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OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF ABSOLUTELY CONTINUOUS FUNCTIONS IN TERMS OF p-NORMS OF DERIVATIVE

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Abstract. In this paper, some inequalities for the generalized Riemann-Liouville fractional integrals of absolutely continuous functions in terms of the Lebesgue p-norms of the derivative are obtained. Applications for mid-point and trapezoid inequalities are provided as well. Some examples for the Hadamard fractional integrals are also given.

1. Introduction

Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)\,dt$$

for $a<x\leq b$ and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)\,dt$$

for $a\leq x<b$, where Γ is the *Gamma function*. For $\alpha=0$, they are defined as

$$J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)\text{ for }x\in(a,b).$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[5], [18]-[29] and the references therein.

In order to extend the above results for generalized Riemann-Liouville fractional integrals, we need the following preparations.

Let (a,b) with $-\infty\leq a<b\leq\infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\operatorname{Re}(\alpha)>0$. Also, let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Following [20, p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ by

$$I_{a+,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{g'(t)f(t)\,dt}{[g(x)-g(t)]^{1-\alpha}},\ a<x\leq b \tag{1.1}$$

and

$$I_{b-,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\frac{g'(t)f(t)\,dt}{[g(t)-g(x)]^{1-\alpha}},\ a\leq x<b. \tag{1.2}$$

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For $g(t) = t$ we have the classical *Riemann-Liouville fractional integrals* introduced above while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [20, p. 111]

$$H_{a+}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t)\,dt}{t},\ 0 \le a < x \le b \tag{1.3}$$

and

$$H_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \frac{f(t)\,dt}{t},\ 0 \le a < x < b. \tag{1.4}$$

One can consider the function $g(t) = -t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha}f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}\,t^{\alpha+1}},\ 0 \le a < x \le b \tag{1.5}$$

and

$$R_{b-}^{\alpha}f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha}\,t^{\alpha+1}},\ 0 \le a < x < b. \tag{1.6}$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the *"β-Exponential fractional integrals"*

$$E_{a+,\beta}^{\alpha}f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x \frac{\exp(\beta t)\, f(t)\,dt}{[\exp(\beta x) - \exp(\beta t)]^{1-\alpha}},\ a < x \le b \tag{1.7}$$

and

$$E_{b-,\beta}^{\alpha}f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b \frac{\exp(\beta t)\, f(t)\,dt}{[\exp(\beta t) - \exp(\beta x)]^{1-\alpha}},\ a \le x < b. \tag{1.8}$$

Motivated by the above results, we obtain in this paper some inequalities for the generalized Riemann-Liouville fractional integrals of absolutely continuous functions in terms of the Lebesgue p-norms of the derivative. Applications for mid-point and trapezoid inequalities are also provided. Some examples for the Hadamard fractional integrals are given as well.

2. Some Preliminary Facts

We have the following representation, see also [16]:

Lemma 1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b).*

(i) *For any $x \in (a,b)$ we have*

$$\begin{aligned} &I_{a+,g}^{\alpha}f(x) + I_{b-,g}^{\alpha}f(x) \\ &= \frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha} f(a) + (g(b)-g(x))^{\alpha} f(b)\right] \\ &+ \frac{1}{\Gamma(\alpha+1)}\left[\int_a^x (g(x)-g(t))^{\alpha} f'(t)\,dt - \int_x^b (g(t)-g(x))^{\alpha} f'(t)\,dt\right]. \end{aligned} \tag{2.1}$$

(ii) *For any* $x \in (a,b)$ *we have*

$$
\begin{aligned}
(2.2)\qquad & I_{x-,g}^{\alpha} f(a) + I_{x+,g}^{\alpha} f(b) \\
&= \frac{1}{\Gamma(\alpha+1)} \left[(g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right] f(x) \\
&+ \frac{1}{\Gamma(\alpha+1)} \left[\int_x^b (g(b)-g(t))^{\alpha} f'(t)\,dt - \int_a^x (g(t)-g(a))^{\alpha} f'(t)\,dt \right].
\end{aligned}
$$

Proof. (i) Since $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function on $[a,b]$, then the Lebesgue integrals

$$
\int_a^x (g(x)-g(t))^{\alpha} f'(t)\,dt \text{ and } \int_x^b (g(t)-g(x))^{\alpha} f'(t)\,dt
$$

exist and integrating by parts, we have

$$
\begin{aligned}
(2.3)\qquad & \frac{1}{\Gamma(\alpha+1)} \int_a^x (g(x)-g(t))^{\alpha} f'(t)\,dt \\
&= \frac{1}{\Gamma(\alpha)} \int_a^x (g(x)-g(t))^{\alpha-1} g'(t) f(t)\,dt - \frac{1}{\Gamma(\alpha+1)} (g(x)-g(a))^{\alpha} f(a) \\
&= I_{a+,g}^{\alpha} f(x) - \frac{1}{\Gamma(\alpha+1)} (g(x)-g(a))^{\alpha} f(a)
\end{aligned}
$$

for $a < x \le b$ and

$$
\begin{aligned}
(2.4)\qquad & \frac{1}{\Gamma(\alpha+1)} \int_x^b (g(t)-g(x))^{\alpha} f'(t)\,dt \\
&= \frac{1}{\Gamma(\alpha+1)} (g(b)-g(x))^{\alpha} f(b) - \frac{1}{\Gamma(\alpha)} \int_x^b (g(t)-g(x))^{\alpha-1} g'(t) f(t)\,dt \\
&= \frac{1}{\Gamma(\alpha+1)} (g(b)-g(x))^{\alpha} f(b) - I_{b-,g}^{\alpha} f(x)
\end{aligned}
$$

for $a \le x < b$.

From (2.3), we then have

$$
\begin{aligned}
I_{a+,g}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha+1)} (g(x)-g(a))^{\alpha} f(a) \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_a^x (g(x)-g(t))^{\alpha} f'(t)\,dt
\end{aligned}
$$

for $a < x \le b$ and from (2.4) we have

$$
\begin{aligned}
I_{b-,g}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha+1)} (g(b)-g(x))^{\alpha} f(b) \\
&- \frac{1}{\Gamma(\alpha+1)} \int_x^b (g(t)-g(x))^{\alpha} f'(t)\,dt,
\end{aligned}
$$

for $a \le x < b$, which by addition give (2.1).

(ii) We have

$$
I_{x+,g}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(b)-g(t))^{\alpha-1} g'(t) f(t)\,dt
$$

for $a \leq x < b$ and

$$I_{x-,g}^{\alpha} f(a) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(t) - g(a))^{\alpha-1} g'(t) f(t) \, dt$$

for $a < x \leq b$.

Since $f : [a,b] \to \mathbb{C}$ is an absolutely continuous function $[a,b]$, then the Lebesgue integrals

$$\int_a^x (g(t) - g(a))^{\alpha} f'(t) \, dt \text{ and } \int_x^b (g(b) - g(t))^{\alpha} f'(t) \, dt$$

exist and integrating by parts, we have

$$\begin{aligned} (2.5) \quad & \frac{1}{\Gamma(\alpha+1)} \int_a^x (g(t) - g(a))^{\alpha} f'(t) \, dt \\ &= \frac{1}{\Gamma(\alpha+1)} (g(x) - g(a))^{\alpha} f(x) - \frac{1}{\Gamma(\alpha)} \int_a^x (g(t) - g(a))^{\alpha-1} g'(t) f(t) \, dt \\ &= \frac{1}{\Gamma(\alpha+1)} (g(x) - g(a))^{\alpha} f(x) - I_{x-,g}^{\alpha} f(a) \end{aligned}$$

for $a < x \leq b$ and

$$\begin{aligned} (2.6) \quad & \frac{1}{\Gamma(\alpha+1)} \int_x^b (g(b) - g(t))^{\alpha} f'(t) \, dt \\ &= \frac{1}{\Gamma(\alpha)} \int_x^b (g(b) - g(t))^{\alpha-1} g'(t) f(t) \, dt - \frac{1}{\Gamma(\alpha+1)} (g(b) - g(x))^{\alpha} f(x) \\ &= I_{x+,g}^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)} (g(b) - g(x))^{\alpha} f(x) \end{aligned}$$

for $a \leq x < b$.

From (2.5) we have

$$\begin{aligned} (2.7) \qquad I_{x-,g}^{\alpha} f(a) &= \frac{1}{\Gamma(\alpha+1)} (g(x) - g(a))^{\alpha} f(x) \\ &\quad - \frac{1}{\Gamma(\alpha+1)} \int_a^x (g(t) - g(a))^{\alpha} f'(t) \, dt \end{aligned}$$

for $a < x \leq b$ and from (2.6)

$$\begin{aligned} (2.8) \qquad I_{x+,g}^{\alpha} f(b) &= \frac{1}{\Gamma(\alpha+1)} (g(b) - g(x))^{\alpha} f(x) \\ &\quad + \frac{1}{\Gamma(\alpha+1)} \int_x^b (g(b) - g(t))^{\alpha} f'(t) \, dt, \end{aligned}$$

for $a \leq x < b$, which by addition produce (2.2). □

Corollary 1. *With the assumptions of Lemma 1, we have*

$$
\begin{aligned}
(2.9)\quad & I^{\alpha}_{a+,g} f\left(\frac{a+b}{2}\right) + I^{\alpha}_{b-,g} f\left(\frac{a+b}{2}\right) \\
&= \frac{1}{\Gamma(\alpha+1)}\left[\left(g\left(\frac{a+b}{2}\right) - g(a)\right)^{\alpha} f(a) + \left(g(b) - g\left(\frac{a+b}{2}\right)\right)^{\alpha} f(b)\right] \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_a^{\frac{a+b}{2}} \left(g\left(\frac{a+b}{2}\right) - g(t)\right)^{\alpha} f'(t)\,dt \\
&- \frac{1}{\Gamma(\alpha+1)} \int_{\frac{a+b}{2}}^{b} \left(g(t) - g\left(\frac{a+b}{2}\right)\right)^{\alpha} f'(t)\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
(2.10)\quad & I^{\alpha}_{\frac{a+b}{2}-,g} f(a) + I^{\alpha}_{\frac{a+b}{2}+,g} f(b) \\
&= \frac{1}{\Gamma(\alpha+1)}\left[\left(g\left(\frac{a+b}{2}\right) - g(a)\right)^{\alpha} + \left(g(b) - g\left(\frac{a+b}{2}\right)\right)^{\alpha}\right] f\left(\frac{a+b}{2}\right) \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_{\frac{a+b}{2}}^{b} (g(b) - g(t))^{\alpha} f'(t)\,dt \\
&- \frac{1}{\Gamma(\alpha+1)} \int_a^{\frac{a+b}{2}} (g(t) - g(a))^{\alpha} f'(t)\,dt.
\end{aligned}
$$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a, b \in I$ by

$$M_g(a,b) := g^{-1}\left(\frac{g(a) + g(b)}{2}\right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a,b) = A(a,b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0,\infty)$ and $g(t) = \ln t$, then $M_g(a,b) = G(a,b) := \sqrt{ab}$, the *geometric mean*. If $I = (0,\infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a,b) = H(a,b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0,\infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a,b) = M_p(a,b) := \left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*. Finally, if $I = \mathbb{R}$ and $g(t) = \exp t$, then

$$M_g(a,b) = LME(a,b) := \ln\left(\frac{\exp a + \exp b}{2}\right),$$

the *LogMeanExp function*.

Corollary 2. *With the assumptions of Lemma 1, we have*

$$
\begin{aligned}
(2.11)\quad & I^{\alpha}_{a+,g} f(M_g(a,b)) + I^{\alpha}_{b-,g} f(M_g(a,b)) \\
&= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} (g(b) - g(a))^{\alpha} \frac{f(a) + f(b)}{2} \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_a^{M_g(a,b)} (g(M_g(a,b)) - g(t))^{\alpha} f'(t)\,dt \\
&- \frac{1}{\Gamma(\alpha+1)} \int_{M_g(a,b)}^{b} (g(t) - g(M_g(a,b)))^{\alpha} f'(t)\,dt
\end{aligned}
$$

and

$$I^{\alpha}_{M_g(a,b)-,g}f(a)+I^{\alpha}_{M_g(a,b)+,g}f(b) \tag{2.12}$$
$$=\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}f(M_g(a,b))$$
$$+\frac{1}{\Gamma(\alpha+1)}\int_{M_g(a,b)}^{b}(g(b)-g(t))^{\alpha}f'(t)\,dt$$
$$-\frac{1}{\Gamma(\alpha+1)}\int_{a}^{M_g(a,b)}(g(t)-g(a))^{\alpha}f'(t)\,dt.$$

3. The Results

We use the following *Lebesgue p-norms*

$$\|h\|_{[c,d],\infty}:=\operatorname*{essup}_{t\in[c,d]}|h(t)|<\infty \text{ provided } h\in L_{\infty}[c,d]$$

and

$$\|h\|_{[c,d],p}:=\left(\int_c^d|h(t)|^p\,dt\right)^{1/p}<\infty \text{ provided } h\in L_p[c,d],\ p\geq 1.$$

In the follow, wherever we mention these norms we assume that they are finite.

We have:

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For any $x\in(a,b)$ we have*

$$\left|I^{\alpha}_{a+,g}f(x)+I^{\alpha}_{b-,g}f(x)-\frac{(g(x)-g(a))^{\alpha}f(a)+(g(b)-g(x))^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \tag{3.1}$$

$$\leq\frac{1}{\Gamma(\alpha+1)}\begin{cases}\|f'(t)\|_{[a,x],\infty}\|(g(x)-g)^{\alpha}\|_{[a,x],1}+\|f'(t)\|_{[x,b],\infty}\|(g-g(x))^{\alpha}\|_{[x,b],1};\\[2mm] \|f'(t)\|_{[a,x],p}\|(g(x)-g)^{\alpha}\|_{[a,x],q}+\|f'(t)\|_{[x,b],p}\|(g-g(x))^{\alpha}\|_{[x,b],q};\\[2mm] \|f'(t)\|_{[a,x],1}(g(x)-g(a))^{\alpha}+\|f'(t)\|_{[x,b],1}(g(b)-g(x))^{\alpha}\end{cases}$$

$$\leq\frac{1}{\Gamma(\alpha+1)}\begin{cases}\|f'(t)\|_{[a,b],\infty}\||g(x)-g|^{\alpha}\|_{[a,b],1};\\[2mm] \|f'(t)\|_{[a,b],p}\||g(x)-g|^{\alpha}\|_{[a,b],q};\\[2mm] \|f'(t)\|_{[a,b],1}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\end{cases}$$

and

$$\left| I^{\alpha}_{x-,g} f(a) + I^{\alpha}_{x+,g} f(b) - \frac{(g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha}}{\Gamma(\alpha+1)} f(x) \right| \tag{3.2}$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \|f'(t)\|_{[a,x],\infty} \|(g-g(a))^{\alpha}\|_{[a,x],1} + \|f'(t)\|_{[x,b],\infty} \|(g(b)-g)^{\alpha}\|_{[x,b],1}; \\ \\ \|f'(t)\|_{[a,x],p} \|(g-g(a))^{\alpha}\|_{[a,x],q} + \|f'(t)\|_{[x,b],p} \|(g(b)-g)^{\alpha}\|_{[x,b],q}; \\ \\ \|f'(t)\|_{[a,x],1} (g(x)-g(a))^{\alpha} + \|f'(t)\|_{[x,b],1} (g(b)-g(x))^{\alpha}, \end{cases}$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \|f'(t)\|_{[a,b],\infty} \left[\|(g-g(a))^{\alpha}\|_{[a,x],1} + \|(g(b)-g)^{\alpha}\|_{[x,b],1} \right]; \\ \\ \|f'(t)\|_{[a,b],p} \left(\|(g-g(a))^{\alpha}\|^{q}_{[a,x],q} + \|(g(b)-g)^{\alpha}\|^{q}_{[x,b],q} \right)^{1/q}; \\ \\ \|f'(t)\|_{[a,b],1} \left[\frac{1}{2}(g(b)-g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha}, \end{cases}$$

