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Hermite – Hadamard – Fejér type inequalities for h - preinvex functions via fractional integrals

MARIAN MATŁOKA

Department of Applied Mathematics
Poznań University of Economics and Business
Al. Niepodległości 10, 61-875 Poznań, Poland
marian.matloka@ue.poznan.pl

Abstract: In this paper, first we have established Hermite - Hadamard - Fejér inequalities involving Riemann - Liouville integrals for h - preinvex function. Second, some Hermite -Hadamard - Fejér type integral inequalities for the fractional integrals are obtained.

Keywords: Riemann-Liouville integrals, Hermite - Hadamard - Fejér inequalities, h - preinvex function.

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1. Introduction

The following definition is well known in the literature: a function $f: I \to R$, $\emptyset \neq I \subset R$, is said to be convex on I if the inequality

$$f(tx + (1-t)y) \leq t\, f(x) + (1-t)y \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Many important inequalities have been established for the class of convex functions, but the most famous is the Hermite - Hadamard inequality. This double inequality is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

where $f: [a,b] \to R$ is a convex function.

In 1978, Breckner introduced an s - convex function as a generalization of a convex function [1]. Such a function is defined in the following way: a function $f: [0,\infty] \to R$ is said to be s-convex in the second sense if

$$f(tx + (1-t)y) \leq t^s\, f(x) + (1-t)^s f(y) \tag{1.3}$$

holds for all $x, y \in [0,\infty)$, $t \in [0,1]$ and for fixed $s \in (0,1]$.

In [2], Dragomir and Fitzpatrick proved the following variant of the Hermite - Hadamard inequality for s – convex functions:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.4}$$

In the paper [3] Varošanec introduced a large class of non-negative functions, the so - called h - convex functions. This class is defined in the following way: a non-negative function $f: I \to R$, $\emptyset \neq I \subset R$ is an interval, is called h – convex if

$$f(tx + (1-t)y) \leq h(1-t)(f(x) + h(t)f(y) \tag{1.5}$$

holds for all $x, y \in I$, $t \in (0,1)$, where $h: J \to R$ is a non-negative function, $h \not\equiv 0$ and J is an interval, $(0,1) \subseteq J$.

In [4] Sarikaya, Saglam and Yildirim proved that for h – convex function the following variant of the Harmite - Hadamard inequality is fulfilled:

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le [f(a)+f(b)]\cdot \int_0^1 h(t)\,dt. \tag{1.6}$$

In [5] Bombardelli and Varošanec proved that for an h – convex function the following variant of the Harmite - Hadamard - Fejér inequality holds:

$$\frac{\int_a^b w(x)\,dx}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)\,w(x)dx$$

$$\le (b-a)\big(f(a)+f(b)\big)\int_0^1 h(t)\,w(ta+(1-t)b)dt, \tag{1.7}$$

where $w: [a,b] \to R,\ w \ge 0$ and symmetric with respect to $\frac{a+b}{2}$.

In 1988, Weir and Mond introduced the concept of preinvex functions [6]. Now, we recall some notions in invexity analysis which will be used throughout the article.

A set $S \subseteq R^n$ is said to be invex with respect to the map $\eta: S\times S \to R^n$, if for every $x, y \in S$ and $t \in [0,1]$

$$y + t\,\eta(x,y) \in S.$$

Let $S \subseteq R^n$ be an invex set with respect to $\eta: S\times S \to R^n$. Then, the function $f: S \to R$ is said to be preinvex with respect to η if for every $x, y \in S$ and $t \in [0,1]$

$$f\big(y + t\,\eta(x,y)\big) \le tf(x) + (1-t)f(y). \tag{1.8}$$

We also need the following assumption regarding the function η which is due to Mohan and Neogy [7].

Condition C. Let $S \subseteq R^n$ be an open invex subset with respect to η. For any $x, y \in S$ and $t \in [0,1]$,

$$\eta\big(y, y + t\,\eta(x,y)\big) = -t\,\eta(x,y),$$

$$\eta\big(x, y + t\,\eta(x,y)\big) = (1-t)\eta(x,y).$$

In 2009, Noor [8] proved the Harmite – Hadamard inequality for preinvex function under the assumption that the Condition C is fulfilled

$$f\left(a+\frac{1}{2}\,\eta(b,a)\right) \leq \frac{1}{\eta(b,a)} \int_{a}^{a+\eta(b,a)} f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.9}$$

In 2013, Matłoka [9] introduced the concept of h - preinvex function. Such a function is defined in the following way: the non-negative function f on the invex set S is said to be h - preinvex with respect to η if

$$f\big(y+t\,\eta(x,y)\big) \leq h(1-t)f(y)+h(t)f(x)$$

holds for all $x, y \in S$ and $t \in [0,1]$.

