Received 01/06/17

ON SOME TRAPEZOID TYPE INEQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

SILVESTRU SEVER DRAGOMIR[1,2]

ABSTRACT. In this paper, some inequalities of trapezoid type for the generalized Riemann-Liouville fractional integrals of absolutely continuous functions in terms of the Lebesgue p-norm of the derivatives are obtained. Some examples for the Hadamard and Harmonic fractional integrals are also given.

1. INTRODUCTION

In 1999, Cerone & Dragomir [6] established the following *generalized trapezoid inequality* for absolutely continuous functions $f:[a,b]\to\mathbb{C}$ and $x\in[a,b]$

$$\left|(x-a)f(a)+f(b)(b-x)-\int_a^b f(t)\,dt\right| \tag{1.1}$$

$$\leq\begin{cases}\left[\frac{1}{4}(b-a)+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty} \text{ if } f'\in L_\infty[a,b];\\[2ex] \frac{1}{(q+1)^{1/q}}\left[(b-x)^{q+1}+(x-a)^{q+1}\right]^{1/q}\|f'\|_{[a,b],p}\\ \text{if } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \text{ and } f'\in L_p[a,b];\\[2ex] \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\|f'\|_{[a,b],1}\end{cases}$$

where the *Lebesgue p-norms* are defined as

$$\|h\|_{[c,d],\infty}:=\operatorname*{essup}_{t\in[c,d]}|h(t)|<\infty \text{ provided } h\in L_\infty[c,d]$$

and

$$\|h\|_{[c,d],p}:=\left(\int_c^d|h(t)|^p\,dt\right)^{1/p}<\infty \text{ provided } h\in L_p[c,d],\ p\geq 1.$$

For $x=\frac{a+b}{2}$ we get the sharp *trapezoid inequalities:*

1991 *Mathematics Subject Classification.* 26D15, 26D10, 26D07, 26A33.

Key words and phrases. Riemann-Liouville fractional integrals, Hadamard fractional integrals, Absolutely continuous functions, Trapezoid type inequalities.

RGMIA Res. Rep. Coll. 20 (2017), Art. 68, 12 pp.

$$\left| \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt \right| \tag{1.2}$$
$$\leq \begin{cases} \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty} & \text{if } f' \in L_\infty[a,b]; \\ \\ \frac{1}{2(q+1)^{1/q}}(b-a)^{1+1/q}\|f'\|_{[a,b],p} \\ \text{if } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1 \text{ and } f' \in L_p[a,b]; \\ \\ \frac{1}{2}(b-a)\|f'\|_{[a,b],1}. \end{cases}$$

For other trapezoid type inequalities see [7]-[11] and the references therein.

In order to extend these results for fractional integrals we need the following definitions.

Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt$$

for $a<x\leq b$ and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\,dt$$

for $a\leq x<b$, where Γ is the *Gamma function*. For $\alpha=0$, they are defined as $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$ for $x\in(a,b)$.

For several inequalities for Riemann-Liouville fractional integrals see [1]-[5], [12]-[26] and the references therein.

Let (a,b) with $-\infty\leq a<b\leq\infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\operatorname{Re}(\alpha)>0$. Also, let g be a *strictly increasing function* on (a,b), having a continuous derivative g' on (a,b). Following [17, p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ by

$$I_{a+,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)f(t)\,dt}{[g(x)-g(t)]^{1-\alpha}},\ a<x\leq b \tag{1.3}$$

and

$$I_{b-,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)f(t)\,dt}{[g(t)-g(x)]^{1-\alpha}},\ a\leq x<b. \tag{1.4}$$

For $g(t)=t$ we have the classical *Riemann-Liouville fractional integrals* introduced above while for the logarithmic function $g(t)=\ln t$ we have the *Hadamard fractional integrals* [17, p. 111]

$$H_{a+}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0\leq a<x\leq b \tag{1.5}$$

and

$$H_{b-}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0\leq a<x<b. \tag{1.6}$$

One can consider the function $g(t) = -t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha} t^{\alpha+1}},\ 0 \le a < x \le b \tag{1.7}$$

and

$$R_{b-}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha} t^{\alpha+1}},\ 0 \le a < x < b. \tag{1.8}$$

Recall also the concept of *generalized mean* generated by a function. If g is a function which maps an interval I of the real line to the real numbers and is both continuous and injective then we can define the *g-mean of two numbers* $a, b \in I$ by

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a,b) = A(a,b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0,\infty)$ and $g(t) = \ln t$, then $M_g(a,b) = G(a,b) := \sqrt{ab}$, the *geometric mean*. If $I = (0,\infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a,b) = H(a,b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0,\infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a,b) = M_p(a,b) := \left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*.

Motivated by the above results, in this paper we establish some inequalities of trapezoid type for the generalized Riemann-Liouville fractional integrals of absolutely continuous functions in terms of the Lebesgue p-norm of the derivatives. Some examples for the Hadamard and Harmonic fractional integrals are also given.

2. Main Results

We need the following equalities that are of interest in themselves as well.

Lemma 1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then for any $x \in (a,b)$ we have*

$$\begin{aligned} &I_{a+,g}^{\alpha} f(x) + I_{b-,g}^{\alpha} f(x) \\ &= \frac{1}{\Gamma(\alpha+1)} \left[(g(x)-g(a))^{\alpha} f(a) + (g(b)-g(x))^{\alpha} f(b)\right] \\ &+ \frac{1}{\Gamma(\alpha+1)} \left[\int_a^x (g(x)-g(t))^{\alpha} f'(t)\,dt - \int_x^b (g(t)-g(x))^{\alpha} f'(t)\,dt\right]. \end{aligned} \tag{2.1}$$

In particular, we have

$$\begin{aligned} &I_{a+,g}^{\alpha} f(M_g(a,b)) + I_{b-,g}^{\alpha} f(M_g(a,b)) \\ &= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} (g(b)-g(a))^{\alpha} \frac{f(a)+f(b)}{2} \\ &+ \frac{1}{\Gamma(\alpha+1)} \int_a^{M_g(a,b)} (g(M_g(a,b))-g(t))^{\alpha} f'(t)\,dt \\ &- \frac{1}{\Gamma(\alpha+1)} \int_{M_g(a,b)}^b (g(t)-g(M_g(a,b)))^{\alpha} f'(t)\,dt. \end{aligned} \tag{2.2}$$

Proof. Since $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function on $[a,b]$, then the Lebesgue integrals

$$\int_a^x (g(x)-g(t))^\alpha f'(t)\,dt \text{ and } \int_x^b (g(t)-g(x))^\alpha f'(t)\,dt$$

exist and integrating by parts, we have

$$\begin{aligned}(2.3)\quad &\frac{1}{\Gamma(\alpha+1)}\int_a^x (g(x)-g(t))^\alpha f'(t)\,dt\\ &=\frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1} g'(t)f(t)\,dt-\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^\alpha f(a)\\ &=I_{a+,g}^\alpha f(x)-\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^\alpha f(a)\end{aligned}$$

for $a<x\le b$ and

$$\begin{aligned}(2.4)\quad &\frac{1}{\Gamma(\alpha+1)}\int_x^b (g(t)-g(x))^\alpha f'(t)\,dt\\ &=\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^\alpha f(b)-\frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1} g'(t)f(t)\,dt\\ &=\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^\alpha f(b)-I_{b-,g}^\alpha f(x)\end{aligned}$$

for $a\le x<b$.

From (2.3), we then have

$$\begin{aligned}I_{a+,g}^\alpha f(x)&=\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^\alpha f(a)\\ &\quad+\frac{1}{\Gamma(\alpha+1)}\int_a^x (g(x)-g(t))^\alpha f'(t)\,dt\end{aligned}$$

for $a<x\le b$ and from (2.4) we have

$$\begin{aligned}I_{b-,g}^\alpha f(x)&=\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^\alpha f(b)\\ &\quad-\frac{1}{\Gamma(\alpha+1)}\int_x^b (g(t)-g(x))^\alpha f'(t)\,dt,\end{aligned}$$

for $a\le x<b$, which by addition give (2.1). □

We have:

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b).*

(i) If $\frac{f'}{g'} \in L_\infty [a,b]$, *then for any* $x \in (a,b)$ *we have*

$$\left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{(g(x)-g(a))^{\alpha} f(a) + (g(b)-g(x))^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \tag{2.5}$$
$$\leq \frac{1}{\Gamma(\alpha+2)} \left[\left\| \frac{f'}{g'} \right\|_{[a,x],\infty} (g(x)-g(a))^{\alpha+1} + \left\| \frac{f'}{g'} \right\|_{[x,b],\infty} (g(b)-g(x))^{\alpha+1} \right]$$
$$\leq \frac{1}{\Gamma(\alpha+2)} \left\| \frac{f'}{g'} \right\|_{[a,b],\infty} \left[(g(x)-g(a))^{\alpha+1} + (g(b)-g(x))^{\alpha+1} \right].$$

(ii) If $\frac{f'}{(g')^{\frac{1}{q}}} \in L_p [a,b]$, *where* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then for any* $x \in (a,b)$ *we have*

$$\left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{(g(x)-g(a))^{\alpha} f(a) + (g(b)-g(x))^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \tag{2.6}$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}}$$
$$\times \left[\left\| \frac{f'}{(g')^{\frac{1}{q}}} \right\|_{[a,x],p} (g(x)-g(a))^{\alpha+\frac{1}{q}} + \left\| \frac{f'}{(g')^{\frac{1}{q}}} \right\|_{[x,b],p} (g(b)-g(x))^{\alpha+\frac{1}{q}} \right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}} \left\| \frac{f'}{(g')^{\frac{1}{q}}} \right\|_{[a,b],p} \left[(g(x)-g(a))^{q\alpha+1} + (g(b)-g(x))^{q\alpha+1} \right]^{1/q}.$$

Proof. (i) Using the identity (2.1) and the properties of modulus, we have successively

$$\left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{(g(x)-g(a))^{\alpha} f(a) + (g(b)-g(x))^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \tag{2.7}$$
$$\leq \frac{1}{\Gamma(\alpha+1)} \left[\left| \int_a^x (g(x)-g(t))^{\alpha} f'(t)\,dt \right| + \left| \int_x^b (g(t)-g(x))^{\alpha} f'(t)\,dt \right| \right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)} \left[\int_a^x (g(x)-g(t))^{\alpha} |f'(t)|\,dt + \int_x^b (g(t)-g(x))^{\alpha} |f'(t)|\,dt \right]$$
$$= \frac{1}{\Gamma(\alpha+1)} \int_a^x (g(x)-g(t))^{\alpha} \left| \frac{f'(t)}{g'(t)} \right| g'(t)\,dt$$
$$+ \frac{1}{\Gamma(\alpha+1)} \int_x^b (g(t)-g(x))^{\alpha} \left| \frac{f'(t)}{g'(t)} \right| g'(t)\,dt$$
$$=: \frac{1}{\Gamma(\alpha+1)} C(x)$$

for $x \in (a,b)$.

We have

$$\int_a^x (g(x)-g(t))^{\alpha} \left| \frac{f'(t)}{g'(t)} \right| g'(t)\,dt \leq \left\| \frac{f'}{g'} \right\|_{[a,x],\infty} \frac{(g(x)-g(a))^{\alpha+1}}{\alpha+1}$$

and

$$\int_x^b (g(t)-g(x))^{\alpha}\left|\frac{f'(t)}{g'(t)}\right| g'(t)\,dt \le \left\|\frac{f'}{g'}\right\|_{[x,b],\infty} \frac{(g(b)-g(x))^{\alpha+1}}{\alpha+1}.$$

Therefore,

$$\begin{aligned}
C(x) &\le \left\|\frac{f'}{g'}\right\|_{[a,x],\infty} \frac{(g(x)-g(a))^{\alpha+1}}{\alpha+1} + \left\|\frac{f'}{g'}\right\|_{[x,b],\infty} \frac{(g(b)-g(x))^{\alpha+1}}{\alpha+1} \\
&\le \max\left\{\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}, \left\|\frac{f'}{g'}\right\|_{[x,b],\infty}\right\}\left[\frac{(g(x)-g(a))^{\alpha+1}+(g(b)-g(x))^{\alpha+1}}{\alpha+1}\right] \\
&= \frac{1}{\alpha+1}\left\|\frac{f'}{g'}\right\|_{[a,b],\infty}\left[(g(x)-g(a))^{\alpha+1}+(g(b)-g(x))^{\alpha+1}\right]
\end{aligned}$$

for $x \in (a,b)$.

By making use of (2.7) we then get the desired result (2.5).

By Hölder's weighted integral inequality

$$\left|\int_c^d u(t)\,v(t)\,w(t)\,dt\right| \le \left(\int_c^d |u(t)|^p w(t)\,dt\right)^{1/p}\left(\int_c^d |v(t)|^q w(t)\,dt\right)^{1/q}$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $w \ge 0$ a.e. on $[c,d]$, we also have

$$\begin{aligned}
&\int_a^x (g(x)-g(t))^{\alpha}\left|\frac{f'(t)}{g'(t)}\right| g'(t)\,dt \\
&\le \left(\int_a^x \left|\frac{f'(t)}{g'(t)}\right|^p g'(t)\,dt\right)^{1/p}\left(\int_a^x (g(x)-g(t))^{\alpha q} g'(t)\,dt\right)^{1/q} \\
&= \left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,x],p} \frac{(g(x)-g(a))^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{1/q}}
\end{aligned}$$

and similarly,

$$\int_x^b (g(t)-g(x))^{\alpha}\left|\frac{f'(t)}{g'(t)}\right| g'(t)\,dt \le \left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[x,b],p} \frac{(g(b)-g(x))^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{1/q}}.$$

Therefore, by Hölder's discrete inequality

$$mn+uv \le (m^p+u^p)^{1/p}(n^q+v^q)^{1/q}$$

where $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m,\, n,\, u,\, v\geq 0$, we have further

$$
\begin{aligned}
C(x) &\leq \left[\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,x],p}\frac{(g(x)-g(a))^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{1/q}}+\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[x,b],p}\frac{(g(b)-g(x))^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{1/q}}\right]\\
&\leq \frac{1}{(\alpha q+1)^{1/q}}\left[\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,x],p}^{p}+\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[x,b],p}^{p}\right]^{1/p}\\
&\times\left[\left((g(x)-g(a))^{\alpha+\frac{1}{q}}\right)^{q}+\left((g(b)-g(x))^{\alpha+\frac{1}{q}}\right)^{q}\right]^{1/q}\\
&=\frac{1}{(\alpha q+1)^{1/q}}\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,b],p}\left[(g(x)-g(a))^{q\alpha+1}+(g(b)-g(x))^{q\alpha+1}\right]^{1/q}.
\end{aligned}
$$

By making use of (2.7) we then get the desired result (2.6). □

Corollary 1. *With the assumptions of Theorem 1, we have*

$$
\begin{aligned}
(2.8)\quad &\left|I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))-\frac{(g(b)-g(a))^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}\right|\\
&\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[\left\|\frac{f'}{g'}\right\|_{[a,M_g(a,b)],\infty}+\left\|\frac{f'}{g'}\right\|_{[M_g(a,b),b],\infty}\right](g(b)-g(a))^{\alpha+1}\\
&\leq \frac{1}{2^{\alpha}\Gamma(\alpha+2)}\left\|\frac{f'}{g'}\right\|_{[a,b],\infty}(g(b)-g(a))^{\alpha+1}
\end{aligned}
$$

provided $\frac{f'}{g'}\in L_{\infty}[a,b]$ *and*

$$
\begin{aligned}
(2.9)\quad &\left|I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))-\frac{(g(b)-g(a))^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}\right|\\
&\leq \frac{(g(b)-g(a))^{\alpha+\frac{1}{q}}}{2^{\alpha+\frac{1}{q}}\Gamma(\alpha+1)(\alpha q+1)^{1/q}}\left[\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,M_g(a,b)],p}+\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[M_g(a,b),b],p}\right]\\
&\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)(\alpha q+1)^{1/q}}\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,b],p}(g(b)-g(a))^{\alpha+\frac{1}{q}},
\end{aligned}
$$

provided $\frac{f'}{(g')^{\frac{1}{q}}}\in L_p[a,b]$, *where* $p,\, q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

3. Applications

If we take $g(t)=t$, $t\in[a,b]$ in the above inequalities, then we get the following results for the classical Riemann-Liouville fractional integrals of the absolutely continuous function $f:[a,b]\to\mathbb{C}$

$$\begin{aligned}\left|J_{a+}^{\alpha}f\left(x\right)+J_{b-}^{\alpha}f\left(x\right)-\frac{\left(x-a\right)^{\alpha}f\left(a\right)+\left(b-x\right)^{\alpha}f\left(b\right)}{\Gamma\left(\alpha+1\right)}\right| \\ \leq\frac{1}{\Gamma\left(\alpha+2\right)}\left[\left\Vert f'\right\Vert_{[a,x],\infty}\left(x-a\right)^{\alpha+1}+\left\Vert f'\right\Vert_{[x,b],\infty}\left(b-x\right)^{\alpha+1}\right] \\ \leq\frac{1}{\Gamma\left(\alpha+2\right)}\left\Vert f'\right\Vert_{[a,b],\infty}\left[\left(x-a\right)^{\alpha+1}+\left(b-x\right)^{\alpha+1}\right]\end{aligned}\tag{3.1}$$

for any $x\in(a,b)$, provided $f'\in L_{\infty}[a,b]$. In particular, we have

$$\begin{aligned}\left|J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)-\frac{\left(b-a\right)^{\alpha}}{2^{\alpha-1}\Gamma\left(\alpha+1\right)}\frac{f\left(a\right)+f\left(b\right)}{2}\right| \\ \leq\frac{1}{2^{\alpha+1}\Gamma\left(\alpha+2\right)}\left[\left\Vert f'\right\Vert_{\left[a,\frac{a+b}{2}\right],\infty}+\left\Vert f'\right\Vert_{\left[\frac{a+b}{2},b\right],\infty}\right]\left(b-a\right)^{\alpha+1} \\ \leq\frac{1}{2^{\alpha}\Gamma\left(\alpha+2\right)}\left\Vert f'\right\Vert_{[a,b],\infty}\left(b-a\right)^{\alpha+1}.\end{aligned}\tag{3.2}$$

If $f'\in L_{p}[a,b]$, where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then for any $x\in(a,b)$ we have

$$\begin{aligned}\left|J_{a+}^{\alpha}f\left(x\right)+J_{b-}^{\alpha}f\left(x\right)-\frac{\left(x-a\right)^{\alpha}f\left(a\right)+\left(b-x\right)^{\alpha}f\left(b\right)}{\Gamma\left(\alpha+1\right)}\right| \\ \leq\frac{1}{\Gamma\left(\alpha+1\right)\left(\alpha q+1\right)^{1/q}}\left[\left\Vert f'\right\Vert_{[a,x],p}\left(x-a\right)^{\alpha+\frac{1}{q}}+\left\Vert f'\right\Vert_{[x,b],p}\left(b-x\right)^{\alpha+\frac{1}{q}}\right] \\ \leq\frac{1}{\Gamma\left(\alpha+1\right)\left(\alpha q+1\right)^{1/q}}\left\Vert f'\right\Vert_{[a,b],p}\left[\left(x-a\right)^{q\alpha+1}+\left(b-x\right)^{q\alpha+1}\right]^{1/q}.\end{aligned}\tag{3.3}$$

In particular, we have

$$\begin{aligned}\left|J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)-\frac{\left(b-a\right)^{\alpha}}{2^{\alpha-1}\Gamma\left(\alpha+1\right)}\frac{f\left(a\right)+f\left(b\right)}{2}\right| \\ \leq\frac{\left(b-a\right)^{\alpha+\frac{1}{q}}}{2^{\alpha+\frac{1}{q}}\Gamma\left(\alpha+1\right)\left(\alpha q+1\right)^{1/q}}\left[\left\Vert f'\right\Vert_{\left[a,\frac{a+b}{2}\right],p}+\left\Vert f'\right\Vert_{\left[\frac{a+b}{2},b\right],p}\right] \\ \leq\frac{1}{2^{\alpha}\Gamma\left(\alpha+1\right)\left(\alpha q+1\right)^{1/q}}\left\Vert f'\right\Vert_{[a,b],p}\left(b-a\right)^{\alpha+\frac{1}{q}}.\end{aligned}\tag{3.4}$$

The case $\alpha=1$ produces the following inequalities for the Riemann integral that improve the trapezoid inequalities from Introduction:

$$\begin{aligned}\left|\left(x-a\right)f\left(a\right)+f\left(b\right)\left(b-x\right)-\int_{a}^{b}f\left(t\right)dt\right| \\ \leq\frac{1}{2}\left[\left(x-a\right)^{2}\left\Vert f'\right\Vert_{[a,x],\infty}+\left(b-x\right)^{2}\left\Vert f'\right\Vert_{[x,b],\infty}\right] \\ \leq\left[\frac{1}{4}\left(b-a\right)+\left(x-\frac{a+b}{2}\right)^{2}\right]\left\Vert f'\right\Vert_{[a,b],\infty}\end{aligned}\tag{3.5}$$

for any $x \in [a,b]$ and in particular

$$\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)^2\left[\|f'\|_{[a,\frac{a+b}{2}],\infty}+\|f'\|_{[\frac{a+b}{2},b],\infty}\right] \le \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty}, \tag{3.6}$$

where $f' \in L_\infty[a,b]$.

If $f' \in L_p[a,b]$, where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then for any $x \in (a,b)$ we have

$$\begin{aligned}\left|(x-a)f(a)+f(b)(b-x)-\int_a^b f(t)\,dt\right| &\le \frac{1}{(q+1)^{1/q}}\left[\|f'\|_{[a,x],p}(x-a)^{1+\frac{1}{q}}+\|f'\|_{[x,b],p}(b-x)^{1+\frac{1}{q}}\right]\\ &\le \frac{1}{(q+1)^{1/q}}\|f'\|_{[a,b],p}\left[(x-a)^{q+1}+(b-x)^{q+1}\right]^{1/q}.\end{aligned} \tag{3.7}$$

and, in particular

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right| &\le \frac{1}{2^{1+1/q}(q+1)^{1/q}}(b-a)^{1+\frac{1}{q}}\left[\|f'\|_{[a,\frac{a+b}{2}],p}+\|f'\|_{[\frac{a+b}{2},b],p}\right]\\ &\le \frac{1}{2(q+1)^{1/q}}(b-a)^{1+\frac{1}{q}}\|f'\|_{[a,b],p}.\end{aligned} \tag{3.8}$$

Now, if we take $g(t)=\ln t,\ t\in(a,b)\subset(0,\infty)$ in inequalities from the above section, then we get the following results for the Hadamard fractional integrals H_{a+}^{α} and H_{b-}^{α}:

If $ef' \in L_\infty[a,b]$, where $e(t)=t$ is the identity function, then for any $x\in(a,b)$ we have

$$\begin{aligned}\left|H_{a+}^{\alpha}f(x)+H_{b-}^{\alpha}f(x)-\frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}f(a)+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| &\le \frac{1}{\Gamma(\alpha+2)}\left[\|ef'\|_{[a,x],\infty}\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha+1}+\|ef'\|_{[x,b],\infty}\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha+1}\right]\\ &\le \frac{1}{\Gamma(\alpha+2)}\|ef'\|_{[a,b],\infty}\left[\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha+1}+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha+1}\right]\end{aligned} \tag{3.9}$$

and, in particular

$$\begin{aligned}\left|H_{a+}^{\alpha}f\left(\sqrt{ab}\right)+H_{b-}^{\alpha}f\left(\sqrt{ab}\right)-\frac{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}\right| &\le \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[\|ef'\|_{[a,\sqrt{ab}],\infty}+\|ef'\|_{[\sqrt{ab},b],\infty}\right]\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha+1}\\ &\le \frac{1}{2^{\alpha}\Gamma(\alpha+2)}\|ef'\|_{[a,b],\infty}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha+1}.\end{aligned} \tag{3.10}$$

If $e^{1/q}f' \in L_p[a,b]$, where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then for any $x\in(a,b)$ we have

$$\left| H_{a+}^{\alpha}f(x)+H_{b-}^{\alpha}f(x)-\frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}f(a)+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \tag{3.11}$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}}$$
$$\times\left[\left\|e^{1/q}f'\right\|_{[a,x],p}\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha+\frac{1}{q}}+\left\|e^{1/q}f'\right\|_{[x,b],p}\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha+\frac{1}{q}}\right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}}\left\|e^{1/q}f'\right\|_{[a,b],p}\left[\left[\ln\left(\frac{x}{a}\right)\right]^{q\alpha+1}+\left[\ln\left(\frac{b}{x}\right)\right]^{q\alpha+1}\right]^{1/q}$$

and, in particular

$$\left| H_{a+}^{\alpha}f\left(\sqrt{ab}\right)+H_{b-}^{\alpha}f\left(\sqrt{ab}\right)-\frac{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}\right| \tag{3.12}$$
$$\leq \frac{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha+\frac{1}{q}}}{2^{\alpha+\frac{1}{q}}\Gamma(\alpha+1)(\alpha q+1)^{1/q}}\left[\left\|e^{1/q}f'\right\|_{[a,\sqrt{ab}],p}+\left\|e^{1/q}f'\right\|_{[\sqrt{ab},b],p}\right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)(\alpha q+1)^{1/q}}\left\|e^{1/q}f'\right\|_{[a,b],p}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha+\frac{1}{q}}.$$

If we take the function $g(t)=-t^{-1}$, $t\in(a,b)\subset(0,\infty)$ in (2.5)-(2.6), then we have the following inequalities for Harmonic fractional integrals and $x\in(a,b)$

$$\left| R_{a+}^{\alpha}f(x)+R_{b-}^{\alpha}f(x)-\frac{\left(\frac{x-a}{xa}\right)^{\alpha}f(a)+\left(\frac{b-x}{bx}\right)^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \tag{3.13}$$
$$\leq \frac{1}{\Gamma(\alpha+2)}\left[\left\|e^{2}f'\right\|_{[a,x],\infty}\left(\frac{x-a}{xa}\right)^{\alpha+1}+\left\|e^{2}f'\right\|_{[x,b],\infty}\left(\frac{b-x}{bx}\right)^{\alpha+1}\right]$$
$$\leq \frac{1}{\Gamma(\alpha+2)}\left\|e^{2}f'\right\|_{[a,b],\infty}\left[\left(\frac{x-a}{xa}\right)^{\alpha+1}+\left(\frac{b-x}{bx}\right)^{\alpha+1}\right]$$

if $e^{2}f'\in L_{\infty}[a,b]$, and

$$\left| R_{a+}^{\alpha}f(x)+R_{b-}^{\alpha}f(x)-\frac{\left(\frac{x-a}{xa}\right)^{\alpha}f(a)+\left(\frac{b-x}{bx}\right)^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \tag{3.14}$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}}$$
$$\times\left[\left\|e^{2/q}f'\right\|_{[a,x],p}\left(\frac{x-a}{xa}\right)^{\alpha+\frac{1}{q}}+\left\|e^{2/q}f'\right\|_{[x,b],p}\left(\frac{b-x}{bx}\right)^{\alpha+\frac{1}{q}}\right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)(\alpha q+1)^{1/q}}\left\|e^{2/q}f'\right\|_{[a,b],p}\left[\left(\frac{x-a}{xa}\right)^{q\alpha+1}+\left(\frac{b-x}{bx}\right)^{q\alpha+1}\right]^{1/q}$$

if $e^{2/q}f'\in L_p[a,b]$ for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Consider the Harmonic mean of two positive numbers $H(a,b) := \frac{2ab}{a+b}$, then by (2.8) and (2.9) we have

$$\left| R_{a+}^{\alpha} f(H(a,b)) + R_{b-}^{\alpha} f(H(a,b)) - \frac{\left(\frac{b-a}{ba}\right)^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} \frac{f(a)+f(b)}{2} \right| \tag{3.15}$$
$$\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)} \left[\left\| e^2 f' \right\|_{[a,H(a,b)],\infty} + \left\| e^2 f' \right\|_{[H(a,b),b],\infty} \right] \left(\frac{b-a}{ba} \right)^{\alpha+1}$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+2)} \left\| e^2 f' \right\|_{[a,b],\infty} \left(\frac{b-a}{ba} \right)^{\alpha+1}$$

provided $e^2 f' \in L_{\infty}[a,b]$ and

$$\left| R_{a+}^{\alpha} f(H(a,b)) + R_{b-}^{\alpha} f(H(a,b)) - \frac{\left(\frac{b-a}{ba}\right)^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} \frac{f(a)+f(b)}{2} \right| \tag{3.16}$$
$$\leq \frac{\left(\frac{b-a}{ba}\right)^{\alpha+\frac{1}{q}}}{2^{\alpha+\frac{1}{q}}\Gamma(\alpha+1)(\alpha q+1)^{1/q}} \left[\left\| e^{2/q} f' \right\|_{[a,H(a,b)],p} + \left\| e^{2/q} f' \right\|_{[H(a,b),b],p} \right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)(\alpha q+1)^{1/q}} \left\| e^{2/q} f' \right\|_{[a,b],p} \left(\frac{b-a}{ba} \right)^{\alpha+\frac{1}{q}},$$

if $e^{2/q} f' \in L_p[a,b]$ for $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

References

[1] A. Aglić Aljinović, Montgomery identity and Ostrowski type inequalities for Riemann-Liouville fractional integral. *J. Math.* **2014**, Art. ID 503195, 6 pp.

[2] A. O. Akdemir, Inequalities of Ostrowski's type for m- and (α, m)-logarithmically convex functions via Riemann-Liouville fractional integrals. *J. Comput. Anal. Appl.* **16** (2014), no. 2, 375–383

[3] G. A. Anastassiou, Fractional representation formulae under initial conditions and fractional Ostrowski type inequalities. *Demonstr. Math.* **48** (2015), no. 3, 357–378

[4] G. A. Anastassiou, The reduction method in fractional calculus and fractional Ostrowski type inequalities. *Indian J. Math.* **56** (2014), no. 3, 333–357.

[5] H. Budak, M. Z. Sarikaya, E. Set, Generalized Ostrowski type inequalities for functions whose local fractional derivatives are generalized s-convex in the second sense. *J. Appl. Math. Comput. Mech.* **15** (2016), no. 4, 11–21.

[6] P. Cerone and S. S. Dragomir, Trapezoidal-type rules from an inequalities point of view. *Handbook of analytic-computational methods in applied mathematics*, 65–134, Chapman & Hall/CRC, Boca Raton, FL, 2000. Preprint *RGMIA Res. Rep. Coll.* **2** (1999), No. 6, Art. 8. [Online http://rgmia.org/papers/v2n6/TrapRulesIneqRep.pdf].

[7] P. Cerone, S. S. Dragomir and C. E. M. Pearce, A generalized trapezoid inequality for functions of bounded variation. *Turkish J. Math.* **24** (2000), no. 2, 147–163

[8] S. S. Dragomir, Improvements of Ostrowski and generalised trapezoid inequality in terms of the upper and lower bounds of the first derivative. *Tamkang J. Math.* **34** (2003), no. 3, 213–222.

[9] S. S. Dragomir, Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation. *Arch. Math.* (Basel) **91** (2008), no. 5, 450–460.

[10] S. S. Dragomir, Refinements of the generalized trapezoid inequality in terms of the cumulative variation and applications. Cubo **17** (2015), no. 2, 31–48.

[11] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results, *Australian J. Math. Anal. Appl.*, Volume **14**, Issue 1, Article 1, pp. 1-287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[12] S. S. Dragomir, Ostrowski type inequalities for Riemann-Liouville fractional integrals of bounded variation, Hölder and Lipschitzian functions, Preprint *RGMIA Res. Rep. Coll.* **20** (2017), Art. 48. [Online http://rgmia.org/papers/v20/v20a48.pdf].

[13] S. S. Dragomir, Ostrowski and trapezoid type inequalities for Riemann-Liouville fractional integrals of absolutely continuous functions with bounded derivatives, Preprint *RGMIA Res. Rep. Coll.* **20** (2017), Art 53. [http://rgmia.org/papers/v20/v20a53.pdf].

[14] S. S. Dragomir, Ostrowski type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation, Preprint *RGMIA Res. Rep. Coll.* **20** (2017), Art 58. [Online http://rgmia.org/papers/v20/v20a58.pdf].

[15] A. Guezane-Lakoud and F. Aissaoui, New fractional inequalities of Ostrowski type. *Transylv. J. Math. Mech.* **5** (2013), no. 2, 103–106

[16] A. Kashuri and R. Liko, Ostrowski type fractional integral inequalities for generalized (s, m, φ)-preinvex functions. *Aust. J. Math. Anal. Appl.* **13** (2016), no. 1, Art. 16, 11 pp.

[17] A. Kilbas, A; H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006. xvi+523 pp. ISBN: 978-0-444-51832-3; 0-444-51832-0

[18] M. A. Noor, K. I.Noor and S. Iftikhar, Fractional Ostrowski inequalities for harmonic h-preinvex functions. *Facta Univ. Ser. Math. Inform.* **31** (2016), no. 2, 417–445

[19] M. Z. Sarikaya and H. Filiz, Note on the Ostrowski type inequalities for fractional integrals. *Vietnam J. Math.* **42** (2014), no. 2, 187–190

[20] M. Z. Sarikaya and H. Budak, Generalized Ostrowski type inequalities for local fractional integrals. *Proc. Amer. Math. Soc.* **145** (2017), no. 4, 1527–1538.

[21] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Comput. Math. Appl.* 63 (2012), no. 7, 1147–1154.

[22] M. Tunç, On new inequalities for h-convex functions via Riemann-Liouville fractional integration, *Filomat* **27**:4 (2013), 559–565.

[23] M. Tunç, Ostrowski type inequalities for m- and (α, m)-geometrically convex functions via Riemann-Louville fractional integrals. *Afr. Mat.* **27** (2016), no. 5-6, 841–850.

[24] H. Yildirim and Z. Kirtay, Ostrowski inequality for generalized fractional integral and related inequalities, *Malaya J. Mat.*, **2**(3)(2014), 322-329.

[25] C. Yildiz, E, Özdemir and Z. S. Muhamet, New generalizations of Ostrowski-like type inequalities for fractional integrals. *Kyungpook Math. J.* **56** (2016), no. 1, 161–172.

[26] H. Yue, Ostrowski inequality for fractional integrals and related fractional inequalities. *Transylv. J. Math. Mech.* **5** (2013), no. 1, 85–89.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa