J^\alpha_{x+} f(b) - \frac{(x-a)^\alpha + (b-x)^\alpha}{\Gamma(\alpha+1)} f(x) \right| \tag{3.1}$$
$$\leq \frac{1}{\Gamma(\alpha+2)} \left[\|f'\|_{[a,x],\infty} (x-a)^{\alpha+1} + \|f'\|_{[x,b],\infty} (b-x)^{\alpha+1} \right]$$
$$\leq \frac{1}{\Gamma(\alpha+2)} \|f'\|_{[a,b],\infty} \left[(x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right]$$

for any $x\in(a,b)$, provided $f'\in L_\infty[a,b]$. In particular, we have

$$\left| J^\alpha_{\frac{a+b}{2}-} f(a) + J^\alpha_{\frac{a+b}{2}+} f(b) - \frac{(b-a)^\alpha}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \right| \tag{3.2}$$
$$\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)} \left[\|f'\|_{[a,\frac{a+b}{2}],\infty} + \|f'\|_{[\frac{a+b}{2},b],\infty} \right] (b-a)^{\alpha+1}$$
$$\leq \frac{1}{2^\alpha\Gamma(\alpha+2)} \|f'\|_{[a,b],\infty} (b-a)^{\alpha+1}.$$

If $f'\in L_p[a,b]$, where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then for any $x\in(a,b)$ we have

$$\left| J^\alpha_{x-} f(a) + J^\alpha_{x+} f(b) - \frac{(x-a)^\alpha + (b-x)^\alpha}{\Gamma(\alpha+1)} f(x) \right| \tag{3.3}$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}} \left[\|f'\|_{[a,x],p} (x-a)^{\alpha+\frac{1}{q}} + \|f'\|_{[x,b],p} (b-x)^{\alpha+\frac{1}{q}} \right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}} \|f'\|_{[a,b],p} \left[(x-a)^{q\alpha+1} + (b-x)^{q\alpha+1} \right]^{1/q}.$$

In particular, we have

$$\left|J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)-\frac{(b-a)^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)\right| \tag{3.4}$$
$$\leq \frac{(b-a)^{\alpha+\frac{1}{q}}}{2^{\alpha+\frac{1}{q}}\Gamma(\alpha+1)(\alpha q+1)^{1/q}}\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],p}+\|f'\|_{\left[\frac{a+b}{2},b\right],p}\right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)(\alpha q+1)^{1/q}}\|f'\|_{[a,b],p}(b-a)^{\alpha+\frac{1}{q}}.$$

The case $\alpha=1$ produces the following inequalities for the Riemann integral, see also 1.2

$$\left|\int_a^b f(t)\,dt-f(x)(b-a)\right| \tag{3.5}$$
$$\leq \frac{1}{2}\left[(x-a)^2\|f'\|_{[a,x],\infty}+(b-x)^2\|f'\|_{[x,b],\infty}\right]$$
$$\leq \left[\frac{1}{4}(b-a)+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty}$$

for any $x\in[a,b]$ and

$$\left|\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)(b-a)\right| \tag{3.6}$$
$$\leq \frac{1}{8}(b-a)^2\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right]\leq\frac{1}{4}(b-a)\|f'\|_{[a,b],\infty},$$

where $f'\in L_{\infty}[a,b]$.

If $f'\in L_p[a,b]$, where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then for any $x\in(a,b)$ we have

$$\left|\int_a^b f(t)\,dt-f(x)(b-a)\right| \tag{3.7}$$
$$\leq \frac{1}{(q+1)^{1/q}}\left[\|f'\|_{[a,x],p}(x-a)^{1+\frac{1}{q}}+\|f'\|_{[x,b],p}(b-x)^{1+\frac{1}{q}}\right]$$
$$\leq \frac{1}{(q+1)^{1/q}}\|f'\|_{[a,b],p}\left[(x-a)^{q+1}+(b-x)^{q+1}\right]^{1/q}$$

and, in particular

$$\left|\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)(b-a)\right| \tag{3.8}$$
$$\leq \frac{1}{2^{1+1/q}(q+1)^{1/q}}(b-a)^{1+\frac{1}{q}}\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],p}+\|f'\|_{\left[\frac{a+b}{2},b\right],p}\right]$$
$$\leq \frac{1}{2(q+1)^{1/q}}(b-a)^{1+\frac{1}{q}}\|f'\|_{[a,b],p}.$$

Now, if we take $g(t)=\ln t$, $t\in(a,b)\subset(0,\infty)$ in inequalities from the above section, then we get the following results for the Hadamard fractional integrals H^{α}_{a+} and H^{α}_{b-}:

If $ef' \in L_\infty [a,b]$, where $e(t) = t$ is the identity function, then for any $x \in (a,b)$ we have

$$\left| H^\alpha_{x-} f(a) + H^\alpha_{x+} f(b) - \frac{\left[\ln\left(\frac{x}{a}\right)\right]^\alpha + \left[\ln\left(\frac{b}{x}\right)\right]^\alpha}{\Gamma(\alpha+1)} f(x) \right| \tag{3.9}$$
$$\leq \frac{1}{\Gamma(\alpha+2)} \left[\|ef'\|_{[a,x],\infty} \left[\ln\left(\frac{x}{a}\right)\right]^{\alpha+1} + \|ef'\|_{[x,b],\infty} \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha+1} \right]$$
$$\leq \frac{1}{\Gamma(\alpha+2)} \|ef'\|_{[a,b],\infty} \left[\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha+1} + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha+1} \right]$$

and, in particular

$$\left| H^\alpha_{\sqrt{ab}-} f(a) + H^\alpha_{\sqrt{ab}+} f(b) - \frac{\left[\ln\left(\frac{b}{a}\right)\right]^\alpha}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\sqrt{ab}\right) \right| \tag{3.10}$$
$$\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)} \left[\|ef'\|_{[a,\sqrt{ab}],\infty} + \|ef'\|_{[\sqrt{ab},b],\infty} \right] \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha+1}$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+2)} \|ef'\|_{[a,b],\infty} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha+1}.$$

If $e^{1/q} f' \in L_p [a,b]$, where $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for any $x \in (a,b)$ we have

$$\left| H^\alpha_{x-} f(a) + H^\alpha_{x+} f(b) - \frac{\left[\ln\left(\frac{x}{a}\right)\right]^\alpha + \left[\ln\left(\frac{b}{x}\right)\right]^\alpha}{\Gamma(\alpha+1)} f(x) \right| \tag{3.11}$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}}$$
$$\times \left[\left\|e^{1/q} f'\right\|_{[a,x],p} \left[\ln\left(\frac{x}{a}\right)\right]^{\alpha+\frac{1}{q}} + \left\|e^{1/q} f'\right\|_{[x,b],p} \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha+\frac{1}{q}} \right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}} \left\|e^{1/q} f'\right\|_{[a,b],p} \left[\left[\ln\left(\frac{x}{a}\right)\right]^{q\alpha+1} + \left[\ln\left(\frac{b}{x}\right)\right]^{q\alpha+1} \right]^{1/q}$$

and, in particular

$$\left| H^\alpha_{\sqrt{ab}-} f(a) + H^\alpha_{\sqrt{ab}+} f(b) - \frac{\left[\ln\left(\frac{b}{a}\right)\right]^\alpha}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\sqrt{ab}\right) \right| \tag{3.12}$$
$$\leq \frac{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha+\frac{1}{q}}}{2^{\alpha+\frac{1}{q}}\Gamma(\alpha+1)(\alpha q+1)^{1/q}} \left[\left\|e^{1/q} f'\right\|_{[a,\sqrt{ab}],p} + \left\|e^{1/q} f'\right\|_{[\sqrt{ab},b],p} \right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)(\alpha q+1)^{1/q}} \left\|e^{1/q} f'\right\|_{[a,b],p} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha+\frac{1}{q}}.$$

If we take the function $g(t) = -t^{-1}$, $t \in (a,b) \subset (0,\infty)$ in the inequalities from the previous section, then we have the following inequalities for Harmonic fractional integrals and $x \in (a,b)$

$$\left| R_{x-}^{\alpha} f(a) + R_{x+}^{\alpha} f(b) - \frac{\left(\frac{x-a}{xa}\right)^{\alpha} + \left(\frac{b-x}{bx}\right)^{\alpha}}{\Gamma(\alpha+1)} f(x) \right| \tag{3.13}$$
$$\leq \frac{1}{\Gamma(\alpha+2)} \left[\left\| e^2 f' \right\|_{[a,x],\infty} \left(\frac{x-a}{xa} \right)^{\alpha+1} + \left\| e^2 f' \right\|_{[x,b],\infty} \left(\frac{b-x}{bx} \right)^{\alpha+1} \right]$$
$$\leq \frac{1}{\Gamma(\alpha+2)} \left\| e^2 f' \right\|_{[a,b],\infty} \left[\left(\frac{x-a}{xa} \right)^{\alpha+1} + \left(\frac{b-x}{bx} \right)^{\alpha+1} \right]$$

if $e^2 f' \in L_{\infty}[a,b]$, and

$$\left| R_{x-}^{\alpha} f(a) + R_{x+}^{\alpha} f(b) - \frac{\left(\frac{x-a}{xa}\right)^{\alpha} + \left(\frac{b-x}{bx}\right)^{\alpha}}{\Gamma(\alpha+1)} f(x) \right| \tag{3.14}$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}}$$
$$\times \left[\left\| e^{2/q} f' \right\|_{[a,x],p} \left(\frac{x-a}{xa} \right)^{\alpha+\frac{1}{q}} + \left\| e^{2/q} f' \right\|_{[x,b],p} \left(\frac{b-x}{bx} \right)^{\alpha+\frac{1}{q}} \right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}} \left\| e^{2/q} f' \right\|_{[a,b],p} \left[\left(\frac{x-a}{xa} \right)^{q\alpha+1} + \left(\frac{b-x}{bx} \right)^{q\alpha+1} \right]^{1/q}$$

if $e^{2/q} f' \in L_p[a,b]$ for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Consider the Harmonic mean of two positive numbers $H(a,b) := \frac{2ab}{a+b}$, then by (3.13) and (3.14) we have

$$\left| R_{H(a,b)-}^{\alpha} f(a) + R_{H(a,b)+}^{\alpha} f(b) - \frac{\left(\frac{x-a}{xa}\right)^{\alpha} + \left(\frac{b-x}{bx}\right)^{\alpha}}{\Gamma(\alpha+1)} f(H(a,b)) \right| \tag{3.15}$$
$$\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)} \left[\left\| e^2 f' \right\|_{[a,H(a,b)],\infty} + \left\| e^2 f' \right\|_{[H(a,b),b],\infty} \right] \left(\frac{b-a}{ba} \right)^{\alpha+1}$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+2)} \left\| e^2 f' \right\|_{[a,b],\infty} \left(\frac{b-a}{ba} \right)^{\alpha+1}$$

provided $e^2 f' \in L_{\infty}[a,b]$ and

$$\left| R_{H(a,b)-}^{\alpha} f(a) + R_{H(a,b)+}^{\alpha} f(b) - \frac{\left(\frac{x-a}{xa}\right)^{\alpha} + \left(\frac{b-x}{bx}\right)^{\alpha}}{\Gamma(\alpha+1)} f(H(a,b)) \right| \tag{3.16}$$
$$\leq \frac{\left(\frac{b-a}{ba}\right)^{\alpha+\frac{1}{q}}}{2^{\alpha+\frac{1}{q}}\Gamma(\alpha+1)(\alpha q+1)^{1/q}} \left[\left\| e^{2/q} f' \right\|_{[a,H(a,b)],p} + \left\| e^{2/q} f' \right\|_{[H(a,b),b],p} \right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)(\alpha q+1)^{1/q}} \left\| e^{2/q} f' \right\|_{[a,b],p} \left(\frac{b-a}{ba} \right)^{\alpha+\frac{1}{q}},$$

if $e^{2/q} f' \in L_p[a,b]$ for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

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