If $h(t)=t^s$ then the function is called s - preinvex.

In the same paper Matłoka proved the Hermite-Hadamard inequality for h - preinvex functions:

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(a+\frac{1}{2}\eta\,(b,a)\right) \leq \frac{1}{\eta(b,a)} \int_{a}^{a+\eta\,(b,a)} f(x)\,dx$$

$$\leq [f(a)+f(b)] \cdot \int_0^1 h(t)\,dt \tag{1.10}$$

In 2014, Matłoka [10] proved the following Hermite - Hadamard - Fejér inequality for h - preinvex function:

$$\frac{\int_a^{a+\eta\,(b,a)} w(x)\,dx}{2h\left(\frac{1}{2}\right)} f\left(a+\frac{1}{2}\eta\,(b,a)\right) \leq \int_{a}^{a+\eta\,(b,a)} f(x)\,w(x)dx$$

$$\leq \eta(b,a)\big(f(a)+f(b)\big) \cdot \int_0^1 h(t)w\big(a+t\,\eta(b,a)\big)\,dt. \tag{1.11}$$

In 2013, Sarikaya, Set, Yaldiz and Basak [11] established the following Hermite-Hadamard inequalities for Riemann-Liouville fractional integral

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b,a)^{\alpha}}\left[\mathrm{I}_{a^+}^{\alpha} f(b)+\mathrm{I}_{b^-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{1.12}$$

where f is convex function and the symbols $\mathrm{I}_{a^+}^{\alpha} f$ and $\mathrm{I}_{b^-}^{\alpha} f$ denote the left - sided and right sided Riemann-Liouville fractional integral of the order $\alpha \in R^+$ that are defined by

$$\mathrm{I}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt \qquad (0 \le a < x \le b),$$

and

$$\mathrm{I}_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt \qquad (0 \le a \le x < b),$$

respectively. Here $\Gamma(\cdot)$ is the gamma function.

In the present we give new inequalities of Hermite - Hadamard - Fejér for h - preinvex functions.

2. Hermite - Hadamard - Fejér inequalities via fractional integrals

Hermite - Hadamard - Fejér inequalities can be represented in fractional integral forms as follows.

Theorem 2.1. Suppose $f: [a,\ a+\eta(b,a)] \to R$ is an h - preinvex function, Condition C for η holds and $\eta(b,a) > 0$, $h\left(\frac{1}{2}\right) > 0$ and $w: [a,\ a+\eta(b,a)] \to R$, $w \ge 0$ is symmetric with respect to $a + \frac{1}{2}\eta\ (b,a)$. Then the following inequalities hold:

$$\frac{\Gamma(\alpha)}{2 \cdot h\left(\frac{1}{2}\right) \cdot \eta(b,a)^{\alpha}} f\left(a + \frac{1}{2}\eta\ (b,a)\right) \left[\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha}\, w(a) + \mathrm{I}_{a^+}^{\alpha} w\big(a+\eta(b,a)\big)\right]$$

$$\le \frac{\Gamma(\alpha)}{\eta(b,a)^{\alpha}} \left[\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha}\, w(a) f(a) + \mathrm{I}_{a^+}^{\alpha} w\big(a+\eta(b,a)\big) f(a+\eta(b,a))\right]$$

$$\le [f(a)+f(b)] \cdot \int_0^1 t^{\alpha-1} [h(t) + h(1-t)]\, w\big(a + t\,\eta\ (b,a)\big) dt, \tag{2.1}$$

Proof. From the definition of an h - preinvex function and from Condition C for η it follows that:

$$f\left(a+\frac{1}{2}\eta\ (b,a)\right) \le h\left(\frac{1}{2}\right)\left[f\big(a+t\,\eta\ (b,a)\big)+f\big(a+(1-t)\eta\ (b,a)\big)\right].$$

Multiplying both sides of the above inequality by

$t^{\alpha-1}w\big(a+t\,\eta\ (b,a)\big)=t^{\alpha-1}w\big(a+(1-t)\ \eta\ (b,a)\big)$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 f\left(a+\frac{1}{2}\eta\ (b,a)\right)t^{\alpha-1}w\big(a+t\,\eta\ (b,a)\big)dt$$

$$\le h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\alpha-1}f\big(a+t\,\eta\ (b,a)\big)\ w\big(a+t\,\eta\ (b,a)\big)dt\right.$$

$$\left.+\int_0^1 t^{\alpha-1}f\big(a+(1-t)\eta\ (b,a)\big)\ w\big(a+(1-t)\eta\ (b,a)\big)dt\right].$$

Since

$$\int_0^1 t^{\alpha-1}\ w\big(a+t\,\eta\ (b,a)\big)dt=\frac{\Gamma(\alpha)}{\eta(b,a)^{\alpha}}\ \mathrm{I}^{\alpha}_{(\ a+\eta(b,a))^-}\ w(a),$$

$$\int_0^1 t^{\alpha-1}f\big(a+t\,\eta\ (b,a)\big)\ w\big(a+t\,\eta\ (b,a)\big)dt=\frac{\Gamma(\alpha)}{\eta(b,a)^{\alpha}}\ \mathrm{I}^{\alpha}_{(\ a+\eta(b,a))^-}\ f(a)w(a)$$

and

$$\int_0^1 t^{\alpha-1}f\big(a+(1-t)\eta\ (b,a)\big)w\big(a+(1-t)\eta(b,a)\big)dt$$

$$=\frac{\Gamma(\alpha)}{\eta(b,a)^{\alpha}}\mathrm{I}^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)w\big(a+\eta(b,a)\big),$$

we have

$$\frac{\Gamma(\alpha)}{\eta(b,a)^{\alpha}}\ f\left(a+\frac{1}{2}\eta\ (b,a)\right)\ \mathrm{I}^{\alpha}_{(\ a+\eta(b,a))^-}\ w(a)$$

$$\leq h\left(\frac{1}{2}\right)\frac{\Gamma(\alpha)}{\eta(b,a)^{\alpha}}\left[\mathrm{I}^{\alpha}_{(a+\eta(b,a))^-} f(a)w(a) + \mathrm{I}^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)w\big(a+\eta(b,a)\big)\right]. \tag{2.2}$$

Similarly, we also have

$$\frac{\Gamma(\alpha)}{\eta(b,a)^{\alpha}}\, f\left(a+\frac{1}{2}\eta\,(b,a)\right)\,\mathrm{I}^{\alpha}_{a^+}\, w(a+\eta(b,a))$$

$$\leq h\left(\frac{1}{2}\right)\frac{\Gamma(\alpha)}{\eta(b,a)^{\alpha}}\left[\mathrm{I}^{\alpha}_{(a+\eta(b,a))^-} f(a)w(a) + \mathrm{I}^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)w\big(a+\eta(b,a)\big)\right] \tag{2.3}$$

Thus, from (2.2) and (2.3) we obtain the first inequality of (2.1).

For the proof of the second inequality we first note that f is an h - preinvex function, then for $t \in [0,1]$, it yields

$$f\big(a+t\,\eta\,(b,a)\big) \leq h(1-t)f(a) + h(t)f(b)$$

and

$$f\big(a+(1-t)\,\eta\,(b,a)\big) \leq h(t)f(a) + h(1-t)f(b).$$

By adding these inequalities we have

$$f\big(a+t\,\eta\,(b,a)\big) + f\big(a+(1-t)\,\eta\,(b,a)\big) \leq [f(a)+f(b)]\cdot[h(1-t)+h(t)].$$

Then multiplying both sides by $t^{\alpha-1}\,w\big(a+t\,\eta\,(b,a)\big) = t^{\alpha-1}\,w\big(a+(1-t)\,\eta\,(b,a)\big)$

and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1}f\big(a+t\,\eta\,(b,a)\big)\,w\big(a+t\,\eta\,(b,a)\big)dt$$

$$+\int_0^1 t^{\alpha-1}f\big(a+(1-t)\,\eta\,(b,a)\big)\,w\big(a+(1-t)\,\eta\,(b,a)\big)dt$$

$$\leq [f(a)+f(b)]\cdot\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]\,w\big(a+t\,\eta\,(b,a)\big)dt$$

i.e.

$$\frac{\Gamma(\alpha)}{\eta(b,a)^{\alpha}}\left[\mathrm{I}^{\alpha}_{(\,a+\eta(b,a))^{-}}\,w(a)f(a)+\mathrm{I}^{\alpha}_{a^{+}}w\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big)\right]$$

$$\leq [f(a)+f(b)]\cdot\int_{0}^{1} t^{\alpha-1}[h(t)+h(1-t)]\,w\big(a+t\,\eta\,(b,a)\big)dt.$$

The proof is completed.

Corollary 2.1. In Theorem 2.1, if $\alpha=1$, then inequalities (2.1) become inequalities (1.11).

Corollary 2.2. In Theorem 2.1, if we take $\eta\,(b,a)=b-a, w(x)\equiv 1$ and $h(t)=t$, which means that f is convex function, then inequalities (2.1) become inequalities (1.12).

Corollary 2.3. In Theorem 2.1, if $\alpha=1$ and $w(x)\equiv 1$, then we get inequalities (1.10).

Corollary 2.4. In Theorem 2.1, if $\alpha=1$ and $w(x)\equiv 1$ and $h(t)=t$, then we get inequalities (1.9).

Corollary 2.5. In Theorem 2.1, if $\alpha=1$ and $\eta\,(b,a)=b-a$, then inequalities (2.1) become inequalities (1.7).

Corollary 2.6. In Theorem 2.1, if $\alpha=1$ and $\eta\,(b,a)=b-a$ and $w(x)\equiv 1$, then we get inequalities (1.6).

Corollary 2.6. In Theorem 2.1, if $\alpha=1$ and $\eta\,(b,a)=b-a$ and $w(x)\equiv 1$ and $h(t)=t^{s}$, then we get inequalities (1.4).

Corollary 2.5. In Theorem 2.1, if $\alpha=1$ and $\eta\,(b,a)=b-a$ and $w(x)\equiv 1$ and $h(t)=t$, then we get inequalities (1.2).

Corollary 2.6. In Theorem 2.1, if we let $w(x)\equiv 1$ and $h(t)=t^{s}$, then inequalities (2.1) become the following inequalities for s - preinvex function

$$2^{s}f\left(a+\frac{1}{2}\eta\,(b,a)\right)\leq\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha}}\left[\mathrm{I}^{\alpha}_{(\,a+\eta(b,a))^{-}}\,f(a)+\mathrm{I}^{\alpha}_{a^{+}}f\big(a+\eta(b,a)\big)\right]$$

$$\leq\alpha\,[f(a)+f(b)]\cdot\left[\frac{1}{\alpha+s}+\frac{\Gamma(\alpha)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right].$$

Corollary 2.7. In Theorem 2.1, if we let $w(x) \equiv 1$, $h(t) = t^s$ and $\eta\,(b,a) = b - a$ then inequalities (2.1) become the following inequalities for s - convex function

$$2^s f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[\mathrm{I}_{b^-}^{\alpha} f(a) + \mathrm{I}_{a^+}^{\alpha} f(b)\right]$$

$$\le \alpha\,[f(a)+f(b)] \cdot \left[\frac{1}{\alpha+s} + \frac{\Gamma(\alpha)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right].$$

Corollary 2.8. In Theorem 2.1, if we let $w(x) \equiv 1$, $h(t) = t$ then inequalities (2.1) become the following inequalities for preinvex function

$$f\left(a + \frac{1}{2}\eta\,(b,a)\right) \le \frac{\Gamma(\alpha+1)}{2\eta(b,a)^\alpha}\left[\mathrm{I}_{(\,a+\eta(b,a))^-}^{\alpha} f(a) + \mathrm{I}_{a^+}^{\alpha} f\big(a+\eta(b,a)\big)\right] \le \frac{f(a)+f(b)}{2}.$$

3. Hermite - Hadamard - Fejér type inequalities via fractional integrals

In order to prove our results we need the following identity.

Lemma 3.1. Let $K \subseteq R$ be an open invex subset with respect to $\eta\colon\ K \times K \to R$ and $a, b \in K$ with $\eta(b,a) > 0$. Suppose that $f\colon\ K \to R$ is differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $w\colon\ K \to [0,\infty)$ is differentiable, then the following equality holds:

$$\int_0^1 [(1-t)^\alpha - t^\alpha]\, w\big(a + t\,\eta\,(b,a)\big) \cdot f'\big(a + t\,\eta\,(b,a)\big)\, dt$$

$$= \frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\Big[\mathrm{I}_{a^+}^{\alpha}\, w\big(a+\eta(b,a)\big) w\big(a+\eta(b,a)\big) + \mathrm{I}_{(\,a+\eta(b,a))^-}^{\alpha}\, w(a) f(a)$$

$$-\mathrm{I}_{a^+}^{\alpha}\, w'\big(a+\eta(b,a)\big)\, f\big(a+\eta(b,a)\big) + \mathrm{I}_{(\,a+\eta(b,a))^-}^{\alpha+1}\, w'(a) f(a)\Big]$$

$$-\frac{1}{\eta(b,a)}\left[f\big(a+\eta(b,a)\big)\, w\big(a+\eta(b,a)\big) + w(a) f(a)\right]. \qquad (3.1)$$

Proof. Integrating by parts

$$\int_0^1 [(1-t)^\alpha - t^\alpha]\, w\big(a+t\,\eta\,(b,a)\big)\cdot f'\big(a+t\,\eta\,(b,a)\big)\,dt$$

$$=\frac{1}{\eta(b,a)} f\big(a+t\,\eta(b,a)\big)\,w\big(a+t\,\eta(b,a)\big)[(1-t)^\alpha - t^\alpha]\;\Big|_0^1$$

$$+\frac{\alpha}{\eta(b,a)}\int_0^1 [(1-t)^{\alpha-1} + t^{\alpha-1}]\, w\big(a+t\,\eta\,(b,a)\big)\cdot f\big(a+t\,\eta\,(b,a)\big)\,dt$$

$$-\int_0^1 [(1-t)^\alpha - t^\alpha]\, w'\big(a+t\,\eta\,(b,a)\big)\cdot f\big(a+t\,\eta\,(b,a)\big)\,dt$$

$$=-\frac{1}{\eta(b,a)}\big[w(a)f(a)+f\big(a+\eta\,(b,a)\big)\,w\big(a+\eta\,(b,a)\big)\big]$$

$$+\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\Big[\mathrm{I}^{\alpha}_{a^+}\, w\big(a+\eta(b,a)\big)\, f\big(a+\eta(b,a)\big)+\mathrm{I}^{\alpha}_{(a+\eta(b,a))^-}\, w(a)f(a)$$

$$-\mathrm{I}^{\alpha+1}_{a^+} w'\big(a+\eta(b,a)\big)\, f\big(a+\eta(b,a)\big)+\mathrm{I}^{\alpha+1}_{(a+\eta(b,a))^-}\, w'(a)f(a)\Big]$$

which completes the proof.

Using this Lemma, we can obtain the following fractional integral inequalities.

Theorem 3.1. Let $K\subseteq R$ be an open invex subset with respect to $\eta\colon\, K\times K\to R$ and $a,b\in K$ with $\eta(b,a)>0$. Suppose that $f\colon\, K\to R$ is a differentiable mapping on K and w: $K\to[0,\infty)$ is differentiable and symmetric to $a+\frac{1}{2}\eta\,(b,a)$. If $|f'|$ is h – preinvex on K, we have the following inequality:

$$\Bigg|\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\Big[\mathrm{I}^{\alpha}_{a^+}\, w\big(a+\eta(b,a)\big)\, f\big(a+\eta(b,a)\big)+\mathrm{I}^{\alpha}_{(a+\eta(b,a))^-}\, w(a)f(a)$$

$$-\mathrm{I}^{\alpha+1}_{a^+}\, w'\big(a+\eta(b,a)\big)\, f\big(a+\eta(b,a)\big)+\mathrm{I}^{\alpha+1}_{(a+\eta(b,a))^-}\, w'(a)f(a)\Big]$$

$$-\frac{1}{\eta(b,a)}\left[f\big(a+\eta\,(b,a)\big)\,w\big(a+\eta\,(b,a)\big)+f(a)\,w(a)\right]\Bigg|$$

$$\leq [|f'(a)|+|f'(b)|]\cdot\int_0^1 t^{\alpha}\,w\big(a+t\,\eta\,(b,a)\big)[h(t)+h(1-t)]\,dt. \tag{3.2}$$

Proof. Using Lemma 3.1 and the h – preinvexity of $|f'|$, we have

$$\Bigg|\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\left[\mathrm{I}_{a^+}^{\alpha}\,w\big(a+\eta(b,a)\big)\,f\big(a+\eta(b,a)\big)+\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha}\,w(a)f(a)\right.$$

$$\left.-\,\mathrm{I}_{a^+}^{\alpha+1}\,w'\big(a+\eta(b,a)\big)\,f\big(a+\eta(b,a)\big)+\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha+1}\,w'(a)f(a)\right]$$

$$-\frac{1}{\eta(b,a)}\left[f\big(a+\eta\,(b,a)\big)\,w\big(a+\eta\,(b,a)\big)+f(a)\,w(a)\right]\Bigg|$$

$$\leq\int_0^1|(1-t)^{\alpha}-t^{\alpha}|\,w\big(a+t\,\eta\,(b,a)\big)\big|f'\big(a+\eta(b,a)\big)\big|\,dt$$

$$\leq\int_0^1[(1-t)^{\alpha}+t^{\alpha}]\,w\big(a+t\,\eta\,(b,a)\big)\,[h(1-t)|f'(a)|+h(t)|f'(b)|\,]\,dt$$

$$=|f'(a)|\cdot\int_0^1(1-t)^{\alpha}\,w\big(a+t\,\eta\,(b,a)\big)h(1-t)dt+|f'(a)|\int_0^1 t^{\alpha}\,w\big(a+t\,\eta\,(b,a)\big)h(1-t)dt$$

$$+|f'(b)|\cdot\int_0^1(1-t)^{\alpha}\,w\big(a+t\,\eta\,(b,a)\big)h(t)dt+|f'(b)|\int_0^1 t^{\alpha}\,w\big(a+t\,\eta\,(b,a)\big)h(t)dt$$

$$=[|f'(a)|+|f'(b)|]\int_0^1 t^{\alpha}\,w\big(a+t\,\eta\,(b,a)\big)\cdot h(t)dt$$

$$+[|f'(a)|+|f'(b)|]\int_0^1 t^{\alpha}\,w\big(a+t\,\eta\,(b,a)\big)\cdot h(1-t)dt$$

$$= [|f'(a)| + |f'(b)|] \cdot \int_0^1 t^\alpha \, w\big(a + t\,\eta\,(b,a)\big)[h(t) + h(1-t)]dt$$

which completes the proof.

Corollary 3.1. If we take $w(x) \equiv 1$, and $h(t) = t$ then inequality (3.2) become the following inequality for preinvex function:

$$\left|\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\left[\mathrm{I}_{a^+}^{\alpha} f\big(a+\eta(b,a)\big) + \mathrm{I}_{(a+\eta(b,a))^-}^{\alpha} f(a) - \frac{1}{\eta(b,a)}\left[f\big(a+\eta(b,a)\big) + f(a)\right]\right|\right.$$

$$\leq \frac{|f'(a)| + |f'(b)|}{\alpha+1}.$$

Corollary 3.2. If we take $w(x) \equiv 1$, and $h(t) = t^s$ then inequality (3.2) become the following inequality for s - preinvex function:

$$\left|\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\left[\mathrm{I}_{a^+}^{\alpha} f\big(a+\eta(b,a)\big) + \mathrm{I}_{(a+\eta(b,a))^-}^{\alpha} f(a) - \frac{1}{\eta(b,a)}\left[f\big(a+\eta(b,a)\big) + f(a)\right]\right|\right.$$

$$\leq [|f'(a)| + |f'(b)|] \cdot \left[\frac{1}{\alpha+s+1} + \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right].$$

Theorem 3.2. Let $K \subseteq R$ be an open invex subset with respect to η: $K \times K \to R$ and $a, b \in K$ with $\eta(b,a) > 0$. Suppose that f: $K \to R$ is a differentiable mapping on K and w: $K \to [0,\infty)$ is differentiable and symmetric to $a + \frac{1}{2}\eta\,(b,a)$. If $|f'|^q$, $q \geq 1$, is h - preinvex on K, then one has:

$$\left|\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\left[\mathrm{I}_{a^+}^{\alpha}\, w\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big) + \mathrm{I}_{(a+\eta(b,a))^-}^{\alpha}\, w(a)f(a)\right.\right.$$

$$\left.-\mathrm{I}_{a^+}^{\alpha+1} w'\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big) + \mathrm{I}_{(a+\eta(b,a))^-}^{\alpha+1}\, w'(a)f(a)\right]$$

$$\left.-\frac{1}{\eta(b,a)}\left[f\big(a+\eta(b,a)\big)w\big(a+\eta(b,a)\big) + f(a)w(a)\right]\right|$$

$$\leq \left(\frac{2}{\alpha+1}\right)^{1-\frac{1}{q}} \left([|f'(a)|^q + |f'(b)|^q] \int_0^1 t^\alpha \left[w\big(a+t\,\eta\,(b,a)\big)\right]^q [h(t)+h(1-t)]dt \right)^{\frac{1}{q}} \qquad (3.3)$$

Proof. By using the Lemma 3.1, h - preinvexity of $|f'|^q$ and the well known power mean inequality, we have

$$\Bigg|\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\big[\mathrm{I}^{\alpha}_{a^+}\, w\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big) + \mathrm{I}^{\alpha}_{(a+\eta(b,a))^-}\, w(a)f(a)$$

$$-\mathrm{I}^{\alpha+1}_{a^+} w'\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big) + \mathrm{I}^{\alpha+1}_{(a+\eta(b,a))^-}\, w'(a)f(a)\big]$$

$$-\frac{1}{\eta(b,a)}\big[f\big(a+\eta(b,a)\big)w\big(a+\eta(b,a)\big) + f(a)w(a)\big]\Bigg|$$

$$\leq \left(\int_0^1 [(1-t)^\alpha + t^\alpha]\, dt\right)^{1-\frac{1}{q}} \cdot$$

$$\cdot \left(\int_0^1 [(1-t)^\alpha + t^\alpha]\left[w\big(a+t\,\eta\,(b,a)\big)\right]^q \left|f'\big(a+t\,\eta\,(b,a)\big)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{2}{\alpha+1}\right)^{1-\frac{1}{q}} \left([|f'(a)|^q + |f'(b)|^q] \int_0^1 t^\alpha \left[w\big(a+t\,\eta\,(b,a)\big)\right]^q [h(t)+h(1-t)]\, dt \right)^{\frac{1}{q}}$$

which completes the proof.

Corollary 3.3. If we take $w(x) \equiv 1$, and $h(t) = t$ then inequality (3.3) become the following inequality for preinvex function

$$\Bigg|\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\big[\mathrm{I}^{\alpha}_{a^+}\, w\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big) + \mathrm{I}^{\alpha}_{(a+\eta(b,a))^-}\, w(a)f(a)$$

$$-\mathrm{I}^{\alpha+1}_{a^+}\, w'\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big) + \mathrm{I}^{\alpha+1}_{(a+\eta(b,a))^-}\, w'(a)f(a)\big]$$

$$-\frac{1}{\eta(b,a)}\left[f\big(a+\eta(b,a)\big)w\big(a+\eta(b,a)\big)+f(a)w(a)\right]\Bigg|$$

$$\leq\left(\frac{2}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{\alpha+1}\right)^{\frac{1}{q}}.$$

Corollary 3.4. If we take $w(x)\equiv 1$, and $h(t)=t^s$ then inequality (3.3) become the following inequality for s - preinvex function

$$\Bigg|\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\left[\mathrm{I}_{a^+}^{\alpha}\,w\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big)+\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha}\,w(a)f(a)\right.$$

$$\left.-\mathrm{I}_{a^+}^{\alpha+1}\,w'\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big)+\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha+1}\,w'(a)f(a)\right]$$

$$-\frac{1}{\eta(b,a)}\left[f\big(a+\eta(b,a)\big)w\big(a+\eta(b,a)\big)+f(a)w(a)\right]\Bigg|$$

$$\leq\left(\frac{2}{\alpha+1}\right)^{1-\frac{1}{q}}\left([|f'(a)|^q+|f'(b)|^q]\cdot\left[\frac{1}{\alpha+s+1}+\frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+2)}\right]\right)^{\frac{1}{q}}.$$

Theorem 3.3. Let $K\subseteq R$ be an open invex subset with respect to η: $K\times K\to R$ and $a,b\in K$ with $\eta(b,a)>0$. Suppose that f: $K\to R$ is a differentiable mapping on K and w: $K\to[0,\infty)$ is differentiable and symmetric to $a+\frac{1}{2}\eta\,(b,a)$. If $|f'|^q$, $q>1$, is h - preinvex on K, then the following inequality for fractional integrals holds:

$$\Bigg|\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\left[\mathrm{I}_{a^+}^{\alpha}\,w\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big)+\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha}\,w(a)f(a)\right.$$

$$\left.-\mathrm{I}_{a^+}^{\alpha+1}\,w'\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big)+\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha+1}\,w'(a)f(a)\right]$$

$$-\frac{1}{\eta(b,a)}\left[f\big(a+\eta(b,a)\big)w\big(a+\eta(b,a)\big)+f(a)w(a)\right]\Bigg|$$

$$\leq\frac{2}{(\alpha\rho+1)^{\frac{1}{p}}}\left([|f'(a)|^q+|f'(b)|^q]\cdot\int_0^1\left[w\big(a+t\,\eta\,(b,a)\big)\right]^q h(t)dt\right)^{\frac{1}{q}},\tag{3.4}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3.1 and using the well known Hőlder inequality, we have

$$\left|\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\left[\mathrm{I}_{a^+}^{\alpha}\, w\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big)+\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha}\, w(a)f(a)\right.\right.$$

$$\left.-\mathrm{I}_{a^+}^{\alpha+1}\, w'\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big)+\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha+1}\, w'(a)f(a)\right]$$

$$\left.-\frac{1}{\eta(b,a)}\left[f\big(a+\eta(b,a)\big)w\big(a+\eta(b,a)\big)+f(a)w(a)\right]\right|$$

$$\leq\left(\int_0^1(1-t)^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left[w\big(a+t\,\eta\,(b,a)\big)\right]^q\left|f'\big(a+t\,\eta\,(b,a)\big)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 t^{\alpha p}\,dt\right)^{\frac{1}{p}}\cdot\left(\int_0^1\left[w\big(a+t\,\eta\,(b,a)\big)\right]^q\left|f'\big(a+t\,\eta\,(b,a)\big)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{2}{(\alpha\,p+1)^{\frac{1}{p}}}\left([|f'(a)|^q+|f'(b)|^q]\cdot\int_0^1\left[w\big(a+t\,\eta\,(b,a)\big)\right]^q h(t)dt\right)^{\frac{1}{q}}.$$

Corollary 3.5. If we take $w(x)\equiv 1$, and $h(t)=t$ then inequality (3.4) become the following inequality for preinvex function

$$\left|\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\left[\mathrm{I}_{a^+}^{\alpha}\, w\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big)+\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha}\, w(a)f(a)\right.\right.$$

$$\left.-\mathrm{I}_{a^+}^{\alpha+1}\, w'\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big)+\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha+1}\, w'(a)f(a)\right]$$

$$\left.-\frac{1}{\eta(b,a)}\left[f\big(a+\eta(b,a)\big)w\big(a+\eta(b,a)\big)+f(a)w(a)\right]\right|$$

$$\leq\frac{2}{(\alpha\,p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.6. If we take $w(x)\equiv 1$, and $h(t)=t^s$ then inequality (3.4) become the following inequality for s - preinvex function

$$\left|\frac{\Gamma(\alpha+1)}{\eta(b,a)^{\alpha+1}}\left[\mathrm{I}_{a^+}^{\alpha}\, w\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big)+\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha}\, w(a)f(a)\right.\right.$$

$$-\mathrm{I}_{a^+}^{\alpha+1}\, w'\big(a+\eta(b,a)\big)f\big(a+\eta(b,a)\big)+\mathrm{I}_{(a+\eta(b,a))^-}^{\alpha+1}\, w'(a)f(a)\Big]$$

$$\left.-\frac{1}{\eta(b,a)}\Big[f\big(a+\eta(b,a)\big)\,w\big(a+\eta(b,a)\big)+f(a)w(a)\Big]\right|$$

$$\leq\frac{2}{(\alpha\,p+1)^{\frac{1}{p}}\,(s+1)^{\frac{1}{q}}}\;\left(|f'(a)|^{q}+|f'(b)|^{q}\right)^{\frac{1}{q}}.$$

REFERENCES

1. Breckner, W.-W.: *Stetigkeitsanssagen für eine Klasse verallgemeinerter Konvexer Funktionen in topologischen linearen Räumen*, Pupl. Inst. Math. (Belgr.) 23, 13-20 (1978).
2. Dragomir, S.-S., Fitzpatrick, S.: *The Hadamard's inequality for s-convex functions in the second sense*, Demonstration Math. 32 (4), 687-696 (1999).
3. Varošanec, S.: *On h-convexity*, J. Math. Anal. Appl. 326, 303-311 (2007).
4. Sarikaya, M.-Z., Saglam, A. Yildirim, H.: *On some Hadamard-type inequalities for h-convex functions*, J. Math. Inequal. 2, 335-341 (2008).
5. Bombardelli, M., Varošanec, S.: *Properties of h-convex functions related to the Hermite - Hadamard - Fejér inequalities*, Comput. Math. Appl. 58, 1869-1877 (2009).
6. Weir, T., Mond, B.: *Preinvex functions in multiobjective optimization*, J. Math. Anal. Appl. 136, 29-38 (1988).
7. Mohan, S.-R., Neogy, S.-K.: *On invex sets and preinvex functions*, J. Math Anal. Appl. 189, 901-908 (1995).
8. Noor, M.-S.: *Hadamard integral inequalities for product of two preinvex functions*, Nonlinear Anal. Forum 14, 167-173 (2009).
9. Matłoka M.: *On some Hadamard – type inequalities for* $(h_1,\ h_2)$ *- preinvex functions on the co-ordinates*.J. Inequal. Appl. doi: 10.1186/242 X -2013-227.
10. Matłoka M.: *Inequalities for h-preinvex functions*. Appl. Math Comput. 234, 52-57 (2014).
11. Sarikaya, M.-Z., Set, E., Yaldiz, H., Basak, N.: *Hermite-Hadamard's inequalities for fractional integrals and related inequalities*, Math. Comput. Model 57, 2403-2407 (2013).