where $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using the identity (2.1) and the properties of modulus, we have

$$\begin{aligned} &\left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{(g(x)-g(a))^{\alpha} f(a) + (g(b)-g(x))^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \\ &\leq \frac{1}{\Gamma(\alpha+1)} \left[\left| \int_a^x (g(x)-g(t))^{\alpha} f'(t)\,dt \right| + \left| \int_x^b (g(t)-g(x))^{\alpha} f'(t)\,dt \right| \right] \\ &\leq \frac{1}{\Gamma(\alpha+1)} \left[\int_a^x (g(x)-g(t))^{\alpha} |f'(t)|\,dt + \int_x^b (g(t)-g(x))^{\alpha} |f'(t)|\,dt \right] \\ &=: C(x). \end{aligned} \tag{3.3}$$

Using Hölder's integral inequality we have for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\int_a^x (g(x)-g(t))^{\alpha} |f'(t)|\,dt \leq \begin{cases} \operatorname{essup}_{t\in[a,x]} |f'(t)| \int_a^x (g(x)-g(t))^{\alpha}\,dt; \\ \\ \left(\int_a^x |f'(t)|^p\,dt\right)^{1/p} \left(\int_a^x (g(x)-g(t))^{\alpha q}\,dt\right)^{1/q}; \\ \\ \operatorname{essup}_{t\in[a,x]} (g(x)-g(t))^{\alpha} \int_a^x |f'(t)|\,dt \end{cases}$$

$$= \begin{cases} \|f'(t)\|_{[a,x],\infty} \|(g(x)-g)^{\alpha}\|_{[a,x],1} \\ \\ \|f'(t)\|_{[a,x],p} \|(g(x)-g)^{\alpha}\|_{[a,x],q} \\ \\ \|f'(t)\|_{[a,x],1} \|g(x)-g\|^{\alpha}_{[a,x],\infty} \end{cases}$$

and, similarly

$$\int_x^b (g(t)-g(x))^\alpha |f'(t)|\,dt \le \begin{cases} \|f'(t)\|_{[x,b],\infty} \|(g-g(x))^\alpha\|_{[x,b],1}; \\ \\ \|f'(t)\|_{[x,b],p} \|(g-g(x))^\alpha\|_{[x,b],q}; \\ \\ \|f'(t)\|_{[x,b],1} \|g-g(x)\|^\alpha_{[x,b],\infty}. \end{cases}$$

Therefore

$$C(x) \le \begin{cases} \|f'(t)\|_{[a,x],\infty} \|(g(x)-g)^\alpha\|_{[a,x],1} + \|f'(t)\|_{[x,b],\infty} \|(g-g(x))^\alpha\|_{[x,b],1}; \\ \\ \|f'(t)\|_{[a,x],p} \|(g(x)-g)^\alpha\|_{[a,x],q} + \|f'(t)\|_{[x,b],p} \|(g-g(x))^\alpha\|_{[x,b],q}; \\ \\ \|f'(t)\|_{[a,x],1} \|g(x)-g\|^\alpha_{[a,x],\infty} + \|f'(t)\|_{[x,b],1} \|g-g(x)\|^\alpha_{[x,b],\infty}; \end{cases}$$

which together with (3.3) proves the first inequality in (3.1).

Now, observe that for any $x \in (a,b)$

$$\begin{aligned} &\|f'(t)\|_{[a,x],\infty} \|(g(x)-g)^\alpha\|_{[a,x],1} + \|f'(t)\|_{[x,b],\infty} \|(g-g(x))^\alpha\|_{[x,b],1} \\ &\le \max\left\{\|f'(t)\|_{[a,x],\infty}, \|f'(t)\|_{[x,b],\infty}\right\} \\ &\times \left[\|(g(x)-g)^\alpha\|_{[a,x],1} + \|(g-g(x))^\alpha\|_{[x,b],1}\right] \\ &= \|f'(t)\|_{[a,b],\infty} \||g(x)-g|^\alpha\|_{[a,b],1} \end{aligned}$$

and, similarly

$$\begin{aligned} &\|f'(t)\|_{[a,x],1} \|g(x)-g\|^\alpha_{[a,x],\infty} + \|f'(t)\|_{[x,b],1} \|g(x)-g\|^\alpha_{[x,b],\infty} \\ &\le \|f'(t)\|_{[a,b],1} \|g(x)-g\|^\alpha_{[a,b],\infty}. \end{aligned}$$

Using Hölder's discrete inequality we also have

$$\begin{aligned} &\|f'(t)\|_{[a,x],p} \|(g(x)-g)^\alpha\|_{[a,x],q} + \|f'(t)\|_{[x,b],p} \|(g-g(x))^\alpha\|_{[x,b],q} \\ &\le \left(\|f'(t)\|^p_{[a,x],p} + \|f'(t)\|^p_{[x,b],p}\right)^{1/p} \\ &\times \left(\|(g(x)-g)^\alpha\|^q_{[a,x],q} + \|(g-g(x))^\alpha\|^q_{[x,b],q}\right)^{1/q} \\ &= \|f'(t)\|_{[a,b],p} \||g(x)-g|^\alpha\|_{[a,b],q} \end{aligned}$$

for any $x \in (a,b)$, which proves the last part of (3.1) by noticing that since g is monotonic increasing, then

$$\|g(x)-g\|^\alpha_{[a,x],\infty} = (g(x)-g(a))^\alpha, \ \|(g-g(x))^\alpha\|_{[x,b],\infty} = (g(b)-g(x))^\alpha$$

and

$$\begin{aligned} \|g(x)-g\|^\alpha_{[a,b],\infty} &= \max\left\{(g(x)-g(a))^\alpha, (g(b)-g(x))^\alpha\right\} \\ &= \left[\max\left\{g(x)-g(a), g(b)-g(x)\right\}\right]^\alpha \\ &= \left[\frac{1}{2}(g(b)-g(a)) + \left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^\alpha. \end{aligned}$$

Using the identity (2.2) we have

$$\left|I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)-\frac{(g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}}{\Gamma(\alpha+1)}f(x)\right| \tag{3.4}$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\left[\left|\int_{x}^{b}(g(b)-g(t))^{\alpha}f'(t)\,dt\right|+\left|\int_{a}^{x}(g(t)-g(a))^{\alpha}f'(t)\,dt\right|\right]$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\left[\int_{x}^{b}(g(b)-g(t))^{\alpha}|f'(t)|\,dt+\int_{a}^{x}(g(t)-g(a))^{\alpha}|f'(t)|\,dt\right]$$
$$=:D(x)$$

for any $x\in(a,b)$.

By Hölder's inequality we have for $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$\int_{x}^{b}(g(b)-g(t))^{\alpha}|f'(t)|\,dt\leq\begin{cases}\|f'(t)\|_{[x,b],\infty}\|(g(b)-g)^{\alpha}\|_{[x,b],1}\,;\\[2mm]\|f'(t)\|_{[x,b],p}\|(g(b)-g)^{\alpha}\|_{[x,b],q}\,;\\[2mm]\|f'(t)\|_{[x,b],1}\|g(b)-g\|_{[x,b],\infty}^{\alpha}\end{cases}$$

and

$$\int_{a}^{x}(g(t)-g(a))^{\alpha}|f'(t)|\,dt\leq\begin{cases}\|f'(t)\|_{[a,x],\infty}\|(g-g(a))^{\alpha}\|_{[a,x],1}\,;\\[2mm]\|f'(t)\|_{[a,x],p}\|(g-g(a))^{\alpha}\|_{[a,x],q}\,;\\[2mm]\|f'(t)\|_{[a,x],1}\|g-g(a)\|_{[a,x],\infty}^{\alpha}\,.\end{cases}$$

Therefore

$$D(x)$$
$$\leq\frac{1}{\Gamma(\alpha+1)}$$
$$\times\begin{cases}\|f'(t)\|_{[a,x],\infty}\|(g-g(a))^{\alpha}\|_{[a,x],1}+\|f'(t)\|_{[x,b],\infty}\|(g(b)-g)^{\alpha}\|_{[x,b],1}\,;\\[2mm]\|f'(t)\|_{[a,x],p}\|(g-g(a))^{\alpha}\|_{[a,x],q}+\|f'(t)\|_{[x,b],p}\|(g(b)-g)^{\alpha}\|_{[x,b],q}\,;\\[2mm]\|f'(t)\|_{[a,x],1}\|g-g(a)\|_{[a,x],\infty}^{\alpha}+\|f'(t)\|_{[x,b],1}\|g(b)-g\|_{[x,b],\infty}^{\alpha}\,,\end{cases}$$

for any $x\in(a,b)$.

Using the discrete Hölder's inequality, we have

$$\|f'(t)\|_{[a,x],\infty}\|(g-g(a))^{\alpha}\|_{[a,x],1}+\|f'(t)\|_{[x,b],\infty}\|(g(b)-g)^{\alpha}\|_{[x,b],1}$$
$$\leq\max\left\{\|f'(t)\|_{[a,x],\infty},\|f'(t)\|_{[x,b],\infty}\right\}$$
$$\times\left[\|(g-g(a))^{\alpha}\|_{[a,x],1}+\|(g(b)-g)^{\alpha}\|_{[x,b],1}\right]$$
$$=\|f'(t)\|_{[a,b],\infty}\left[\|(g-g(a))^{\alpha}\|_{[a,x],1}+\|(g(b)-g)^{\alpha}\|_{[x,b],1}\right],$$

$$
\begin{aligned}
&\left\|f'(t)\right\|_{[a,x],p}\left\|(g-g(a))^{\alpha}\right\|_{[a,x],q}+\left\|f'(t)\right\|_{[x,b],p}\left\|(g(b)-g)^{\alpha}\right\|_{[x,b],q}\\
&\leq\left(\left\|f'(t)\right\|_{[a,x],p}^{p}+\left\|f'(t)\right\|_{[x,b],p}^{p}\right)^{1/p}\\
&\times\left[\left\|(g-g(a))^{\alpha}\right\|_{[a,x],q}^{q}+\left\|(g(b)-g)^{\alpha}\right\|_{[x,b],q}^{q}\right]^{1/q}\\
&=\left\|f'(t)\right\|_{[a,b],p}\left[\left\|(g-g(a))^{\alpha}\right\|_{[a,x],q}^{q}+\left\|(g(b)-g)^{\alpha}\right\|_{[x,b],q}^{q}\right]^{1/q}
\end{aligned}
$$

and, since g is increasing

$$
\begin{aligned}
&\left\|f'(t)\right\|_{[a,x],1}\left\|g-g(a)\right\|_{[a,x],\infty}^{\alpha}+\left\|f'(t)\right\|_{[x,b],1}\left\|g(b)-g\right\|_{[x,b],\infty}^{\alpha}\\
&\leq\left(\left\|f'(t)\right\|_{[a,x],1}+\left\|f'(t)\right\|_{[x,b],1}\right)\max\left\{\left\|g-g(a)\right\|_{[a,x],\infty}^{\alpha},\left\|g(b)-g\right\|_{[x,b],\infty}^{\alpha}\right\}\\
&=\left\|f'(t)\right\|_{[a,b],1}\max\left\{\left\|g-g(a)\right\|_{[a,x],\infty}^{\alpha},\left\|g(b)-g\right\|_{[x,b],\infty}^{\alpha}\right\}\\
&=\left\|f'(t)\right\|_{[a,b],1}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}.
\end{aligned}
$$

These together with (3.4) prove the desired result (3.2). □

We have the following mid-point type inequalities:

Corollary 3. *With the assumptions of Theorem 1 we have for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$*

$$
\begin{aligned}
(3.5)\quad &\left|I_{a+,g}^{\alpha}f\left(\frac{a+b}{2}\right)+I_{b-,g}^{\alpha}f\left(\frac{a+b}{2}\right)\right.\\
&\left.-\frac{\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\alpha}f(a)+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\alpha}f(b)}{\Gamma(\alpha+1)}\right|\\
&\leq\frac{1}{\Gamma(\alpha+1)}\begin{cases}
\left\|f'(t)\right\|_{\left[a,\frac{a+b}{2}\right],\infty}\left\|\left(g\left(\frac{a+b}{2}\right)-g\right)^{\alpha}\right\|_{\left[a,\frac{a+b}{2}\right],1}\\
+\left\|f'(t)\right\|_{\left[\frac{a+b}{2},b\right],\infty}\left\|\left(g-g\left(\frac{a+b}{2}\right)\right)^{\alpha}\right\|_{\left[\frac{a+b}{2},b\right],1};\\[2ex]
\left\|f'(t)\right\|_{\left[a,\frac{a+b}{2}\right],p}\left\|\left(g\left(\frac{a+b}{2}\right)-g\right)^{\alpha}\right\|_{\left[a,\frac{a+b}{2}\right],q}\\
+\left\|f'(t)\right\|_{\left[\frac{a+b}{2},b\right],p}\left\|\left(g-g\left(\frac{a+b}{2}\right)\right)^{\alpha}\right\|_{\left[\frac{a+b}{2},b\right],q};\\[2ex]
\left\|f'(t)\right\|_{\left[a,\frac{a+b}{2}\right],1}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\alpha}\\
+\left\|f'(t)\right\|_{\left[\frac{a+b}{2},b\right],1}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\alpha}
\end{cases}\\
&\leq\frac{1}{\Gamma(\alpha+1)}\begin{cases}
\left\|f'(t)\right\|_{[a,b],\infty}\left\|\left(g\left(\frac{a+b}{2}\right)-g\right)^{\alpha}\right\|_{[a,b],1};\\[2ex]
\left\|f'(t)\right\|_{[a,b],p}\left\|\left(g\left(\frac{a+b}{2}\right)-g\right)^{\alpha}\right\|_{[a,b],q};\\[2ex]
\left\|f'(t)\right\|_{[a,b],1}\left[\frac{1}{2}(g(b)-g(a))+\left|g\left(\frac{a+b}{2}\right)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}
\end{cases}
\end{aligned}
$$

and

$$\begin{aligned}(3.6)\quad &\left| I^{\alpha}_{\frac{a+b}{2}-,g} f(a) + I^{\alpha}_{\frac{a+b}{2}+,g} f(b) \right. \\ &\left. - \frac{\left(g\left(\frac{a+b}{2}\right) - g(a)\right)^{\alpha} + \left(g(b) - g\left(\frac{a+b}{2}\right)\right)^{\alpha}}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \right| \\ &\le \frac{1}{\Gamma(\alpha+1)} \begin{cases} \|f'(t)\|_{\left[a,\frac{a+b}{2}\right],\infty} \|(g-g(a))^{\alpha}\|_{\left[a,\frac{a+b}{2}\right],1} \\ +\|f'(t)\|_{\left[\frac{a+b}{2},b\right],\infty} \|(g(b)-g)^{\alpha}\|_{\left[\frac{a+b}{2},b\right],1}; \\ \\ \|f'(t)\|_{\left[a,\frac{a+b}{2}\right],p} \|(g-g(a))^{\alpha}\|_{\left[a,\frac{a+b}{2}\right],q} \\ +\|f'(t)\|_{\left[\frac{a+b}{2},b\right],p} \|(g(b)-g)^{\alpha}\|_{\left[\frac{a+b}{2},b\right],q}; \\ \\ \|f'(t)\|_{\left[a,\frac{a+b}{2}\right],1} \left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\alpha} \\ +\|f'(t)\|_{\left[\frac{a+b}{2},b\right],1} \left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\alpha}, \end{cases} \\ &\le \frac{1}{\Gamma(\alpha+1)} \begin{cases} \|f'(t)\|_{[a,b],\infty} \left[\|(g-g(a))^{\alpha}\|_{\left[a,\frac{a+b}{2}\right],1} + \|(g(b)-g)^{\alpha}\|_{\left[\frac{a+b}{2},b\right],1} \right]; \\ \\ \|f'(t)\|_{[a,b],p} \left(\|(g-g(a))^{\alpha}\|^{q}_{\left[a,\frac{a+b}{2}\right],q} + \|(g(b)-g)^{\alpha}\|^{q}_{\left[\frac{a+b}{2},b\right],q} \right)^{1/q}; \\ \\ \|f'(t)\|_{[a,b],1} \left[\frac{1}{2}(g(b)-g(a)) + \left| g\left(\frac{a+b}{2}\right) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha}, \end{cases}\end{aligned}$$

We also have the simpler inequalities for the g-mean of the numbers a, b :

Corollary 4. *With the assumptions of Theorem 1 we have*

$$\begin{aligned}(3.7)\quad &\left| I^{\alpha}_{a+,g} f(M_g(a,b)) + I^{\alpha}_{b-,g} f(M_g(a,b)) - \frac{(g(b)-g(a))^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} \frac{f(a)+f(b)}{2} \right| \\ &\le \frac{1}{\Gamma(\alpha+1)} \begin{cases} \|f'(t)\|_{[a,M_g(a,b)],\infty} \|(g(M_g(a,b))-g)^{\alpha}\|_{[a,M_g(a,b)],1} \\ +\|f'(t)\|_{[M_g(a,b),b],\infty} \|(g-g(M_g(a,b)))^{\alpha}\|_{[M_g(a,b),b],1}; \\ \\ \|f'(t)\|_{[a,M_g(a,b)],p} \|(g(M_g(a,b))-g)^{\alpha}\|_{[a,M_g(a,b)],q} \\ +\|f'(t)\|_{[M_g(a,b),b],p} \|(g-g(M_g(a,b)))^{\alpha}\|_{[M_g(a,b),b],q}; \\ \\ \frac{1}{2^{\alpha}} \|f'(t)\|_{[a,b],1} (g(b)-g(a))^{\alpha} \end{cases} \\ &\le \frac{1}{\Gamma(\alpha+1)} \begin{cases} \|f'(t)\|_{[a,b],\infty} \|(g(M_g(a,b))-g)^{\alpha}\|_{[a,b],1}; \\ \\ \|f'(t)\|_{[a,b],p} \|(g(M_g(a,b))-g)^{\alpha}\|_{[a,b],q}; \\ \\ \frac{1}{2^{\alpha}} \|f'(t)\|_{[a,b],1} (g(b)-g(a))^{\alpha} \end{cases}\end{aligned}$$

and

$$\left|I^{\alpha}_{M_g(a,b)-,g}f(a)+I^{\alpha}_{M_g(a,b)+,g}f(b)-\frac{(g(b)-g(a))^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}f(M_g(a,b))\right| \tag{3.8}$$

$$\leq\frac{1}{\Gamma(\alpha+1)}\begin{cases}\|f'(t)\|_{[a,M_g(a,b)],\infty}\|(g-g(a))^{\alpha}\|_{[a,M_g(a,b)],1}\\+\|f'(t)\|_{[M_g(a,b),b],\infty}\|(g(b)-g)^{\alpha}\|_{[M_g(a,b),b],1}\,;\\ \\ \|f'(t)\|_{[a,M_g(a,b)],p}\|(g-g(a))^{\alpha}\|_{[a,M_g(a,b)],q}\\+\|f'(t)\|_{[M_g(a,b),b],p}\|(g(b)-g)^{\alpha}\|_{[M_g(a,b),b],q}\,;\\ \\ \frac{1}{2^{\alpha}}\|f'(t)\|_{[a,b],1}(g(b)-g(a))^{\alpha}\,,\end{cases}$$

$$\leq\frac{1}{\Gamma(\alpha+1)}\begin{cases}\|f'(t)\|_{[a,b],\infty}\left[\|(g-g(a))^{\alpha}\|_{[a,M_g(a,b)],1}+\|(g(b)-g)^{\alpha}\|_{[M_g(a,b),b],1}\right];\\ \\ \|f'(t)\|_{[a,b],p}\left(\|(g-g(a))^{\alpha}\|^{q}_{[a,M_g(a,b)],q}+\|(g(b)-g)^{\alpha}\|^{q}_{[M_g(a,b),b],q}\right)^{1/q};\\ \\ \frac{1}{2^{\alpha}}\|f'(t)\|_{[a,b],1}(g(b)-g(a))^{\alpha}\,,\end{cases}$$

where $p,\ q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

4. An Application for Hadamard Fractional Integrals

If we take in the above section $g(t)=\ln t,\ t\in(a,b)\subset(0,\infty)$ then we can get various inequalities for Hadamard fractional integrals H^{α}_{a+} and H^{α}_{b-}. We have

$$\left|H^{\alpha}_{a+}f(x)+H^{\alpha}_{b-}f(x)-\frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}f(a)+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \tag{4.1}$$

$$\leq\frac{1}{\Gamma(\alpha+1)}\begin{cases}\|f'(t)\|_{[a,x],\infty}\|(\ln x-\ln)^{\alpha}\|_{[a,x],1}+\|f'(t)\|_{[x,b],\infty}\|(\ln-\ln x)^{\alpha}\|_{[x,b],1}\,;\\ \\ \|f'(t)\|_{[a,x],p}\|(\ln x-\ln)^{\alpha}\|_{[a,x],q}+\|f'(t)\|_{[x,b],p}\|(\ln-\ln x)^{\alpha}\|_{[x,b],q}\,;\\ \\ \|f'(t)\|_{[a,x],1}\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}+\|f'(t)\|_{[x,b],1}\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}\end{cases}$$

$$\leq\frac{1}{\Gamma(\alpha+1)}\begin{cases}\|f'(t)\|_{[a,b],\infty}\||\ln x-\ln|^{\alpha}\|_{[a,b],1}\,;\\ \\ \|f'(t)\|_{[a,b],p}\||\ln x-\ln|^{\alpha}\|_{[a,b],q}\,;\\ \\ \|f'(t)\|_{[a,b],1}\left[\frac{1}{2}\ln\left(\frac{b}{a}\right)+\left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha}\end{cases}$$

and

$$(4.2)\quad \left| H_{x-}^{\alpha} f(a) + H_{x+}^{\alpha} f(b) - \frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} + \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}{\Gamma(\alpha+1)} f(x) \right|$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \|f'(t)\|_{[a,x],\infty} \|(\ln - \ln a)^{\alpha}\|_{[a,x],1} + \|f'(t)\|_{[x,b],\infty} \|(\ln b - \ln)^{\alpha}\|_{[x,b],1}; \\ \\ \|f'(t)\|_{[a,x],p} \|(\ln - \ln a)^{\alpha}\|_{[a,x],q} + \|f'(t)\|_{[x,b],p} \|(\ln b - \ln)^{\alpha}\|_{[x,b],q}; \\ \\ \|f'(t)\|_{[a,x],1} \left[\ln\left(\frac{x}{a}\right)\right]^{\alpha} + \|f'(t)\|_{[x,b],1} \left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}, \end{cases}$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \|f'(t)\|_{[a,b],\infty} \left[\|(\ln - \ln a)^{\alpha}\|_{[a,x],1} + \|(\ln b - \ln)^{\alpha}\|_{[x,b],1}\right]; \\ \\ \|f'(t)\|_{[a,b],p} \left(\|(\ln - \ln a)^{\alpha}\|_{[a,x],q}^{q} + \|(\ln b - \ln)^{\alpha}\|_{[x,b],q}^{q}\right)^{1/q}; \\ \\ \|f'(t)\|_{[a,b],1} \left[\frac{1}{2}\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{x}{G(a,b)}\right)\right|\right]^{\alpha}, \end{cases}$$

for any $x \in (a,b)$, where $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

If we take in these inequalities $x = G(a,b)$, then we get

$$(4.3)\quad \left| H_{a+}^{\alpha} f(G(a,b)) + H_{b-}^{\alpha} f(G(a,b)) - \frac{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} \frac{f(a) + f(b)}{2} \right|$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \|f'(t)\|_{[a,G(a,b)],\infty} \|(\ln G(a,b) - \ln)^{\alpha}\|_{[a,G(a,b)],1} \\ + \|f'(t)\|_{[G(a,b),b],\infty} \|(\ln - \ln G(a,b))^{\alpha}\|_{[G(a,b),b],1}; \\ \\ \|f'(t)\|_{[a,G(a,b)],p} \|(\ln G(a,b) - \ln)^{\alpha}\|_{[a,G(a,b)],q} \\ + \|f'(t)\|_{[G(a,b),b],p} \|(\ln - \ln G(a,b))^{\alpha}\|_{[G(a,b),b],q}; \\ \\ \frac{1}{2^{\alpha}} \|f'(t)\|_{[a,b],1} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \end{cases}$$

$$\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \|f'(t)\|_{[a,b],\infty} \||\ln G(a,b) - \ln|^{\alpha}\|_{[a,b],1}; \\ \\ \|f'(t)\|_{[a,b],p} \||\ln G(a,b) - \ln|^{\alpha}\|_{[a,b],q}; \\ \\ \frac{1}{2^{\alpha}} \|f'(t)\|_{[a,b],1} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha} \end{cases}$$

and

$$
(4.4)\quad \left| H^{\alpha}_{G(a,b)-} f(a) + H^{\alpha}_{G(a,b)+} f(b) - \frac{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} f(G(a,b)) \right|
$$

$$
\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \|f'(t)\|_{[a,G(a,b)],\infty} \|(\ln - \ln a)^{\alpha}\|_{[a,G(a,b)],1} \\ + \|f'(t)\|_{[G(a,b),b],\infty} \|(\ln b - \ln)^{\alpha}\|_{[G(a,b),b],1}; \\ \\ \|f'(t)\|_{[a,G(a,b)],p} \|(\ln - \ln a)^{\alpha}\|_{[a,G(a,b)],q} \\ + \|f'(t)\|_{[G(a,b),b],p} \|(\ln b - \ln)^{\alpha}\|_{[G(a,b),b],q}; \\ \\ \frac{1}{2^{\alpha}} \|f'(t)\|_{[a,b],1} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}, \end{cases}
$$

$$
\leq \frac{1}{\Gamma(\alpha+1)} \begin{cases} \|f'(t)\|_{[a,b],\infty} \left[\|(\ln - \ln a)^{\alpha}\|_{[a,G(a,b)],1} + \|(\ln b - \ln)^{\alpha}\|_{[G(a,b),b],1} \right]; \\ \\ \|f'(t)\|_{[a,b],p} \left(\|(\ln - \ln a)^{\alpha}\|^{q}_{[a,G(a,b)],q} + \|(\ln b - \ln)^{\alpha}\|^{q}_{[G(a,b),b],q} \right)^{1/q}; \\ \\ \frac{1}{2^{\alpha}} \|f'(t)\|_{[a,b],1} \left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}, \end{cases}
$$

where $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

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