Received 06/06/17

OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF g-LIPSCHITZIAN FUNCTIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we establish some Ostrowski and trapezoid type inequalities for the Riemann-Liouville fractional integrals of functions that are *g-Lipschitzian*. Applications for the *g-mean of two numbers* are provided as well. Some particular cases for Hadamard and Harmonic fractional integrals are also provided.

1. Introduction

Let (a,b) with $-\infty \leq a < b \leq \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\operatorname{Re}(\alpha) > 0$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Following [19, p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ by

$$I_{a+,g}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)\,f(t)\,dt}{[g(x)-g(t)]^{1-\alpha}},\ a < x \leq b \tag{1.1}$$

and

$$I_{b-,g}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)\,f(t)\,dt}{[g(t)-g(x)]^{1-\alpha}},\ a \leq x < b. \tag{1.2}$$

For $g(t) = t$ we have the classical *Riemann-Liouville fractional integrals*

$$J_{a+}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}},\ a < x \leq b \tag{1.3}$$

and

$$J_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha}},\ a \leq x < b, \tag{1.4}$$

while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [19, p. 111]

$$H_{a+}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0 \leq a < x \leq b \tag{1.5}$$

and

$$H_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0 \leq a < x < b. \tag{1.6}$$

1991 *Mathematics Subject Classification.* 26D15, 26D10, 26D07, 26A33.

Key words and phrases. Riemann-Liouville fractional integrals, Functions of bounded variation, Lipshitzian functions, Trapezoid type inequalities.

RGMIA Res. Rep. Coll. 20 (2017), Art. 75, 15 pp

One can consider the function $g(t)=-t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_{a}^{x}\frac{f(t)\,dt}{(x-t)^{1-\alpha}t^{\alpha+1}},\ 0\leq a<x\leq b \tag{1.7}$$

and

$$R_{b-}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_{x}^{b}\frac{f(t)\,dt}{(t-x)^{1-\alpha}t^{\alpha+1}},\ 0\leq a<x<b. \tag{1.8}$$

Also, for $g(t)=\exp(\beta t)$, $\beta>0$, we can consider the *"β-Exponential fractional integrals"*

$$E_{a+,\beta}^{\alpha}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_{a}^{x}\frac{\exp(\beta t)f(t)\,dt}{[\exp(\beta x)-\exp(\beta t)]^{1-\alpha}},\ a<x\leq b \tag{1.9}$$

and

$$E_{b-,\beta}^{\alpha}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_{x}^{b}\frac{\exp(\beta t)f(t)\,dt}{[\exp(\beta t)-\exp(\beta x)]^{1-\alpha}},\ a\leq x<b. \tag{1.10}$$

In the recent paper [14] we obtained the following Ostrowski type inequalities for functions of bounded variation:

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For any $x\in(a,b)$ we have the inequalities*

$$\left|I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)f(x)\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}\frac{g'(t)\bigvee_{t}^{x}(f)\,dt}{[g(x)-g(t)]^{1-\alpha}}+\int_{x}^{b}\frac{g'(t)\bigvee_{x}^{t}(f)\,dt}{[g(t)-g(x)]^{1-\alpha}}\right]$$

$$\leq\frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}\bigvee_{a}^{x}(f)+[g(b)-g(x)]^{\alpha}\bigvee_{x}^{b}(f)\right]$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_{a}^{b}(f);\\ \left((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p}\right)^{1/p}\left(\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right)\left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right],\end{cases}$$

and

$$\left|I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)-\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)f(x)\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)\bigvee_t^x(f)\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_x^b\frac{g'(t)\bigvee_x^t(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\left[[g(x)-g(a)]^{\alpha}\bigvee_a^x(f)+[g(b)-g(x)]^{\alpha}\bigvee_x^b(f)\right]$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_a^b(f);\\ \left((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right].\end{cases}$$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a,\ b\in I$ as

$$M_g(a,b):=g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

The following particular case for *g-mean* is of interest [14].

Corollary 1. *With the assumptions of Theorem 1 we have*

$$\left|I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))-\frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}f(M_g(a,b))\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)\bigvee_t^{M_g(a,b)}(f)\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}}+\int_{M_g(a,b)}^b\frac{g'(t)\bigvee_{M_g(a,b)}^t(f)\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}}\right]$$
$$\leq\frac{1}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_a^b(f);$$

and

$$\left|I^{\alpha}_{M_g(a,b)-,g}f(a)+I^{\alpha}_{M_g(a,b)+,g}f(b)-\frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}f(M_g(a,b))\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)}\frac{g'(t)\bigvee_t^{M_g(a,b)}(f)\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_{M_g(a,b)}^b\frac{g'(t)\bigvee_x^t(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]$$
$$\leq\frac{1}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_a^b(f).$$

Remark 1. *If we take in Theorem 1* $x=\frac{a+b}{2}$, *then we obtain similar mid-point inequalities, however the details are not presented here. Some applications for the Hadamard fractional integrals are also provided in* [14].

The following trapezoid type inequalities for functions of bounded variation also hold [15]:

Theorem 2. *Let* $f:[a,b]\to\mathbb{C}$ *be a complex valued function of bounded variation on the real interval* $[a,b]$, *and* g *be a strictly increasing function on* (a,b), *having a continuous derivative* g' *on* (a,b). *Then we have the inequalities*

$$\left|I^{\alpha}_{a+,g}f(x)+I^{\alpha}_{b-,g}f(x)-\frac{[g(x)-g(a)]^{\alpha}f(a)+[g(b)-g(x)]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)\bigvee_a^t(f)\,dt}{[g(x)-g(t)]^{1-\alpha}}+\int_x^b\frac{g'(t)\bigvee_t^b(f)\,dt}{[g(t)-g(x)]^{1-\alpha}}\right]$$

$$\leq\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}\bigvee_a^x(f)+(g(b)-g(x))^{\alpha}\bigvee_x^b(f)\right]$$
$$\leq\frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_a^b(f);\\ \left((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right)\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}$$

and

$$\left|I^{\alpha}_{x-,g}f(a)+I^{\alpha}_{x+,g}f(b)-\frac{[g(x)-g(a)]^{\alpha}f(a)+[g(b)-g(x)]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right|$$
$$\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^x\frac{g'(t)\bigvee_a^t(f)\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_x^b\frac{g'(t)\bigvee_t^b(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]$$

$$\leq \frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}\bigvee_{a}^{x}(f)+(g(b)-g(x))^{\alpha}\bigvee_{x}^{b}(f)\right]$$

$$\leq \frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_{a}^{b}(f);\\ \left((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p}\right)^{1/p}\left(\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \left((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right)\left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right]\end{cases}$$

for any $x\in(a,b)$

(ii) We also have

$$\left|\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\frac{f(b)+f(a)}{2}\right|$$
$$\leq \frac{1}{2\Gamma(\alpha)}\left[\int_{a}^{b}\frac{g'(t)\bigvee_{t}^{b}(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}+\int_{a}^{b}\frac{g'(t)\bigvee_{a}^{t}(f)\,dt}{[g(t)-g(a)]^{1-\alpha}}\right]$$
$$\leq \frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\bigvee_{a}^{b}(f).$$

In particular, we have [15]:

Corollary 2. *With the assumptions of Theorem 2 we have*

$$\left|I_{a+,g}^{\alpha}f(M_{g}(a,b))+I_{b-,g}^{\alpha}f(M_{g}(a,b))-\frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_{a}^{M_{g}(a,b)}\frac{g'(t)\bigvee_{a}^{t}(f)\,dt}{[g(M_{g}(a,b))-g(t)]^{1-\alpha}}+\int_{M_{g}(a,b)}^{b}\frac{g'(t)\bigvee_{t}^{b}(f)\,dt}{[g(t)-g(M_{g}(a,b))]^{1-\alpha}}\right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_{a}^{b}(f)$$

and

$$\left|I_{M_{g}(a,b)-,g}^{\alpha}f(a)+I_{M_{g}(a,b)+,g}^{\alpha}f(b)-\frac{f(a)+f(b)}{2^{\alpha}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_{a}^{M_{g}(a,b)}\frac{g'(t)\bigvee_{a}^{t}(f)\,dt}{[g(t)-g(a)]^{1-\alpha}}+\int_{M_{g}(a,b)}^{b}\frac{g'(t)\bigvee_{t}^{b}(f)\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_{a}^{b}(f).$$

For several Ostrowski and trapezoid type inequalities for Riemann-Liouville fractional integrals see [1]-[5], [17]-[29] and the references therein.

Motivated by the above results, in this paper we establish some Ostrowski and trapezoid type inequalities for the generalized Riemann-Liouville fractional integrals of generalized Lipschitzian functions. Applications for the *g-mean of two numbers* are provided as well. Some particular cases for Hadamard and Harmonic fractional integrals are also provided.

2. Some Preliminary Facts

We have the following two parameter identities, see also [15] :

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be Lebesgue integrable on $[a,b]$, g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b) and λ, μ some complex parameters:*

(i) For any $x\in(a,b)$ we have the representation

$$I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha+1)}\left(\lambda\left[g(x)-g(a)\right]^{\alpha}+\mu\left[g(b)-g(x)\right]^{\alpha}\right)+\frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}\frac{g'(t)\left[f(t)-\lambda\right]dt}{\left[g(x)-g(t)\right]^{1-\alpha}}+\int_{x}^{b}\frac{g'(t)\left[f(t)-\mu\right]dt}{\left[g(t)-g(x)\right]^{1-\alpha}}\right]\tag{2.1}$$

and

$$I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)=\frac{1}{\Gamma(\alpha+1)}\left(\lambda\left[g(x)-g(a)\right]^{\alpha}+\mu\left[g(b)-g(x)\right]^{\alpha}\right)+\frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}\frac{g'(t)\left[f(t)-\lambda\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}}+\int_{x}^{b}\frac{g'(t)\left[f(t)-\mu\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}}\right].\tag{2.2}$$

(ii) We have

$$\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}=\frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\frac{\lambda+\mu}{2}+\frac{1}{2\Gamma(\alpha)}\left[\int_{a}^{b}\frac{g'(t)\left[f(t)-\lambda\right]dt}{\left[g(b)-g(t)\right]^{1-\alpha}}+\int_{a}^{b}\frac{g'(t)\left[f(t)-\mu\right]dt}{\left[g(t)-g(a)\right]^{1-\alpha}}\right].\tag{2.3}$$

Proof. (i) We observe that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{g'(t)\left[f(t)-\lambda\right]dt}{\left[g(x)-g(t)\right]^{1-\alpha}}\\&=I_{a+,g}^{\alpha}f(x)-\lambda\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{g'(t)\,dt}{\left[g(x)-g(t)\right]^{1-\alpha}}\\&=I_{a+,g}^{\alpha}f(x)-\frac{\left[g(x)-g(a)\right]^{\alpha}}{\alpha\Gamma(\alpha)}\lambda=I_{a+,g}^{\alpha}f(x)-\frac{\left[g(x)-g(a)\right]^{\alpha}}{\Gamma(\alpha+1)}\lambda\end{aligned}\tag{2.4}$$

for $a<x\le b$ and, similarly,

$$\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\frac{g'(t)\left[f(t)-\mu\right]dt}{\left[g(t)-g(x)\right]^{1-\alpha}}=I_{b-,g}^{\alpha}f(x)-\frac{\left[g(b)-g(x)\right]^{\alpha}}{\Gamma(\alpha+1)}\mu\tag{2.5}$$

for $a\le x<b$.

If $x\in(a,b)$, then by adding the equalities (2.4) and (2.5) we get the representation (2.1).

By the definition of fractional integrals we have

$$I_{x+,g}^{\alpha}f(b):=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\frac{g'(t)\,f(t)\,dt}{\left[g(b)-g(t)\right]^{1-\alpha}},\ a\le x<b$$

and

$$I_{x-,g}^{\alpha}f(a):=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{g'(t)\,f(t)\,dt}{\left[g(t)-g(a)\right]^{1-\alpha}},\ a<x\le b.$$

Then

$$\frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)[f(t)-\lambda]\,dt}{[g(b)-g(t)]^{1-\alpha}} = I_{x+,g}^{\alpha}f(b) - \frac{[g(b)-g(x)]^{\alpha}}{\Gamma(\alpha+1)}\lambda \tag{2.6}$$

for $a \le x < b$ and

$$\frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)[f(t)-\mu]\,dt}{[g(t)-g(a)]^{1-\alpha}} = I_{x-,g}^{\alpha}f(a) - \frac{[g(x)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\mu \tag{2.7}$$

for $a < x \le b$.

If $x \in (a,b)$, then by adding the equalities (2.6) and (2.7) we get the representation (2.1).

(ii) If we take $x = b$ in (2.4) we get

$$\frac{1}{\Gamma(\alpha)}\int_a^b \frac{g'(t)[f(t)-\lambda]\,dt}{[g(b)-g(t)]^{1-\alpha}} = I_{a+,g}^{\alpha}f(b) - \frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\lambda \tag{2.8}$$

while from $x = a$ in (2.5) we get

$$\frac{1}{\Gamma(\alpha)}\int_a^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(t)-g(a)]^{1-\alpha}} = I_{b-,g}^{\alpha}f(a) - \frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}\mu. \tag{2.9}$$

If we add (2.8) with (2.9) and divide by 2 we get (2.3). □

Remark 2. *If we take in (2.1) and (2.2)* $x = M_g(a,b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$*, then we get*

$$\begin{aligned} I_{a+,g}^{\alpha}f(M_g(a,b)) + I_{b-,g}^{\alpha}f(M_g(a,b)) &= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\left(\frac{\lambda+\mu}{2}\right) \\ &+ \frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)} \frac{g'(t)[f(t)-\lambda]\,dt}{[g(M_g(a,b))-g(t)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(t)-g(M_g(a,b))]^{1-\alpha}}\right] \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} I_{M_g(a,b)-,g}^{\alpha}f(a) + I_{M_g(a,b)+,g}^{\alpha}f(b) &= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\left(\frac{\lambda+\mu}{2}\right) \\ &+ \frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)} \frac{g'(t)[f(t)-\lambda]\,dt}{[g(t)-g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)[f(t)-\mu]\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]. \end{aligned} \tag{2.11}$$

The above lemma provides various identities of interest by taking particular values for the parameters λ and μ, out of which we give only a few:

Corollary 3. *With the assumptions of Lemma 1 we have:*

$$\begin{aligned} I_{a+,g}^{\alpha}f(x) + I_{b-,g}^{\alpha}f(x) &= \frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}\right)f(x) \\ &+ \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)[f(t)-f(x)]\,dt}{[g(x)-g(t)]^{1-\alpha}} + \int_x^b \frac{g'(t)[f(t)-f(x)]\,dt}{[g(t)-g(x)]^{1-\alpha}}\right] \end{aligned} \tag{2.12}$$

for any $x \in (a,b)$.

The proof is obvious by taking $\lambda = \mu = f(x)$ in Lemma 1. These identity was obtained in [14]. If we take in (2.12) $x = M_g(a,b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, then we get the corresponding identity that was obtained in [14].

Corollary 4. *With the assumptions of Lemma 1 we have:*

$$\begin{aligned} I^{\alpha}_{x-,g}f(a) + I^{\alpha}_{x+,g}f(b) &= \frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha} f(a) + [g(b)-g(x)]^{\alpha} f(b)\right) \\ &\quad + \frac{1}{\Gamma(\alpha)}\left[\int_a^x \frac{g'(t)[f(t)-f(a)]\,dt}{[g(t)-g(a)]^{1-\alpha}} + \int_x^b \frac{g'(t)[f(t)-f(b)]\,dt}{[g(b)-g(t)]^{1-\alpha}}\right], \end{aligned} \tag{2.13}$$

for any $x \in (a,b)$.

We also have

$$\begin{aligned} \frac{I^{\alpha}_{b-,g}f(a) + I^{\alpha}_{a+,g}f(b)}{2} &= \frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\frac{f(b)+f(a)}{2} \\ &\quad + \frac{1}{2\Gamma(\alpha)}\left[\int_a^b \frac{g'(t)[f(t)-f(b)]\,dt}{[g(b)-g(t)]^{1-\alpha}} + \int_a^b \frac{g'(t)[f(t)-f(a)]\,dt}{[g(t)-g(a)]^{1-\alpha}}\right]. \end{aligned} \tag{2.14}$$

The proof of (2.13) is obvious by taking $\lambda = f(a)$, $\mu = f(b)$ in Lemma 1. The proof of (2.14) follows by Lemma 1 on taking $\lambda = f(b)$ and $\mu = f(a)$.

Remark 3. *If we take in (2.13) $x = M_g(a,b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$, then we get*

$$\begin{aligned} I^{\alpha}_{M_g(a,b)-,g}f(a) + I^{\alpha}_{M_g(a,b)+,g}f(b) &= \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\left(\frac{f(a)+f(b)}{2}\right) \\ &\quad + \frac{1}{\Gamma(\alpha)}\left[\int_a^{M_g(a,b)} \frac{g'(t)[f(t)-f(a)]\,dt}{[g(t)-g(a)]^{1-\alpha}} + \int_{M_g(a,b)}^b \frac{g'(t)[f(t)-f(b)]\,dt}{[g(b)-g(t)]^{1-\alpha}}\right]. \end{aligned} \tag{2.15}$$

3. Inequalities for Generalized Lipschitz Condition

Following [21], for two functions $f, g : [a,b] \to \mathbb{C}$ we say that f is of *g-Lipschitz type* with constant $K > 0$ if

$$|f(y) - f(x)| \le K|g(y) - g(x)| \tag{3.1}$$

for any $x, y \in [a,b]$. This condition can be weakened by assuming that (3.1) holds for almost every $x, y \in [a,b]$.

If functions f and g are real valued and both continuous on the closed interval $[a,b]$, and differentiable on the open interval (a,b), then according to Cauchy's mean value theorem, there exists some $c \in (a,b)$, such that

$$[f(b) - f(a)]\,g'(c) = [g(b) - g(a)]\,f'(c).$$

Now, if $f, g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$, differentiable on (a,b) and $g'(t) \ne 0$ for any $t \in (a,b)$, then by assuming that $L := \sup_{t\in(a,b)}\left|\frac{f'(t)}{g'(t)}\right| < \infty$, we get that f is of g-Lipschitz type with constant $L > 0$. This can provide many examples of such pairs by taking various particular functions g that are strictly monotonic on (a,b). For instance, if we take $g : [a,b] \subset (0,\infty) \to (0,\infty)$, $g(t) = t^p$, $p > 0$ and if

$f:[a,b]\to\mathbb{R}$ is differentiable on (a,b) and such that $\sup_{t\in(a,b)}\left|\frac{f'(t)}{t^{p-1}}\right| =: L_p < \infty$, then the function f is $(\cdot)^p$-Lipschitzian with the constant L_p. If we assume that $\sup_{t\in(a,b)}|tf'(t)| =: L_{-1} < \infty$, then the function f is ln-Lipschitzian with the constant L_{-1}.

Theorem 3. *Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f:[a,b]\to\mathbb{C}$ is of g-Lipschitz type with constant $K>0$ then we have*

$$\left|I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}\left([g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}\right)f(x)\right| \leq \frac{1}{(\alpha+1)\Gamma(\alpha)}K\left[[g(x)-g(a)]^{\alpha+1}+[g(b)-g(x)]^{\alpha+1}\right] \tag{3.2}$$

and

$$\left|I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)-\frac{\left([g(x)-g(a)]^{\alpha}f(a)+[g(b)-g(x)]^{\alpha}f(b)\right)}{\Gamma(\alpha+1)}\right| \leq \frac{1}{(\alpha+1)\Gamma(\alpha)}K\left[[g(x)-g(a)]^{\alpha+1}+[g(b)-g(x)]^{\alpha+1}\right] \tag{3.3}$$

for any $x\in(a,b)$.

We also have

$$\left|\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}[g(b)-g(a)]^{\alpha}\frac{f(b)+f(a)}{2}\right| \leq \frac{1}{(\alpha+1)\Gamma(\alpha)}K[g(b)-g(a)]^{\alpha+1}. \tag{3.4}$$

Proof. Using the identity (2.12) we have

$$\begin{aligned}
&\left|I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)-\frac{[g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}}{\Gamma(\alpha+1)}f(x)\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\left|\int_a^x \frac{g'(t)[f(t)-f(x)]\,dt}{[g(x)-g(t)]^{1-\alpha}}\right|+\left|\int_x^b \frac{g'(t)[f(t)-f(x)]\,dt}{[g(t)-g(x)]^{1-\alpha}}\right|\right] \\
&\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^x \left|\frac{g'(t)[f(t)-f(x)]}{[g(x)-g(t)]^{1-\alpha}}\right|dt+\int_x^b \left|\frac{g'(t)[f(t)-f(x)]}{[g(t)-g(x)]^{1-\alpha}}\right|dt\right] \\
&= \frac{1}{\Gamma(\alpha)}\int_a^x \left|\frac{f(t)-f(x)}{g(x)-g(t)}\right| g'(t)[g(x)-g(t)]^{\alpha}\,dt \\
&+ \frac{1}{\Gamma(\alpha)}\int_x^b \left|\frac{f(t)-f(x)}{g(x)-g(t)}\right| g'(t)[g(t)-g(x)]^{\alpha}\,dt \\
&=: C(x)
\end{aligned} \tag{3.5}$$

for $x\in(a,b)$.

Using the fact that $f:[a,b]\to\mathbb{C}$ is of g-*Lipschitz type* with constant $K>0$ then

$$\left|\frac{f(t)-f(x)}{g(x)-g(t)}\right|\leq K \text{ for } a\leq t<x$$

and

$$\left|\frac{f(t)-f(x)}{g(x)-g(t)}\right|\leq K \text{ for } x<t\leq b.$$

These imply that

$$C(x) \leq \frac{1}{\Gamma(\alpha)} K \left[\int_a^x g'(t) [g(x) - g(t)]^\alpha dt + \int_x^b g'(t) [g(t) - g(x)]^\alpha dt \right]$$
$$= \frac{1}{\Gamma(\alpha)} K \left[\frac{[g(x) - g(a)]^{\alpha+1} + [g(b) - g(x)]^{\alpha+1}}{\alpha+1} \right]$$

and the inequality (3.2) is proved.

If we use the equality (2.13) we have in a similar way that

$$I \left| {}^{\alpha}_{x-,g} f(a) + I^{\alpha}_{x+,g} f(b) - \frac{([g(x) - g(a)]^\alpha f(a) + [g(b) - g(x)]^\alpha f(b))}{\Gamma(\alpha+1)} \right| \tag{3.6}$$
$$\leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x \left| \frac{g'(t)[f(t) - f(a)]}{[g(t) - g(a)]^{1-\alpha}} \right| dt + \int_x^b \left| \frac{g'(t)[f(t) - f(b)]}{[g(b) - g(t)]^{1-\alpha}} \right| dt \right]$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_a^x \left| \frac{f(t) - f(a)}{g(t) - g(a)} \right| g'(t) [g(t) - g(a)]^\alpha dt$$
$$+ \frac{1}{\Gamma(\alpha)} \int_x^b \left| \frac{f(t) - f(b)}{g(t) - g(b)} \right| g'(t) [g(b) - g(t)]^\alpha dt$$
$$\leq \frac{K}{\Gamma(\alpha)} \left[\int_a^x g'(t) [g(t) - g(a)]^\alpha dt + \int_x^b g'(t) [g(b) - g(t)]^\alpha dt \right]$$
$$= \frac{K}{\Gamma(\alpha)} \left[\frac{[g(x) - g(a)]^{\alpha+1} + [g(b) - g(x)]^{\alpha+1}}{\alpha+1} \right]$$

for $x \in (a, b)$, and the inequality (3.3) is proved.

Finally, by the equality we have

$$\frac{I^{\alpha}_{b-,g} f(a) + I^{\alpha}_{a+,g} f(b)}{2} - \frac{1}{\Gamma(\alpha+1)} [g(b) - g(a)]^\alpha \frac{f(b) + f(a)}{2} \tag{3.7}$$
$$\leq \frac{1}{2\Gamma(\alpha)} \left[\int_a^b \left| \frac{g'(t)[f(t) - f(b)]}{[g(b) - g(t)]^{1-\alpha}} \right| dt + \int_a^b \left| \frac{g'(t)[f(t) - f(a)]}{[g(t) - g(a)]^{1-\alpha}} \right| dt \right]$$
$$= \frac{1}{2\Gamma(\alpha)} \int_a^b \left| \frac{f(b) - f(t)}{g(b) - g(t)} \right| g'(t) [g(b) - g(t)]^\alpha dt$$
$$+ \frac{1}{2\Gamma(\alpha)} \int_a^b \left| \frac{f(t) - f(a)}{g(t) - g(a)} \right| g'(t) [g(t) - g(a)]^\alpha dt$$
$$\leq \frac{1}{2\Gamma(\alpha)} K \left[\int_a^b g'(t) [g(b) - g(t)]^\alpha dt + \int_a^b g'(t) [g(t) - g(a)]^\alpha dt \right]$$
$$= \frac{1}{2\Gamma(\alpha)} K \left[\frac{[g(b) - g(a)]^{\alpha+1}}{\alpha+1} + \frac{[g(b) - g(a)]^{\alpha+1}}{\alpha+1} \right]$$
$$= \frac{1}{\Gamma(\alpha)} K \frac{[g(b) - g(a)]^{\alpha+1}}{\alpha+1}$$

and the inequality (3.4) is proved. □

We have the following mid-point type inequalities:

Corollary 5. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}(3.8)\quad & \left| I^{\alpha}_{a+,g} f\left(\frac{a+b}{2}\right) + I^{\alpha}_{b-,g} f\left(\frac{a+b}{2}\right) \right. \\ & \left. - \frac{1}{\Gamma(\alpha+1)} \left(\left[g\left(\frac{a+b}{2}\right) - g(a) \right]^{\alpha} + \left[g(b) - g\left(\frac{a+b}{2}\right) \right]^{\alpha} \right) f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{1}{(\alpha+1)\Gamma(\alpha)} K \left[\left[g\left(\frac{a+b}{2}\right) - g(a) \right]^{\alpha+1} + \left[g(b) - g\left(\frac{a+b}{2}\right) \right]^{\alpha+1} \right]\end{aligned}$$

and

$$\begin{aligned}(3.9)\quad & \left| I^{\alpha}_{\frac{a+b}{2}-,g} f(a) + I^{\alpha}_{\frac{a+b}{2}+,g} f(b) \right. \\ & \left. - \frac{\left(\left[g\left(\frac{a+b}{2}\right) - g(a) \right]^{\alpha} f(a) + \left[g(b) - g\left(\frac{a+b}{2}\right) \right]^{\alpha} f(b) \right)}{\Gamma(\alpha+1)} \right| \\ & \leq \frac{1}{(\alpha+1)\Gamma(\alpha)} K \left[\left[g\left(\frac{a+b}{2}\right) - g(a) \right]^{\alpha+1} + \left[g(b) - g\left(\frac{a+b}{2}\right) \right]^{\alpha+1} \right].\end{aligned}$$

We have the following inequalities for the g-*mean of two numbers* a, $b \in I$

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

Corollary 6. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}(3.10)\quad & \left| I^{\alpha}_{a+,g} f(M_g(a,b)) + I^{\alpha}_{b-,g} f(M_g(a,b)) - \frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} f(M_g(a,b)) \right| \\ & \leq \frac{1}{2^{\alpha}(\alpha+1)\Gamma(\alpha)} K [g(b)-g(a)]^{\alpha+1}\end{aligned}$$

and

$$\begin{aligned}(3.11)\quad & \left| I^{\alpha}_{M_g(a,b)-,g} f(a) + I^{\alpha}_{M_g(a,b)+,g} f(b) - \frac{[g(b)-g(a)]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)} \frac{f(a)+f(b)}{2} \right| \\ & \leq \frac{1}{2^{\alpha}(\alpha+1)\Gamma(\alpha)} K [g(b)-g(a)]^{\alpha+1}.\end{aligned}$$

Remark 4. *Assume that* $f : [a,b] \to \mathbb{R}$ *is differentiable on* (a,b) *and* g *is a strictly increasing function on* (a,b), *having a continuous derivative* g' *on* (a,b). *Then we have*

$$\begin{aligned}(3.12)\quad & \left| I^{\alpha}_{a+,g} f(x) + I^{\alpha}_{b-,g} f(x) - \frac{[g(x)-g(a)]^{\alpha} + [g(b)-g(x)]^{\alpha}}{\Gamma(\alpha+1)} f(x) \right| \\ & \leq \frac{1}{(\alpha+1)\Gamma(\alpha)} \sup_{t\in(a,b)} \left(\frac{|f'(t)|}{g'(t)} \right) \left[[g(x)-g(a)]^{\alpha+1} + [g(b)-g(x)]^{\alpha+1} \right]\end{aligned}$$

and

$$\begin{aligned}(3.13)\quad & \left| I^{\alpha}_{x-,g} f(a) + I^{\alpha}_{x+,g} f(b) - \frac{[g(x)-g(a)]^{\alpha} f(a) + [g(b)-g(x)]^{\alpha} f(b)}{\Gamma(\alpha+1)} \right| \\ & \leq \frac{1}{(\alpha+1)\Gamma(\alpha)} \sup_{t\in(a,b)} \left(\frac{|f'(t)|}{g'(t)} \right) \left[[g(x)-g(a)]^{\alpha+1} + [g(b)-g(x)]^{\alpha+1} \right]\end{aligned}$$

for any $x \in (a,b)$.

We also have

$$\left|\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}\left[g(b)-g(a)\right]^{\alpha}\frac{f(b)+f(a)}{2}\right| \le \frac{1}{(\alpha+1)\Gamma(\alpha)}\sup_{t\in(a,b)}\left(\frac{|f'(t)|}{g'(t)}\right)\left[g(b)-g(a)\right]^{\alpha+1}. \tag{3.14}$$

In particular, we have

$$\left|I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))-\frac{\left[g(b)-g(a)\right]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}f(M_g(a,b))\right| \le \frac{1}{2^{\alpha}(\alpha+1)\Gamma(\alpha)}\sup_{t\in(a,b)}\left(\frac{|f'(t)|}{g'(t)}\right)\left[g(b)-g(a)\right]^{\alpha+1} \tag{3.15}$$

and

$$\left|I_{M_g(a,b)-,g}^{\alpha}f(a)+I_{M_g(a,b)+,g}^{\alpha}f(b)-\frac{\left[g(b)-g(a)\right]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}\right| \le \frac{1}{2^{\alpha}(\alpha+1)\Gamma(\alpha)}\sup_{t\in(a,b)}\left(\frac{|f'(t)|}{g'(t)}\right)\left[g(b)-g(a)\right]^{\alpha+1}. \tag{3.16}$$

4. Some Applications

In the following we assume that $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b).

If we take $g(t)=\ln t$, $t\in(a,b)\subset(0,\infty)$ in (3.12)-(3.14), then we have the following inequalities for the Hadamard fractional integrals

$$\left|H_{a+,}^{\alpha}f(x)+H_{b-}^{\alpha}f(x)-\frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}{\Gamma(\alpha+1)}f(x)\right| \le \frac{1}{(\alpha+1)\Gamma(\alpha)}\sup_{t\in(a,b)}(t|f'(t)|)\left[\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha+1}+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha+1}\right] \tag{4.1}$$

and

$$\left|H_{x-}^{\alpha}f(a)+H_{x+}^{\alpha}f(b)-\frac{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}f(a)+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \le \frac{1}{(\alpha+1)\Gamma(\alpha)}\sup_{t\in(a,b)}(t|f'(t)|)\left[\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha+1}+\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha+1}\right] \tag{4.2}$$

for any $x\in(a,b)$.

We also have

$$\left|\frac{H_{b-}^{\alpha}f(a)+H_{a+}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}\frac{f(b)+f(a)}{2}\right| \le \frac{1}{(\alpha+1)\Gamma(\alpha)}\sup_{t\in(a,b)}(t|f'(t)|)\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha+1}. \tag{4.3}$$

In particular, we have by (3.15) and (3.16) that

$$\left|H_{a+}^{\alpha}f(G(a,b))+H_{b-}^{\alpha}f(G(a,b))-\frac{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}f(G(a,b))\right| \le \frac{1}{2^{\alpha}(\alpha+1)\Gamma(\alpha)}\sup_{t\in(a,b)}\left(t\left|f'(t)\right|\right)\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha+1} \tag{4.4}$$

and

$$\left|H_{G(a,b)-}^{\alpha}f(a)+H_{G(a,b)+}^{\alpha}f(b)-\frac{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}\right| \le \frac{1}{2^{\alpha}(\alpha+1)\Gamma(\alpha)}\sup_{t\in(a,b)}\left(t\left|f'(t)\right|\right)\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha+1}. \tag{4.5}$$

If we take the function $g(t)=-t^{-1}$ in (3.12)-(3.14), then we have the following inequalities for Harmonic fractional integrals

$$\left|R_{a+}^{\alpha}f(x)+R_{b-}^{\alpha}f(x)-\frac{\left(\frac{x-a}{xa}\right)^{\alpha}+\left(\frac{b-x}{bx}\right)^{\alpha}}{\Gamma(\alpha+1)}f(x)\right| \le \frac{1}{(\alpha+1)\Gamma(\alpha)}\sup_{t\in(a,b)}\left(t^{2}\left|f'(t)\right|\right)\left[\left(\frac{x-a}{xa}\right)^{\alpha+1}+\left(\frac{b-x}{bx}\right)^{\alpha+1}\right] \tag{4.6}$$

and

$$\left|R_{x-}^{\alpha}f(a)+R_{x+}^{\alpha}f(b)-\frac{\left(\frac{x-a}{xa}\right)^{\alpha}f(a)+\left(\frac{b-x}{bx}\right)^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \le \frac{1}{(\alpha+1)\Gamma(\alpha)}\sup_{t\in(a,b)}\left(t^{2}\left|f'(t)\right|\right)\left[\left(\frac{x-a}{xa}\right)^{\alpha+1}+\left(\frac{b-x}{bx}\right)^{\alpha+1}\right] \tag{4.7}$$

for any $x\in(a,b)$.

We also have

$$\left|\frac{R_{b-}^{\alpha}f(a)+R_{a+}^{\alpha}f(b)}{2}-\frac{1}{\Gamma(\alpha+1)}\left(\frac{b-a}{ba}\right)^{\alpha}\frac{f(b)+f(a)}{2}\right| \le \frac{1}{(\alpha+1)\Gamma(\alpha)}\sup_{t\in(a,b)}\left(t^{2}\left|f'(t)\right|\right)\left[\frac{b-a}{ba}\right]^{\alpha+1}. \tag{4.8}$$

In particular, we have

$$\left|R_{a+}^{\alpha}f(H(a,b))+R_{b-}^{\alpha}f(H(a,b))-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left(\frac{b-a}{ba}\right)^{\alpha}f(H(a,b))\right| \le \frac{1}{2^{\alpha}(\alpha+1)\Gamma(\alpha)}\sup_{t\in(a,b)}\left(t^{2}\left|f'(t)\right|\right)\left(\frac{b-a}{ba}\right)^{\alpha+1} \tag{4.9}$$

and

$$\left|R_{H(a,b)-}^{\alpha}f(a)+R_{H(a,b)+}^{\alpha}f(b)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left(\frac{b-a}{ba}\right)^{\alpha}\frac{f(a)+f(b)}{2}\right| \le \frac{1}{2^{\alpha}(\alpha+1)\Gamma(\alpha)}\sup_{t\in(a,b)}\left(t^{2}\left|f'(t)\right|\right)\left(\frac{b-a}{ba}\right)^{\alpha+1}. \tag{4.10}$$

References

[1] A. Aglić Aljinović, Montgomery identity and Ostrowski type inequalities for Riemann-Liouville fractional integral. *J. Math.* **2014**, Art. ID 503195, 6 pp.

[2] A. O. Akdemir, Inequalities of Ostrowski's type for m- and (α, m)-logarithmically convex functions via Riemann-Liouville fractional integrals. *J. Comput. Anal. Appl.* **16** (2014), no. 2, 375–383

[3] G. A. Anastassiou, Fractional representation formulae under initial conditions and fractional Ostrowski type inequalities. *Demonstr. Math.* **48** (2015), no. 3, 357–378

[4] G. A. Anastassiou, The reduction method in fractional calculus and fractional Ostrowski type inequalities. *Indian J. Math.* **56** (2014), no. 3, 333–357.

[5] H. Budak, M. Z. Sarikaya, E. Set, Generalized Ostrowski type inequalities for functions whose local fractional derivatives are generalized s-convex in the second sense. *J. Appl. Math. Comput. Mech.* **15** (2016), no. 4, 11–21.

[6] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view. *Handbook of analytic-computational methods in applied mathematics*, 135–200, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[7] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications. *Comput. Math. Appl.* **38** (1999), no. 11-12, 33–37.

[8] S. S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation. *Bull. Austral. Math. Soc.* **60** (1999), No. 3, 495–508.

[9] S. S. Dragomir, On the midpoint quadrature formula for mappings with bounded variation and applications. *Kragujevac J. Math.* **22** (2000), 13–19.

[10] S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. Appl.* **4** (2001), No. 1, 59-66. Preprint: *RGMIA Res. Rep. Coll.* **2** (1999), Art. 7, [Online: http://rgmia.org/papers/v2n1/v2n1-7.pdf]

[11] S. S. Dragomir, Refinements of the Ostrowski inequality in terms of the cumulative variation and applications, *Analysis* (Berlin) **34** (2014), No. 2, 223–240. Preprint: *RGMIA Res. Rep. Coll.* **16** (2013), Art. 29 [Online:http://rgmia.org/papers/v16/v16a29.pdf].

[12] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results, *Australian J. Math. Anal. Appl.*, Volume **14**, Issue 1, Article 1, pp. 1-287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[13] S. S. Dragomir, Ostrowski type inequalities for Riemann-Liouville fractional integrals of bounded variation, Hölder and Lipschitzian functions, Preprint *RGMIA Res. Rep. Coll.* **20** (2017), Art. 48. [Online http://rgmia.org/papers/v20/v20a48.pdf].

[14] S. S. Dragomir, Ostrowski type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation, Preprint *RGMIA Res. Rep. Coll.* **20** (2017), Art 58. [Online http://rgmia.org/papers/v20/v20a58.pdf].

[15] S. S. Dragomir, Trapezoid type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation, Preprint *RGMIA Res. Rep. Coll.* **20** (2017), Art 61. [Online http://rgmia.org/papers/v20/v20a61.pdf].

[16] S. S. Dragomir, M. S. Moslehian and Y. J. Cho, Some reverses of the Cauchy-Schwarz inequality for complex functions of self-adjoint operators in Hilbert spaces. *Math. Inequal. Appl.* **17** (2014), no. 4, 1365–1373.

[17] A. Guezane-Lakoud and F. Aissaoui, New fractional inequalities of Ostrowski type. *Transylv. J. Math. Mech.* **5** (2013), no. 2, 103–106

[18] A. Kashuri and R. Liko, Ostrowski type fractional integral inequalities for generalized (s, m, φ)-preinvex functions. *Aust. J. Math. Anal. Appl.* **13** (2016), no. 1, Art. 16, 11 pp.

[19] A. Kilbas, A; H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006. xvi+523 pp. ISBN: 978-0-444-51832-3; 0-444-51832-0

[20] M. A. Noor, K. I.Noor and S. Iftikhar, Fractional Ostrowski inequalities for harmonic h-preinvex functions. *Facta Univ. Ser. Math. Inform.* **31** (2016), no. 2, 417–445.

[21] J. E. Pečarić, S. S. Dragomir, and J. Sándor, On some Grüss type inequalities for isotonic functionals. *Rad Hrvatske Akad. Znan. Umjet.* No. **467** (1994), 41–47. [Online http://dizbi.hazu.hr/object/view/jA8Gc5N7bm].

[22] M. Z. Sarikaya and H. Filiz, Note on the Ostrowski type inequalities for fractional integrals. *Vietnam J. Math.* **42** (2014), no. 2, 187–190

[23] M. Z. Sarikaya and H. Budak, Generalized Ostrowski type inequalities for local fractional integrals. *Proc. Amer. Math. Soc.* **145** (2017), no. 4, 1527–1538.

[24] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Comput. Math. Appl.* 63 (2012), no. 7, 1147–1154.

[25] M. Tunç, On new inequalities for h-convex functions via Riemann-Liouville fractional integration, *Filomat* **27**:4 (2013), 559–565.

[26] M. Tunç, Ostrowski type inequalities for m- and (α, m)-geometrically convex functions via Riemann-Louville fractional integrals. *Afr. Mat.* **27** (2016), no. 5-6, 841–850.

[27] H. Yildirim and Z. Kirtay, Ostrowski inequality for generalized fractional integral and related inequalities, *Malaya J. Mat.*, **2**(3)(2014), 322-329.

[28] C. Yildiz, E, Özdemir and Z. S. Muhamet, New generalizations of Ostrowski-like type inequalities for fractional integrals. *Kyungpook Math. J.* **56** (2016), no. 1, 161–172.

[29] H. Yue, Ostrowski inequality for fractional integrals and related fractional inequalities. *Transylv. J. Math. Mech.* **5** (2013), no. 1, 85–89.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa