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OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF ABSOLUTELY CONTINUOUS FUNCTIONS WITH BOUNDED DERIVATIVES

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ABSTRACT. In this paper we establish some Ostrowski and trapezoid type inequalities for the generalized Riemann-Liouville fractional integrals of absolutely continuous functions with bounded derivatives.

1. INTRODUCTION

Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt$$

for $a<x\le b$ and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt$$

for $a\le x<b$, where Γ is the *Gamma function.* For $\alpha=0$, they are defined as

$$J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)\ \text{ for } x\in(a,b).$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[5], [17]-[28] and the references therein.

In the recent paper [14] we obtained the following result for absolutely continuous functions with bounded derivatives:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $x\in(a,b)$ and there exists the real numbers $m_1(x)$, $M_1(x)$, $m_2(x)$, $M_2(x)$ such that*

$$m_1(x)\le f'(t)\le M_1(x)\ \text{ for a.e. } t\in(a,x) \tag{1.1}$$

and

$$m_2(x)\le f'(t)\le M_2(x)\ \text{ for a.e. } t\in(x,b) \tag{1.2}$$

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then

$$\frac{1}{\Gamma(\alpha+2)}\left[m_2(x)(b-x)^{\alpha+1}-M_1(x)(x-a)^{\alpha+1}\right] \tag{1.3}$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right]-J_{a+}^{\alpha}f(x)-J_{b-}^{\alpha}f(x)$$
$$\leq \frac{1}{\Gamma(\alpha+2)}\left[M_2(x)(b-x)^{\alpha+1}-m_1(x)(x-a)^{\alpha+1}\right]$$

and

$$\frac{1}{\Gamma(\alpha+2)}\left[m_2(x)(b-x)^{\alpha+1}-M_1(x)(x-a)^{\alpha+1}\right] \tag{1.4}$$
$$\leq J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)$$
$$\leq \frac{1}{\Gamma(\alpha+2)}\left[M_2(x)(b-x)^{\alpha+1}-m_1(x)(x-a)^{\alpha+1}\right].$$

In particular, we have the simpler inequalities:

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If there exists the real numbers m, M, such that $m\leq f'(t)\leq M$ for a.e. $t\in(a,b)$, then*

$$\left|\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}(b-a)^{\alpha}-J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)-J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right| \tag{1.5}$$
$$\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(M-m)$$

and

$$\left|J_{\frac{a+b}{2}-}^{\alpha}f(a)+J_{\frac{a+b}{2}+}^{\alpha}f(b)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha}\right| \tag{1.6}$$
$$\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(b-a)^{\alpha+1}(M-m).$$

We also have the following result for convex functions [14]:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function and $x\in(a,b)$, then we have the inequalities*

$$\frac{1}{\Gamma(\alpha+2)}\left[f'_{+}(x)(b-x)^{\alpha+1}-f'_{-}(x)(x-a)^{\alpha+1}\right] \tag{1.7}$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right]-J_{a+}^{\alpha}f(x)-J_{b-}^{\alpha}f(x)$$
$$\leq \frac{1}{\Gamma(\alpha+2)}\left[f'_{-}(b)(b-x)^{\alpha+1}-f'_{+}(a)(x-a)^{\alpha+1}\right]$$

and

$$\frac{1}{\Gamma(\alpha+2)}\left[f'_{+}(x)(b-x)^{\alpha+1}-f'_{-}(x)(x-a)^{\alpha+1}\right] \tag{1.8}$$
$$\leq J_{x-}^{\alpha}f(a)+J_{x+}^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)$$
$$\leq \frac{1}{\Gamma(\alpha+2)}\left[f'_{-}(b)(b-x)^{\alpha+1}-f'_{+}(a)(x-a)^{\alpha+1}\right],$$

where $f'_{\pm}(\cdot)$ are the lateral derivatives of f.

In particular, we have

$$
\begin{aligned}
(1.9)\quad 0 &\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a)^{\alpha+1} \\
&\leq \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}(b-a)^{\alpha}-J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)-J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_{-}(b)-f'_{+}(a)\right](b-a)^{\alpha+1},
\end{aligned}
$$

and

$$
\begin{aligned}
(1.10)\qquad 0 &\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a)^{\alpha+1} \\
&\leq J_{\frac{a+b}{2}-}^{\alpha}f(a)+J_{\frac{a+b}{2}+}^{\alpha}f(b)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha} \\
&\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_{-}(b)-f'_{+}(a)\right](b-a)^{\alpha+1}.
\end{aligned}
$$

In order to extend the above results for generalized Riemann-Liouville fractional integrals, we need the following preparations.

Let (a,b) with $-\infty \leq a < b \leq \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\mathrm{Re}(\alpha) > 0$. Also, let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Following [19, p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ by

$$
(1.11)\qquad I_{a+,g}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{g'(t)f(t)\,dt}{[g(x)-g(t)]^{1-\alpha}},\ a < x \leq b
$$

and

$$
(1.12)\qquad I_{b-,g}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_{x}^{b}\frac{g'(t)f(t)\,dt}{[g(t)-g(x)]^{1-\alpha}},\ a \leq x < b.
$$

For $g(t) = t$ we have the classical *Riemann-Liouville fractional integrals* introduced above while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [19, p. 111]

$$
(1.13)\qquad H_{a+}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_{a}^{x}\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0 \leq a < x \leq b
$$

and

$$
(1.14)\qquad H_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_{x}^{b}\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0 \leq a < x < b.
$$

One can consider the function $g(t) = -t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$
(1.15)\qquad R_{a+}^{\alpha}f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_{a}^{x}\frac{f(t)\,dt}{(x-t)^{1-\alpha}t^{\alpha+1}},\ 0 \leq a < x \leq b
$$

and

$$
(1.16)\qquad R_{b-}^{\alpha}f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_{x}^{b}\frac{f(t)\,dt}{(t-x)^{1-\alpha}t^{\alpha+1}},\ 0 \leq a < x < b.
$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the *"β-Exponential fractional integrals"*

$$E_{a+,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x \frac{\exp(\beta t) f(t)\, dt}{[\exp(\beta x) - \exp(\beta t)]^{1-\alpha}}, \ a < x \leq b \tag{1.17}$$

and

$$E_{b-,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b \frac{\exp(\beta t) f(t)\, dt}{[\exp(\beta t) - \exp(\beta x)]^{1-\alpha}}, \ a \leq x < b. \tag{1.18}$$

Motivated by the above results, we obtain in this paper some inequalities for the generalized Riemann-Liouville fractional integrals of absolutely continuous functions with bounded derivatives and of convex functions. Applications for mid-point and trapezoid inequalities are provided as well.

2. Some Identities

We have the following representation:

Lemma 1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b).*

(i) *For any $x \in (a,b)$ we have*

$$\begin{aligned} & I_{a+,g}^{\alpha} f(x) + I_{b-,g}^{\alpha} f(x) \\ & = \frac{1}{\Gamma(\alpha+1)} \left[(g(x) - g(a))^{\alpha} f(a) + (g(b) - g(x))^{\alpha} f(b) \right] \\ & + \frac{1}{\Gamma(\alpha+1)} \left[\int_a^x (g(x) - g(t))^{\alpha} f'(t)\, dt - \int_x^b (g(t) - g(x))^{\alpha} f'(t)\, dt \right]. \end{aligned} \tag{2.1}$$

(ii) *For any $x \in (a,b)$ we have*

$$\begin{aligned} & I_{x-,g}^{\alpha} f(a) + I_{x+,g}^{\alpha} f(b) \\ & = \frac{1}{\Gamma(\alpha+1)} \left[(g(x) - g(a))^{\alpha} + (g(b) - g(x))^{\alpha} \right] f(x) \\ & + \frac{1}{\Gamma(\alpha+1)} \left[\int_x^b (g(b) - g(t))^{\alpha} f'(t)\, dt - \int_a^x (g(t) - g(a))^{\alpha} f'(t)\, dt \right]. \end{aligned} \tag{2.2}$$

(iii) *We have the trapezoid equality*

$$\begin{aligned} & \frac{I_{b-,g}^{\alpha} f(a) + I_{a+,g}^{\alpha} f(b)}{2} \\ & = \frac{1}{\Gamma(\alpha+1)} (g(b) - g(a))^{\alpha} \frac{f(b) + f(a)}{2} \\ & + \frac{1}{\Gamma(\alpha+1)} \int_a^b \frac{(g(b) - g(t))^{\alpha} - (g(t) - g(a))^{\alpha}}{2} f'(t)\, dt. \end{aligned} \tag{2.3}$$

Proof. (i) Since $f : [a,b] \to \mathbb{C}$ is an absolutely continuous function on $[a,b]$, then the Lebesgue integrals

$$\int_a^x (g(x) - g(t))^{\alpha} f'(t)\, dt \text{ and } \int_x^b (g(t) - g(x))^{\alpha} f'(t)\, dt$$

exist and integrating by parts, we have

$$\begin{aligned} (2.4)\quad & \frac{1}{\Gamma(\alpha+1)}\int_a^x (g(x)-g(t))^{\alpha} f'(t)\,dt \\ &= \frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1} g'(t) f(t)\,dt - \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha} f(a) \\ &= I_{a+,g}^{\alpha} f(x) - \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha} f(a) \end{aligned}$$

for $a<x\le b$ and

$$\begin{aligned} (2.5)\quad & \frac{1}{\Gamma(\alpha+1)}\int_x^b (g(t)-g(x))^{\alpha} f'(t)\,dt \\ &= \frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha} f(b) - \frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1} g'(t) f(t)\,dt \\ &= \frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha} f(b) - I_{b-,g}^{\alpha} f(x) \end{aligned}$$

for $a\le x<b$.

From (2.4), we then have

$$\begin{aligned} I_{a+,g}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha} f(a) \\ &\quad + \frac{1}{\Gamma(\alpha+1)}\int_a^x (g(x)-g(t))^{\alpha} f'(t)\,dt \end{aligned}$$

for $a<x\le b$ and from (2.5) we have

$$\begin{aligned} I_{b-,g}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha} f(b) \\ &\quad - \frac{1}{\Gamma(\alpha+1)}\int_x^b (g(t)-g(x))^{\alpha} f'(t)\,dt, \end{aligned}$$

for $a\le x<b$, which by addition give (2.1).

(ii) We have

$$I_{x+,g}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha)}\int_x^b (g(b)-g(t))^{\alpha-1} g'(t) f(t)\,dt$$

for $a\le x<b$ and

$$I_{x-,g}^{\alpha} f(a) = \frac{1}{\Gamma(\alpha)}\int_a^x (g(t)-g(a))^{\alpha-1} g'(t) f(t)\,dt$$

for $a<x\le b$.

Since $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function $[a,b]$, then the Lebesgue integrals

$$\int_a^x (g(t)-g(a))^{\alpha} f'(t)\,dt \text{ and } \int_x^b (g(b)-g(t))^{\alpha} f'(t)\,dt$$

exist and integrating by parts, we have

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\int_a^x (g(t)-g(a))^{\alpha} f'(t)\,dt \\
&=\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha} f(x)-\frac{1}{\Gamma(\alpha)}\int_a^x (g(t)-g(a))^{\alpha-1} g'(t) f(t)\,dt \\
&=\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha} f(x)-I_{x-,g}^{\alpha} f(a)
\end{aligned} \tag{2.6}$$

for $a<x\leq b$ and

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\int_x^b (g(b)-g(t))^{\alpha} f'(t)\,dt \\
&=\frac{1}{\Gamma(\alpha)}\int_x^b (g(b)-g(t))^{\alpha-1} g'(t) f(t)\,dt-\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha} f(x) \\
&=I_{x+,g}^{\alpha} f(b)-\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha} f(x)
\end{aligned} \tag{2.7}$$

for $a\leq x<b$.

From (2.6) we have

$$\begin{aligned}
I_{x-,g}^{\alpha} f(a)=&\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha} f(x) \\
&-\frac{1}{\Gamma(\alpha+1)}\int_a^x (g(t)-g(a))^{\alpha} f'(t)\,dt
\end{aligned} \tag{2.8}$$

for $a<x\leq b$ and from (2.7)

$$\begin{aligned}
I_{x+,g}^{\alpha} f(b)=&\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha} f(x) \\
&+\frac{1}{\Gamma(\alpha+1)}\int_x^b (g(b)-g(t))^{\alpha} f'(t)\,dt,
\end{aligned} \tag{2.9}$$

for $a\leq x<b$, which by addition produce (2.2).

(iii) For $x=b$ in (2.8) we have

$$\begin{aligned}
I_{b-,g}^{\alpha} f(a)=&\frac{1}{\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha} f(b) \\
&-\frac{1}{\Gamma(\alpha+1)}\int_a^b (g(t)-g(a))^{\alpha} f'(t)\,dt
\end{aligned}$$

while from (2.9) we have for $x=a$ that

$$\begin{aligned}
I_{a+,g}^{\alpha} f(b)=&\frac{1}{\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha} f(a) \\
&+\frac{1}{\Gamma(\alpha+1)}\int_a^b (g(b)-g(t))^{\alpha} f'(t)\,dt.
\end{aligned}$$

If we add these two equalities and divide by 2, we get (2.3). □

Corollary 2. *With the assumptions of Lemma 1, we have*

$$I_{a+,g}^{\alpha}f\left(\frac{a+b}{2}\right)+I_{b-,g}^{\alpha}f\left(\frac{a+b}{2}\right) \tag{2.10}$$
$$=\frac{1}{\Gamma(\alpha+1)}\left[\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\alpha}f(a)+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\alpha}f(b)\right]$$
$$+\frac{1}{\Gamma(\alpha+1)}\int_{a}^{\frac{a+b}{2}}\left(g\left(\frac{a+b}{2}\right)-g(t)\right)^{\alpha}f'(t)\,dt$$
$$-\frac{1}{\Gamma(\alpha+1)}\int_{\frac{a+b}{2}}^{b}\left(g(t)-g\left(\frac{a+b}{2}\right)\right)^{\alpha}f'(t)\,dt$$

and

$$I_{\frac{a+b}{2}-,g}^{\alpha}f(a)+I_{\frac{a+b}{2}+,g}^{\alpha}f(b) \tag{2.11}$$
$$=\frac{1}{\Gamma(\alpha+1)}\left[\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\alpha}+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\alpha}\right]f\left(\frac{a+b}{2}\right)$$
$$+\frac{1}{\Gamma(\alpha+1)}\int_{\frac{a+b}{2}}^{b}(g(b)-g(t))^{\alpha}f'(t)\,dt$$
$$-\frac{1}{\Gamma(\alpha+1)}\int_{a}^{\frac{a+b}{2}}(g(t)-g(a))^{\alpha}f'(t)\,dt.$$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a,\ b\in I$ by

$$M_g(a,b):=g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

Corollary 3. *With the assumptions of Lemma 1, we have*

$$I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b)) \tag{2.12}$$
$$=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\frac{f(a)+f(b)}{2}$$
$$+\frac{1}{\Gamma(\alpha+1)}\int_{a}^{M_g(a,b)}(g(M_g(a,b))-g(t))^{\alpha}f'(t)\,dt$$
$$-\frac{1}{\Gamma(\alpha+1)}\int_{M_g(a,b)}^{b}(g(t)-g(M_g(a,b)))^{\alpha}f'(t)\,dt$$

and

$$I_{M_g(a,b)-,g}^{\alpha}f(a)+I_{M_g(a,b)+,g}^{\alpha}f(b) \tag{2.13}$$
$$=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}f(M_g(a,b))$$
$$+\frac{1}{\Gamma(\alpha+1)}\int_{M_g(a,b)}^{b}(g(b)-g(t))^{\alpha}f'(t)\,dt$$
$$-\frac{1}{\Gamma(\alpha+1)}\int_{a}^{M_g(a,b)}(g(t)-g(a))^{\alpha}f'(t)\,dt.$$

3. Inequalities for Functions with Bounded Derivatives

We have:

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $x\in(a,b)$ and there exists the real numbers $m_1(x)$, $M_1(x)$, $m_2(x)$, $M_2(x)$ such that the conditions (1.1) and (1.2) hold, then*

$$\frac{1}{\Gamma(\alpha+1)}\left[m_2(x)\int_x^b(g(t)-g(x))^{\alpha}dt-M_1(x)\int_a^x(g(x)-g(t))^{\alpha}dt\right] \tag{3.1}$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}f(a)+(g(b)-g(x))^{\alpha}f(b)\right]$$
$$-I_{a+,g}^{\alpha}f(x)-I_{b-,g}^{\alpha}f(x)$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\left[M_2(x)\int_x^b(g(t)-g(x))^{\alpha}dt-m_1(x)\int_a^x(g(x)-g(t))^{\alpha}dt\right]$$

and

$$\frac{1}{\Gamma(\alpha+1)}\left[m_2(x)\int_x^b(g(b)-g(t))^{\alpha}dt-M_1(x)\int_a^x(g(t)-g(a))^{\alpha}dt\right] \tag{3.2}$$
$$\leq I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)$$
$$-\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right]f(x)$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\left[M_2(x)\int_x^b(g(b)-g(t))^{\alpha}dt-m_1(x)\int_a^x(g(t)-g(a))^{\alpha}dt\right].$$

Proof. We have from (2.1) that

$$\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}f(a)+(g(b)-g(x))^{\alpha}f(b)\right] \tag{3.3}$$
$$-I_{a+,g}^{\alpha}f(x)-I_{b-,g}^{\alpha}f(x)$$
$$=\frac{1}{\Gamma(\alpha+1)}\left[\int_x^b(g(t)-g(x))^{\alpha}f'(t)\,dt-\int_a^x(g(x)-g(t))^{\alpha}f'(t)\,dt\right]$$

for any $x\in(a,b)$.

Using the conditions (1.1) and (1.2) we have

$$\begin{aligned} m_2(x)\int_x^b (g(t)-g(x))^\alpha dt &\le \int_x^b (g(t)-g(x))^\alpha f'(t)\,dt \\ &\le M_2(x)\int_x^b (g(t)-g(x))^\alpha dt \end{aligned}$$

and

$$\begin{aligned} m_1(x)\int_a^x (g(x)-g(t))^\alpha dt &\le \int_a^x (g(x)-g(t))^\alpha f'(t)\,dt \\ &\le M_1(x)\int_a^x (g(x)-g(t))^\alpha dt. \end{aligned}$$

These imply that

$$\begin{aligned} & m_2(x)\int_x^b (g(t)-g(x))^\alpha dt - M_1(x)\int_a^x (g(x)-g(t))^\alpha dt \\ & \le \int_x^b (t-x)^\alpha f'(t)\,dt - \int_a^x (x-t)^\alpha f'(t)\,dt \\ & \le M_2(x)\int_x^b (g(t)-g(x))^\alpha dt - m_1(x)\int_a^x (g(x)-g(t))^\alpha dt \end{aligned}$$

that is equivalent to

$$\begin{aligned} & \frac{1}{\Gamma(\alpha+1)}\left[m_2(x)\int_x^b (g(t)-g(x))^\alpha dt - M_1(x)\int_a^x (g(x)-g(t))^\alpha dt\right] \\ & \le \frac{1}{\Gamma(\alpha+1)}\left[\int_x^b (t-x)^\alpha f'(t)\,dt - \int_a^x (x-t)^\alpha f'(t)\,dt\right] \\ & \le \frac{1}{\Gamma(\alpha+1)}\left[M_2(x)\int_x^b (g(t)-g(x))^\alpha dt - m_1(x)\int_a^x (g(x)-g(t))^\alpha dt\right]. \end{aligned}$$

By using the equality (3.3) we get (3.1).

From (2.2) we have

$$\begin{aligned} & I^\alpha_{x-,g} f(a) + I^\alpha_{x+,g} f(b) \\ & - \frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^\alpha + (g(b)-g(x))^\alpha\right] f(x) \\ & = \frac{1}{\Gamma(\alpha+1)}\left[\int_x^b (g(b)-g(t))^\alpha f'(t)\,dt - \int_a^x (g(t)-g(a))^\alpha f'(t)\,dt\right]. \end{aligned} \tag{3.4}$$

In a similar way, we have

$$\begin{aligned} m_2(x)\int_x^b (g(b)-g(t))^\alpha dt &\le \int_x^b (g(b)-g(t))^\alpha f'(t)\,dt \\ &\le M_2(x)\int_x^b (g(b)-g(t))^\alpha dt \end{aligned}$$

and

$$m_1(x)\int_a^x (g(t)-g(a))^\alpha dt \le \int_a^x (g(t)-g(a))^\alpha f'(t)\,dt \le M_1(x)\int_a^x (g(t)-g(a))^\alpha dt,$$

which implies that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+1)}\left[m_2(x)\int_x^b (g(b)-g(t))^\alpha dt - M_1(x)\int_a^x (g(t)-g(a))^\alpha dt\right]\\ &\le \frac{1}{\Gamma(\alpha+1)}\left[\int_x^b (g(b)-g(t))^\alpha f'(t)\,dt - \int_a^x (g(t)-g(a))^\alpha f'(t)\,dt\right]\\ &\le \frac{1}{\Gamma(\alpha+1)}\left[M_2(x)\int_x^b (g(b)-g(t))^\alpha dt - m_1(x)\int_a^x (g(t)-g(a))^\alpha dt\right]\end{aligned}$$

and by (3.4) we get (3.2). □

Corollary 4. *With the assumptions of Theorem 3 and if there exist the real numbers* $m_1(M_g(a,b))$, $M_1(M_g(a,b))$, $m_2(M_g(a,b))$, $M_2(M_g(a,b))$ *such that*

$$m_1(M_g(a,b)) \le f'(t) \le M_1(M_g(a,b)) \text{ for a.e. } t\in(a, M_g(a,b)) \tag{3.5}$$

and

$$m_2(M_g(a,b)) \le f'(t) \le M_2(M_g(a,b)) \text{ for a.e. } t\in(M_g(a,b), b) \tag{3.6}$$

then

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+1)}\left[m_2(M_g(a,b))\int_{M_g(a,b)}^b (g(t)-g(M_g(a,b)))^\alpha dt\right.\\ &\left.-M_1(M_g(a,b))\int_a^{M_g(a,b)} (g(M_g(a,b))-g(t))^\alpha dt\right]\\ &\le \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(g(b)-g(a))^\alpha \frac{f(a)+f(b)}{2}\\ &- I^\alpha_{a+,g} f(M_g(a,b)) - I^\alpha_{b-,g} f(M_g(a,b))\\ &\le \frac{1}{\Gamma(\alpha+1)}\left[M_2(M_g(a,b))\int_{M_g(a,b)}^b (g(t)-g(M_g(a,b)))^\alpha dt\right.\\ &\left.-m_1(M_g(a,b))\int_a^{M_g(a,b)} (g(M_g(a,b))-g(t))^\alpha dt\right]\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}(3.8)\quad &\frac{1}{\Gamma(\alpha+1)}\left[m_2\left(M_g(a,b)\right)\int_{M_g(a,b)}^{b}\left(g(b)-g(t)\right)^{\alpha}dt\right.\\ &\left.-M_1\left(M_g(a,b)\right)\int_{a}^{M_g(a,b)}\left(g(t)-g(a)\right)^{\alpha}dt\right]\\ &\leq I_{M_g(a,b)-,g}^{\alpha}f(a)+I_{M_g(a,b)+,g}^{\alpha}f(b)\\ &-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left(g(b)-g(a)\right)^{\alpha}f\left(M_g(a,b)\right)\\ &\leq\frac{1}{\Gamma(\alpha+1)}\left[M_2\left(M_g(a,b)\right)\int_{M_g(a,b)}^{b}\left(g(b)-g(t)\right)^{\alpha}dt\right.\\ &\left.-m_1\left(M_g(a,b)\right)\int_{a}^{M_g(a,b)}\left(g(t)-g(a)\right)^{\alpha}dt\right].\end{aligned}$$

The case of convex functions is of interest:

Corollary 5. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $x\in(a,b)$, then*

$$\begin{aligned}(3.9)\quad &\frac{1}{\Gamma(\alpha+1)}\left[f'_+(x)\int_x^b\left(g(t)-g(x)\right)^{\alpha}dt-f'_-(x)\int_a^x\left(g(x)-g(t)\right)^{\alpha}dt\right]\\ &\leq\frac{1}{\Gamma(\alpha+1)}\left[\left(g(x)-g(a)\right)^{\alpha}f(a)+\left(g(b)-g(x)\right)^{\alpha}f(b)\right]\\ &-I_{a+,g}^{\alpha}f(x)-I_{b-,g}^{\alpha}f(x)\\ &\leq\frac{1}{\Gamma(\alpha+1)}\left[f'_-(b)\int_x^b\left(g(t)-g(x)\right)^{\alpha}dt-f'_+(a)\int_a^x\left(g(x)-g(t)\right)^{\alpha}dt\right]\end{aligned}$$

and

$$\begin{aligned}(3.10)\quad &\frac{1}{\Gamma(\alpha+1)}\left[f'_+(x)\int_x^b\left(g(b)-g(t)\right)^{\alpha}dt-f'_-(x)\int_a^x\left(g(t)-g(a)\right)^{\alpha}dt\right]\\ &\leq I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)\\ &-\frac{1}{\Gamma(\alpha+1)}\left[\left(g(x)-g(a)\right)^{\alpha}+\left(g(b)-g(x)\right)^{\alpha}\right]f(x)\\ &\leq\frac{1}{\Gamma(\alpha+1)}\left[f'_-(b)\int_x^b\left(g(b)-g(t)\right)^{\alpha}dt-f'_+(a)\int_a^x\left(g(t)-g(a)\right)^{\alpha}dt\right].\end{aligned}$$

Proof. Since f is convex, then the derivative f' exists almost everywhere on $[a,b]$ and

$$f'_+(a)\leq f'(t)\leq f'_-(x)\ \text{ for a.e. }\ t\in(a,x)$$

and

$$f'_+(x)\leq f'(t)\leq f'_-(b)\ \text{ for a.e. }\ t\in(x,b).$$

Now, writing the inequalities (3.1) and (3.2) for $m_1(x)=f'_+(a)$, $M_1(x)=f'_-(x)$, $m_2(x)=f'_+(x)$ and $M_2(x)=f'_-(b)$ we get the desired results (3.9) and (3.10). □

Corollary 6. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and assume that there exists the constants $m<M$ such that*

$$m\leq f'(t)\leq M \text{ for a.e. } t\in[a,b] \tag{3.11}$$

Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then we have the inequalities

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+1)}\left[m\int_x^b (g(t)-g(x))^\alpha dt - M\int_a^x (g(x)-g(t))^\alpha dt\right]\\ &\leq \frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^\alpha f(a)+(g(b)-g(x))^\alpha f(b)\right]\\ &-I^\alpha_{a+,g}f(x)-I^\alpha_{b-,g}f(x)\\ &\leq \frac{1}{\Gamma(\alpha+1)}\left[M\int_x^b (g(t)-g(x))^\alpha dt - m\int_a^x (g(x)-g(t))^\alpha dt\right]\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+1)}\left[m\int_x^b (g(b)-g(t))^\alpha dt - M\int_a^x (g(t)-g(a))^\alpha dt\right]\\ &\leq I^\alpha_{x-,g}f(a)+I^\alpha_{x+,g}f(b)\\ &-\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^\alpha+(g(b)-g(x))^\alpha\right]f(x)\\ &\leq \frac{1}{\Gamma(\alpha+1)}\left[M\int_x^b (g(b)-g(t))^\alpha dt - m\int_a^x (g(t)-g(a))^\alpha dt\right]\end{aligned} \tag{3.13}$$

for $x\in(a,b)$.

Equivalently, we have the inequalities

$$\begin{aligned}&\left|\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^\alpha f(a)+(g(b)-g(x))^\alpha f(b)\right]\right.\\ &-I^\alpha_{a+,g}f(x)-I^\alpha_{b-,g}f(x)\\ &\left.-\frac{1}{2}(M+m)\left[\int_x^b (g(t)-g(x))^\alpha dt-\int_a^x (g(x)-g(t))^\alpha dt\right]\right|\\ &\leq \frac{1}{2\Gamma(\alpha+1)}(M-m)\int_a^b |g(t)-g(x)|^\alpha dt\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}&\left|I^\alpha_{x-,g}f(a)+I^\alpha_{x+,g}f(b)\right.\\ &-\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^\alpha+(g(b)-g(x))^\alpha\right]f(x)\\ &\left.-\frac{1}{2}(M+m)\left[\int_x^b (g(b)-g(t))^\alpha dt-\int_a^x (g(t)-g(a))^\alpha dt\right]\right|\\ &\leq \frac{1}{2\Gamma(\alpha+1)}(M-m)\left[\int_x^b (g(b)-g(t))^\alpha dt+\int_a^x (g(t)-g(a))^\alpha dt\right]\end{aligned} \tag{3.15}$$

for $x\in(a,b)$.

Since g is a strictly increasing function on (a,b), then by the elementary Hölder's inequality we have

$$\begin{aligned}
\int_a^b |g(t)-g(x)|^\alpha dt &= \int_a^x (g(x)-g(t))^\alpha dt + \int_x^b (g(t)-g(x))^\alpha dt \\
&\le \int_a^x (g(x)-g(a))^\alpha dt + \int_x^b (g(b)-g(x))^\alpha dt \\
&= (x-a)(g(x)-g(a))^\alpha + (b-x)(g(b)-g(x))^\alpha \\
&\le \begin{cases} \max\{x-a,b-x\}\left[(g(b)-g(x))^\alpha + (g(x)-g(a))^\alpha\right]; \\ \left[(x-a)^p+(b-x)^p\right]^{1/p}\left[(g(x)-g(a))^{q\alpha}+(g(b)-g(x))^{q\alpha}\right]^{1/q} \\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1; \\ (b-a)\left[\max\{g(x)-g(a),g(b)-g(x)\}\right]^\alpha \end{cases} \\
&= \begin{cases} \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[(g(b)-g(x))^\alpha + (g(x)-g(a))^\alpha\right]; \\ \left[(x-a)^p+(b-x)^p\right]^{1/p}\left[(g(x)-g(a))^{q\alpha}+(g(b)-g(x))^{q\alpha}\right]^{1/q} \\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1; \\ (b-a)\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^\alpha \end{cases}
\end{aligned}$$

and by (3.14) we get the chain of inequalities:

$$\begin{aligned}
(3.16)\quad &\left|\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^\alpha f(a)+(g(b)-g(x))^\alpha f(b)\right]\right. \\
&- I_{a+,g}^\alpha f(x) - I_{b-,g}^\alpha f(x) \\
&\left.-\frac{1}{2}(M+m)\left[\int_x^b (g(t)-g(x))^\alpha dt - \int_a^x (g(x)-g(t))^\alpha dt\right]\right| \\
&\le \frac{1}{2\Gamma(\alpha+1)}(M-m)\int_a^b |g(t)-g(x)|^\alpha dt \\
&\le \frac{1}{2\Gamma(\alpha+1)}(M-m)\left[(x-a)(g(x)-g(a))^\alpha + (b-x)(g(b)-g(x))^\alpha\right] \\
&\le \frac{1}{2\Gamma(\alpha+1)}(M-m) \\
&\times \begin{cases} \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[(g(b)-g(x))^\alpha + (g(x)-g(a))^\alpha\right]; \\ \left[(x-a)^p+(b-x)^p\right]^{1/p}\left[(g(x)-g(a))^{q\alpha}+(g(b)-g(x))^{q\alpha}\right]^{1/q} \\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1; \\ (b-a)\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^\alpha, \end{cases}
\end{aligned}$$

for any $x \in (a,b)$.

We also have

$$\begin{aligned}
&\int_a^x (g(t)-g(a))^\alpha dt + \int_x^b (g(b)-g(t))^\alpha dt \\
&\le \int_a^x (g(x)-g(a))^\alpha dt + \int_x^b (g(b)-g(x))^\alpha dt \\
&= (x-a)(g(x)-g(a))^\alpha + (b-x)(g(b)-g(x))^\alpha .
\end{aligned}$$

Therefore, by (3.15) we have the chain of inequalities

$$\begin{aligned}(3.17)\quad & \Big|I^{\alpha}_{x-,g}f(a)+I^{\alpha}_{x+,g}f(b) \\ & -\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right]f(x) \\ & -\frac{1}{2}(M+m)\left[\int_x^b (g(b)-g(t))^{\alpha}dt-\int_a^x (g(t)-g(a))^{\alpha}dt\right]\Big| \\ & \leq \frac{1}{2\Gamma(\alpha+1)}(M-m)\left[\int_x^b (g(b)-g(t))^{\alpha}dt+\int_a^x (g(t)-g(a))^{\alpha}dt\right] \\ & \leq \frac{1}{2\Gamma(\alpha+1)}(M-m)\left[(x-a)(g(x)-g(a))^{\alpha}+(b-x)(g(b)-g(x))^{\alpha}\right] \\ & \leq \frac{1}{2\Gamma(\alpha+1)}(M-m) \\ & \times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[(g(b)-g(x))^{\alpha}+(g(x)-g(a))^{\alpha}\right]; \\ \left[(x-a)^p+(b-x)^p\right]^{1/p}\left[(g(x)-g(a))^{q\alpha}+(g(b)-g(x))^{q\alpha}\right]^{1/q} \\ \text{where } p,q>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1; \\ (b-a)\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\end{cases}\end{aligned}$$

for any $x\in(a,b)$.

Remark 1. *If we take $x=M_g(a,b)$ in (3.14) and (3.15), then we get the simpler inequalities*

$$\begin{aligned}(3.18)\quad & \Big|\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\frac{f(a)+f(b)}{2} \\ & -I^{\alpha}_{a+,g}f(M_g(a,b))-I^{\alpha}_{b-,g}f(M_g(a,b))-\frac{1}{2}(M+m) \\ & \times\left[\int_{M_g(a,b)}^b (g(t)-g(M_g(a,b)))^{\alpha}dt-\int_a^{M_g(a,b)} (g(M_g(a,b))-g(t))^{\alpha}dt\right]\Big| \\ & \leq \frac{1}{2\Gamma(\alpha+1)}(M-m)\int_a^b |g(t)-g(M_g(a,b))|^{\alpha}dt\end{aligned}$$

and

$$\begin{aligned}(3.19)\quad & \Big|I^{\alpha}_{M_g(a,b)-,g}f(a)+I^{\alpha}_{M_g(a,b)+,g}f(b) \\ & -\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}f(M_g(a,b)) \\ & -\frac{1}{2}(M+m)\left[\int_{M_g(a,b)}^b (g(b)-g(t))^{\alpha}dt-\int_a^{M_g(a,b)} (g(t)-g(a))^{\alpha}dt\right]\Big| \\ & \leq \frac{1}{2\Gamma(\alpha+1)}(M-m) \\ & \times\left[\int_{M_g(a,b)}^b (g(b)-g(t))^{\alpha}dt+\int_a^{M_g(a,b)} (g(t)-g(a))^{\alpha}dt\right].\end{aligned}$$

We also observe that, if we assume that the function $f:[a,b]\to\mathbb{R}$ is convex then we can take in the inequalities (3.14)-(3.19) $m=f'_{+}(a)$ and $M=f'_{-}(b)$ provided these quantities are finite. The details are omitted.

The following trapezoid type inequality also holds:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$ and assume that there exists the constants $m<M$ such that the condition (3.11) is satisfied. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then we have the inequalities*

$$\begin{aligned}&\left|\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\frac{f(b)+f(a)}{2}\right.\\&\left.-\frac{1}{2\Gamma(\alpha+1)}\frac{m+M}{2}\left[\int_a^b(g(b)-g(t))^{\alpha}dt-\int_a^b(g(t)-g(a))^{\alpha}dt\right]\right|\\&\leq\frac{1}{2\Gamma(\alpha+1)}\frac{M-m}{2}\int_a^b\left|(g(b)-g(t))^{\alpha}-(g(t)-g(a))^{\alpha}\right|dt.\end{aligned}\tag{3.20}$$

Proof. Observe that, by (2.3) we have

$$\begin{aligned}&\frac{1}{\Gamma(\alpha+1)}\int_a^b\frac{(g(b)-g(t))^{\alpha}-(g(t)-g(a))^{\alpha}}{2}\left(f'(t)-\frac{m+M}{2}\right)dt\\&=\frac{1}{\Gamma(\alpha+1)}\int_a^b\frac{(g(b)-g(t))^{\alpha}-(g(t)-g(a))^{\alpha}}{2}f'(t)\,dt\\&-\frac{m+M}{2}\frac{1}{\Gamma(\alpha+1)}\int_a^b\frac{(g(b)-g(t))^{\alpha}-(g(t)-g(a))^{\alpha}}{2}dt\\&=\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\frac{f(b)+f(a)}{2}\\&-\frac{m+M}{2}\frac{1}{\Gamma(\alpha+1)}\int_a^b\frac{(g(b)-g(t))^{\alpha}-(g(t)-g(a))^{\alpha}}{2}dt.\end{aligned}\tag{3.21}$$

If we take the modulus in this equality, we get

$$\begin{aligned}&\left|\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\frac{f(b)+f(a)}{2}\right.\\&\left.-\frac{m+M}{2}\frac{1}{\Gamma(\alpha+1)}\int_a^b\frac{(g(b)-g(t))^{\alpha}-(g(t)-g(a))^{\alpha}}{2}dt\right|\\&\leq\frac{1}{\Gamma(\alpha+1)}\int_a^b\left|\frac{(g(b)-g(t))^{\alpha}-(g(t)-g(a))^{\alpha}}{2}\right|\left|f'(t)-\frac{m+M}{2}\right|dt\\&\leq\frac{1}{\Gamma(\alpha+1)}\frac{M-m}{2}\int_a^b\left|\frac{(g(b)-g(t))^{\alpha}-(g(t)-g(a))^{\alpha}}{2}\right|dt,\end{aligned}$$

which proves (3.20). □

Remark 2. *If we take $g(t)=t$ into above inequalities we recapture some of the results for the traditional Riemann-Liouville integrals stated in the introduction.*

4. More Particular Inequalities

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If we assume more properties for this function, then we can get further simpler bounds. For instance, assume that $g:[a,b]\to\mathbb{R}$ is *r-H-Hölder continuous* on $[a,b]$ with $r\in(0,1]$ and $K>0$ namely

$$|f(t)-f(s)|\le K|t-s|^{r} \tag{4.1}$$

for any $t,\ s\in[a,b]$. If $r=1$ and $K=L$ we call the function *L-Lipschitzian* on $[a,b]$.

If $g:[a,b]\to\mathbb{R}$ is *r-H-Hölder continuous* on $[a,b]$ with $r\in(0,1]$ and $K>0$, then

$$\begin{aligned}\int_a^b|g(t)-g(x)|^{\alpha}dt&\le K\int_a^b|t-x|^{\alpha r}dt=K\left[\int_a^x(x-t)^{\alpha r}dt+\int_x^b(t-x)^{\alpha r}dt\right]\\&=K\left[\frac{(x-a)^{\alpha r+1}+(b-x)^{\alpha r+1}}{\alpha r+1}\right]\end{aligned}$$

for $x\in[a,b]$.

From (3.14) we then get the simpler inequality

$$\begin{aligned}&\Bigg|\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}f(a)+(g(b)-g(x))^{\alpha}f(b)\right]\\&-I_{a+,g}^{\alpha}f(x)-I_{b-,g}^{\alpha}f(x)\\&-\frac{1}{2}(M+m)\left[\int_x^b(g(t)-g(x))^{\alpha}dt-\int_a^x(g(x)-g(t))^{\alpha}dt\right]\Bigg|\\&\le\frac{1}{2(\alpha r+1)\Gamma(\alpha+1)}(M-m)K\left[(x-a)^{\alpha r+1}+(b-x)^{\alpha r+1}\right],\end{aligned} \tag{4.2}$$

for $x\in(a,b)$, provided $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function on $[a,b]$ and that there exists the constants $m<M$ such that the condition (3.11) is satisfied.

If g is L-Lipschitzian on $[a,b]$, then by (4.2) we have

$$\begin{aligned}&\Bigg|\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}f(a)+(g(b)-g(x))^{\alpha}f(b)\right]\\&-I_{a+,g}^{\alpha}f(x)-I_{b-,g}^{\alpha}f(x)\\&-\frac{1}{2}(M+m)\left[\int_x^b(g(t)-g(x))^{\alpha}dt-\int_a^x(g(x)-g(t))^{\alpha}dt\right]\Bigg|\\&\le\frac{1}{2\Gamma(\alpha+2)}(M-m)L\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right],\end{aligned} \tag{4.3}$$

for $x\in(a,b)$.

Now, if we take $x = \frac{a+b}{2}$ in (4.3), then we get the mid-point inequality:

$$\begin{aligned}(4.4)\quad & \left|\frac{1}{\Gamma(\alpha+1)}\left[\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\alpha} f(a)+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\alpha} f(b)\right]\right. \\ & -I_{a+,g}^{\alpha} f\left(\frac{a+b}{2}\right)-I_{b-,g}^{\alpha} f\left(\frac{a+b}{2}\right)-\frac{1}{2}(M+m) \\ & \left.\times\left[\int_{\frac{a+b}{2}}^{b}\left(g(t)-g\left(\frac{a+b}{2}\right)\right)^{\alpha} dt-\int_{a}^{\frac{a+b}{2}}\left(g\left(\frac{a+b}{2}\right)-g(t)\right)^{\alpha} dt\right]\right| \\ & \leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}(M-m) L(b-a)^{\alpha+1}.\end{aligned}$$

If $g:[a,b]\to\mathbb{R}$ is *r-H-Hölder continuous* on $[a,b]$ with $r\in(0,1]$ and $K>0$, then by (3.15) we have

$$\begin{aligned}\int_{x}^{b}(g(b)-g(t))^{\alpha} dt+\int_{a}^{x}(g(t)-g(a))^{\alpha} dt & \leq H\left[\int_{x}^{b}(b-t)^{r\alpha} dt+\int_{a}^{x}(x-a)^{r\alpha} dt\right] \\ & =K\left[\frac{(x-a)^{\alpha r+1}+(b-x)^{\alpha r+1}}{\alpha r+1}\right]\end{aligned}$$

From (3.15) we get the simpler inequality

$$\begin{aligned}(4.5)\quad & \left|I_{x-,g}^{\alpha} f(a)+I_{x+,g}^{\alpha} f(b)\right. \\ & -\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right] f(x) \\ & \left.-\frac{1}{2}(M+m)\left[\int_{x}^{b}(g(b)-g(t))^{\alpha} dt-\int_{a}^{x}(g(t)-g(a))^{\alpha} dt\right]\right| \\ & \leq \frac{1}{2(\alpha r+1)\Gamma(\alpha+1)}(M-m) K\left[(x-a)^{\alpha r+1}+(b-x)^{\alpha r+1}\right]\end{aligned}$$

for $x\in(a,b)$, provided $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function on $[a,b]$ and that there exists the constants $m<M$ such that the condition (3.11) is satisfied.

If g is L-Lipschitzian on $[a,b]$, then by (4.5) we have

$$\begin{aligned}(4.6)\quad & \left|I_{x-,g}^{\alpha} f(a)+I_{x+,g}^{\alpha} f(b)\right. \\ & -\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right] f(x) \\ & \left.-\frac{1}{2}(M+m)\left[\int_{x}^{b}(g(b)-g(t))^{\alpha} dt-\int_{a}^{x}(g(t)-g(a))^{\alpha} dt\right]\right| \\ & \leq \frac{1}{2\Gamma(\alpha+2)}(M-m) L\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right],\end{aligned}$$

for $x\in(a,b)$.

If we take $x = \frac{a+b}{2}$ in (4.6), then we get

$$\begin{aligned}
&\Bigg| I^{\alpha}_{\frac{a+b}{2}-,g} f(a) + I^{\alpha}_{\frac{a+b}{2}+,g} f(b) \\
&- \frac{1}{\Gamma(\alpha+1)} \left[\left(g\left(\frac{a+b}{2}\right) - g(a) \right)^{\alpha} + \left(g(b) - g\left(\frac{a+b}{2}\right) \right)^{\alpha} \right] f\left(\frac{a+b}{2}\right) \\
&- \frac{1}{2}(M+m) \left[\int_{\frac{a+b}{2}}^{b} (g(b)-g(t))^{\alpha} dt - \int_{a}^{\frac{a+b}{2}} (g(t)-g(a))^{\alpha} dt \right] \Bigg| \\
&\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)} (M-m) L (b-a)^{\alpha+1}.
\end{aligned} \tag{4.7}$$

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