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HERMITE-HADAMARD TYPE INEQUALITIES OF FIRST KIND FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS

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Abstract. In this paper we establish some Hermite-Hadamard type inequalities for the Generalized Riemann-Liouville fractional integrals $I_{a+,g}^{\alpha}f$ and $I_{b-,g}^{\alpha}f$, g, where g is a strictly increasing function on (a,b), having a continuous derivative on (a,b) and under the assumption that the composite function $f\circ g^{-1}$ is convex on $(g(a),g(b))$. Some applications for Hadamard and Exponential fractional integrals are also provided.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\leq\frac{f(a)+f(b)}{2},$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality*.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [10], the recent survey paper [9] and the references therein.

Let (a,b) with $-\infty\leq a<b\leq\infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\operatorname{Re}(\alpha)>0$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Following [13, p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ by

$$I_{a+,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{g'(t)\,f(t)\,dt}{[g(x)-g(t)]^{1-\alpha}},\ a<x\leq b \tag{1.1}$$

and

$$I_{b-,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\frac{g'(t)\,f(t)\,dt}{[g(t)-g(x)]^{1-\alpha}},\ a\leq x<b. \tag{1.2}$$

For $g(t)=t$ we have the classical *Riemann-Liouville fractional integrals*

$$J_{a+}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{f(t)\,dt}{(x-t)^{1-\alpha}},\ a<x\leq b \tag{1.3}$$

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and

$$J_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha}},\ a \le x < b, \tag{1.4}$$

while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [13, p. 111]

$$H_{a+}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0 \le a < x \le b \tag{1.5}$$

and

$$H_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0 \le a < x < b. \tag{1.6}$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the *"β-Exponential fractional integrals"*

$$E_{a+,\beta}^{\alpha}f(x) := \frac{\beta}{\Gamma(\alpha)}\int_a^x \frac{\exp(\beta t) f(t)\,dt}{[\exp(\beta x) - \exp(\beta t)]^{1-\alpha}},\ a < x \le b \tag{1.7}$$

and

$$E_{b-,\beta}^{\alpha}f(x) := \frac{\beta}{\Gamma(\alpha)}\int_x^b \frac{\exp(\beta t) f(t)\,dt}{[\exp(\beta t) - \exp(\beta x)]^{1-\alpha}},\ a \le x < b. \tag{1.8}$$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a, b \in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a,b) = A(a,b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0,\infty)$ and $g(t) = \ln t$, then $M_g(a,b) = G(a,b) := \sqrt{ab}$, the *geometric mean*. If $I = (0,\infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a,b) = H(a,b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0,\infty)$ and $g(t) = t^p$, $p \ne 0$, then $M_g(a,b) = M_p(a,b) := \left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I = \mathbb{R}$ and $g(t) = \exp t$, then

$$M_g(a,b) = LME(a,b) := \ln\left(\frac{\exp a + \exp b}{2}\right),$$

the *LogMeanExp function*.

In this paper we establish some Hermite-Hadamard type inequalities for the operator

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{I_{a+,g}^{\alpha}f(x)}{(g(x)-g(a))^{\alpha}} + \frac{I_{b-,g}^{\alpha}f(x)}{(g(b)-g(x))^{\alpha}}\right],\ x \in (a,b)$$

under the assumption that the composite function $f \circ g^{-1}$ is convex on $(g(a), g(b))$. We call these inequalities of first kind, to make a distinction between these and the inequalities for the dual operator

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{x-,g}^{\alpha}f(a)}{(g(x)-g(a))^{\alpha}} + \frac{I_{x+,g}^{\alpha}f(b)}{(g(b)-g(x))^{\alpha}}\right],\ x \in (a,b)$$

that will be considered in another work.

Some applications for Hadamard and Exponential fractional integrals are also provided.

2. Main Results

We have:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f\circ g^{-1}$ is convex on $(g(a),g(b))$, then for any $x\in(a,b)$ we have the inequalities*

$$
\begin{aligned}
&\frac{1}{\alpha}f\circ g^{-1}\left(\frac{\alpha}{\alpha+1}\frac{g(a)+g(b)}{2}+\frac{1}{\alpha+1}g(x)\right)\\
&\le\int_0^1 s^{\alpha-1}f\circ g^{-1}\left(s\frac{g(a)+g(b)}{2}+(1-s)g(x)\right)ds\\
&\le\frac{1}{2}\Gamma(\alpha)\left[\frac{I^{\alpha}_{a+,g}f(x)}{(g(x)-g(a))^{\alpha}}+\frac{I^{\alpha}_{b-,g}f(x)}{(g(b)-g(x))^{\alpha}}\right]\\
&\le\frac{1}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{1}{\alpha}f(x)\right].
\end{aligned}
\tag{2.1}
$$

Proof. Using the change of variable $u=g(t)$, then we have $du=g'(t)\,dt$, $t=g^{-1}(u)$ and

$$I^{\alpha}_{a+,g}f(x)=\frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x)}\frac{f\circ g^{-1}(u)\,du}{[g(x)-u]^{1-\alpha}}$$

for $a<x\le b$ and

$$I^{\alpha}_{b-,g}f(x)=\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)}\frac{f\circ g^{-1}(u)\,du}{[u-g(x)]^{1-\alpha}}$$

for $a\le x<b$.

Further, if we change the variable $u=(1-s)g(a)+sg(x)$, with $s\in[0,1]$, then for $a<x\le b$ we have

$$
\begin{aligned}
&I^{\alpha}_{a+,g}f(x)\\
&=\frac{1}{\Gamma(\alpha)}(g(x)-g(a))^{\alpha}\int_0^1(1-s)^{\alpha-1}f\circ g^{-1}((1-s)g(a)+sg(x))\,ds\\
&=\frac{1}{\Gamma(\alpha)}(g(x)-g(a))^{\alpha}\int_0^1 s^{\alpha-1}f\circ g^{-1}(sg(a)+(1-s)g(x))\,ds
\end{aligned}
$$

where for the last equality we replaced s by $1-s$.

If we change the variable $u=(1-s)g(x)+sg(b)$, with $s\in[0,1]$, then for $a\le x<b$ we also have

$$
\begin{aligned}
I^{\alpha}_{b-,g}f(x)&=\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)}\frac{f\circ g^{-1}(u)\,du}{[u-g(x)]^{1-\alpha}}\\
&=\frac{1}{\Gamma(\alpha)}(g(b)-g(x))^{\alpha}\int_0^1 s^{\alpha-1}f\circ g^{-1}((1-s)g(x)+sg(b))\,ds.
\end{aligned}
$$

Therefore we obtain the equality of interest

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{I^{\alpha}_{a+,g}f(x)}{(g(x)-g(a))^{\alpha}}+\frac{I^{\alpha}_{b-,g}f(x)}{(g(b)-g(x))^{\alpha}}\right] \tag{2.2}$$
$$=\frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\circ g^{-1}(sg(a)+(1-s)g(x))+f\circ g^{-1}((1-s)g(x)+sg(b))\right]ds$$

for any $x\in(a,b)$.

By the convexity of $f\circ g^{-1}$ on $(g(a),g(b))$ we get

$$\frac{f\circ g^{-1}(sg(a)+(1-s)g(x))+f\circ g^{-1}((1-s)g(x)+sg(b))}{2}$$
$$\geq f\circ g^{-1}\left(s\frac{g(a)+g(b)}{2}+(1-s)g(x)\right)$$

for any $s\in[0,1]$.

On multiplying this inequality by $s^{\alpha-1}$ and taking the integral over s, we get

$$\int_0^1 s^{\alpha-1}\frac{f\circ g^{-1}(sg(a)+(1-s)g(x))+f\circ g^{-1}((1-s)g(x)+sg(b))}{2}ds$$
$$\geq\int_0^1 s^{\alpha-1}f\circ g^{-1}\left(s\frac{g(a)+g(b)}{2}+(1-s)g(x)\right)ds,$$

which proves the second inequality in (2.1).

By Jensen's inequality for the convex function $f\circ g^{-1}$ we also have

$$\frac{\int_0^1 s^{\alpha-1}f\circ g^{-1}\left(s\frac{g(a)+g(b)}{2}+(1-s)g(x)\right)ds}{\int_0^1 s^{\alpha-1}ds}$$
$$\geq f\circ g^{-1}\left(\frac{\int_0^1 s^{\alpha-1}\left(s\frac{g(a)+g(b)}{2}+(1-s)g(x)\right)ds}{\int_0^1 s^{\alpha-1}ds}\right)$$
$$=f\circ g^{-1}\left(\frac{\alpha}{\alpha+1}\frac{g(a)+g(b)}{2}+\frac{1}{\alpha+1}g(x)\right)$$

giving that

$$\int_0^1 s^{\alpha-1}f\circ g^{-1}\left(s\frac{g(a)+g(b)}{2}+(1-s)g(x)\right)ds$$
$$\geq\frac{1}{\alpha}f\circ g^{-1}\left(\frac{\alpha}{\alpha+1}\frac{g(a)+g(b)}{2}+\frac{1}{\alpha+1}g(x)\right),$$

which proves the first inequality in (2.1).

By the convexity of $f\circ g^{-1}$ we also have

$$\frac{f\circ g^{-1}(sg(a)+(1-s)g(x))+f\circ g^{-1}((1-s)g(x)+sg(b))}{2}$$
$$\leq\frac{1}{2}\left[s\left(f\circ g^{-1}\right)(g(a))+(1-s)\left(f\circ g^{-1}\right)(g(x))\right]$$
$$+\frac{1}{2}\left[(1-s)\left(f\circ g^{-1}\right)(g(x))+s\left(f\circ g^{-1}\right)(g(b))\right]$$
$$=s\frac{f(a)+f(b)}{2}+(1-s)f(x)$$

for any $x \in (a,b)$ and $s \in [0,1]$.

On multiplying this inequality by $s^{\alpha-1}$ and taking the integral over s, we get

$$\begin{aligned}
&\frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(x\right)\right)+f\circ g^{-1}\left(\left(1-s\right)g\left(x\right)+sg\left(b\right)\right)\right]ds\\
&\le \int_0^1\left[s^{\alpha}\frac{f\left(a\right)+f\left(b\right)}{2}+\left(1-s\right)s^{\alpha-1}f\left(x\right)\right]ds\\
&=\frac{1}{\alpha+1}\left[\frac{f\left(a\right)+f\left(b\right)}{2}+\frac{1}{\alpha}f\left(x\right)\right]
\end{aligned}$$

for any $x \in (a,b)$, which proves the last part of (2.1). □

Corollary 1. *With the assumptions of Theorem 1, we have*

$$\begin{aligned}
f\left(M_g\left(a,b\right)\right) &\le \frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}\left[I^{\alpha}_{a+,g}f(M_g\left(a,b\right))+I^{\alpha}_{b-,g}f(M_g\left(a,b\right))\right]\\
&\le \frac{\alpha}{\alpha+1}\left[\frac{f\left(a\right)+f\left(b\right)}{2}+\frac{1}{\alpha}f\left(M_g\left(a,b\right)\right)\right]\le\frac{f\left(a\right)+f\left(b\right)}{2}.
\end{aligned}\tag{2.3}$$

Proof. If we take $x = M_g(a,b)$ in (2.1), then we get

$$\begin{aligned}
&\frac{1}{\alpha}f\circ g^{-1}\left(\frac{\alpha}{\alpha+1}\frac{g\left(a\right)+g\left(b\right)}{2}+\frac{1}{\left(\alpha+1\right)}g\left(M_g\left(a,b\right)\right)\right)\\
&\le\int_0^1 s^{\alpha-1}f\circ g^{-1}\left(s\frac{g\left(a\right)+g\left(b\right)}{2}+\left(1-s\right)g\left(M_g\left(a,b\right)\right)\right)ds\\
&\le\frac{1}{2}\Gamma\left(\alpha\right)\left[\frac{I^{\alpha}_{a+,g}f(M_g\left(a,b\right))}{\left(g\left(M_g\left(a,b\right)\right)-g\left(a\right)\right)^{\alpha}}+\frac{I^{\alpha}_{b-,g}f(M_g\left(a,b\right))}{\left(g\left(b\right)-g\left(M_g\left(a,b\right)\right)\right)^{\alpha}}\right]\\
&\le\frac{1}{\alpha+1}\left[\frac{f\left(a\right)+f\left(b\right)}{2}+\frac{1}{\alpha}f\left(M_g\left(a,b\right)\right)\right]
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
&\frac{1}{\alpha}f\left(M_g\left(a,b\right)\right)\le 2^{\alpha-1}\Gamma\left(\alpha\right)\left[\frac{I^{\alpha}_{a+,g}f(M_g\left(a,b\right))}{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}+\frac{I^{\alpha}_{b-,g}f(M_g\left(a,b\right))}{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}\right]\\
&\le\frac{1}{\alpha+1}\left[\frac{f\left(a\right)+f\left(b\right)}{2}+\frac{1}{\alpha}f\left(M_g\left(a,b\right)\right)\right].
\end{aligned}\tag{2.4}$$

By the convexity of $f\circ g^{-1}$ we also have

$$\begin{aligned}
f\left(M_g\left(a,b\right)\right)=f\circ g^{-1}\left(\frac{g\left(a\right)+g\left(b\right)}{2}\right)&\le\frac{\left(f\circ g^{-1}\right)\left(g\left(a\right)\right)+f\circ g^{-1}\left(g\left(b\right)\right)}{2}\\
&=\frac{f\left(a\right)+f\left(b\right)}{2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{\alpha+1}\left[\frac{f\left(a\right)+f\left(b\right)}{2}+\frac{1}{\alpha}f\left(M_g\left(a,b\right)\right)\right]&\le\frac{1}{\alpha+1}\left[\frac{f\left(a\right)+f\left(b\right)}{2}+\frac{1}{\alpha}\frac{f\left(a\right)+f\left(b\right)}{2}\right]\\
&=\frac{1}{\alpha}\frac{f\left(a\right)+f\left(b\right)}{2}
\end{aligned}$$

and by (2.4) we get the desired result (2.3). □

Corollary 2. *With the assumptions of Theorem 1, we have the integral inequalities*

$$\begin{aligned}
(2.5)\qquad & \frac{1}{\alpha} f\circ g^{-1}\left(\frac{\alpha}{\alpha+1}\frac{g(a)+g(b)}{2}+\frac{1}{\alpha+1}\frac{1}{b-a}\int_a^b g(x)\,dx\right)\\
&\le \frac{1}{\alpha}\frac{1}{b-a}\int_a^b f\circ g^{-1}\left(\frac{\alpha}{\alpha+1}\frac{g(a)+g(b)}{2}+\frac{1}{\alpha+1}g(x)\right)dx\\
&\le \frac{1}{b-a}\int_a^b\left(\int_0^1 s^{\alpha-1} f\circ g^{-1}\left(s\frac{g(a)+g(b)}{2}+(1-s)\,g(x)\right)ds\right)dx\\
&\le \frac{1}{2}\Gamma(\alpha)\frac{1}{b-a}\int_a^b\left[\frac{I^{\alpha}_{a+,g}f(x)}{(g(x)-g(a))^{\alpha}}+\frac{I^{\alpha}_{b-,g}f(x)}{(g(b)-g(x))^{\alpha}}\right]dx\\
&\le \frac{1}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{1}{\alpha}\frac{1}{b-a}\int_a^b f(x)\,dx\right].
\end{aligned}$$

The first inequality in (2.5) follows by Jensen's inequality for the convex function $f\circ g^{-1}$. The rest follows by taking the integral mean in (2.1).

Remark 1. *In the case of $g(t)=t$, $t\in[a,b]$, and if f is convex on (a,b), then we have by (2.1) and (2.3) the following inequalities for the classical Riemann-Liouville fractional integrals J^{α}_{a+} and J^{α}_{b-} that have been obtained in* [8]

$$\begin{aligned}
(2.6)\qquad \frac{1}{\alpha} f\left(\frac{\alpha}{\alpha+1}\frac{a+b}{2}+\frac{1}{\alpha+1}x\right) &\le \int_0^1 s^{\alpha-1} f\left(s\frac{a+b}{2}+(1-s)\,x\right)ds\\
&\le \frac{1}{2}\Gamma(\alpha)\left[\frac{J^{\alpha}_{a+}f(x)}{(x-a)^{\alpha}}+\frac{J^{\alpha}_{b-}f(x)}{(b-x)^{\alpha}}\right]\\
&\le \frac{1}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{1}{\alpha}f(x)\right]
\end{aligned}$$

for any $x\in(a,b)$ and

$$\begin{aligned}
(2.7)\qquad f\left(\frac{a+b}{2}\right) &\le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(\frac{a+b}{2})+J^{\alpha}_{b-}f(\frac{a+b}{2})\right]\\
&\le \frac{\alpha}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{1}{\alpha}f\left(\frac{a+b}{2}\right)\right]\le\frac{f(a)+f(b)}{2}.
\end{aligned}$$

From (2.5), we have for a convex function f that

$$\begin{aligned}
(2.8)\qquad f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b f\left(\frac{\alpha}{\alpha+1}\frac{a+b}{2}+\frac{1}{\alpha+1}x\right)dx\\
&\le \frac{1}{b-a}\alpha\int_a^b\left(\int_0^1 s^{\alpha-1} f\left(s\frac{a+b}{2}+(1-s)\,x\right)ds\right)dx\\
&\le \frac{1}{2}\Gamma(\alpha+1)\frac{1}{b-a}\int_a^b\left[\frac{J^{\alpha}_{a+}f(x)}{(x-a)^{\alpha}}+\frac{J^{\alpha}_{b-}f(x)}{(b-x)^{\alpha}}\right]dx\\
&\le \frac{\alpha}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{1}{\alpha}\frac{1}{b-a}\int_a^b f(x)\,dx\right]\le\frac{f(a)+f(b)}{2}.
\end{aligned}$$

For some recent papers on Hermite-Hadamard inequalities for the Riemann-Liouville fractional integrals see [1], [3], [4], [11]-[26] and the references therein.

3. Application for GA-Convex Functions

Let $I \subset (0, \infty)$ be an interval; a real-valued function $f : I \to \mathbb{R}$ is said to be *GA-convex* (concave) on I if

$$f\left(x^{1-\lambda} y^{\lambda}\right) \leq (\geq) (1-\lambda) f(x) + \lambda f(y) \tag{3.1}$$

for all x, $y \in I$ and $\lambda \in [0, 1]$.

Since the condition (3.1) can be written as

$$f \circ \exp((1-\lambda)\ln x + \lambda \ln y) \leq (\geq) (1-\lambda) f \circ \exp(\ln x) + \lambda f \circ \exp(\ln y), \tag{3.2}$$

then we observe that $f : I \to \mathbb{R}$ is *GA-convex* (concave) on I if and only if $f \circ \exp$ is convex (concave) on $\ln I := \{\ln z, z \in I\}$. If $I = [a, b]$ then $\ln I = [\ln a, \ln b]$.

It is known that the function $f(x) = \ln(1+x)$ is GA-convex on $(0, \infty)$ [2].

For real and positive values of x, the *Euler gamma* function Γ and its *logarithmic derivative* ψ, the so-called *digamma function*, are defined by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt \text{ and } \psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

It has been shown in [25] that the function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \psi(x) + \frac{1}{2x}$$

is GA-concave on $(0, \infty)$ while the function $g : (0, \infty) \to \mathbb{R}$ defined by

$$g(x) = \psi(x) + \frac{1}{2x} + \frac{1}{12x^2}$$

is GA-convex on $(0, \infty)$.

For some recent inequalities on GA-convex functions see [5]-[7].

If we assume that the function $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ is *GA-convex* on $[a, b]$, then by Theorem 1 for $g : [a, b] \subset (0, \infty) \to \mathbb{R}$, $g(t) = \ln t$, we have the following inequalities for Hadamard fractional integrals

$$\begin{aligned} \frac{1}{\alpha} f\left([G(a,b)]^{\frac{\alpha}{\alpha+1}} x^{\frac{1}{\alpha+1}}\right) &\leq \int_0^1 s^{\alpha-1} f\left([G(a,b)]^s x^{1-s}\right) ds \\ &\leq \frac{1}{2}\Gamma(\alpha)\left[\frac{H_{a+}^{\alpha} f(x)}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}} + \frac{H_{b-}^{\alpha} f(x)}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}\right] \\ &\leq \frac{1}{\alpha+1}\left[\frac{f(a)+f(b)}{2} + \frac{1}{\alpha} f(x)\right], \end{aligned} \tag{3.3}$$

for any $x \in (a, b)$, where $G(a, b) := \sqrt{ab}$ is the geometric mean of a, b.

From (2.3) we have

$$\begin{aligned} f(G(a,b)) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}\left[H_{a+}^{\alpha} f(G(a,b)) + H_{b-}^{\alpha} f(G(a,b))\right] \\ &\leq \frac{\alpha}{\alpha+1}\left[\frac{f(a)+f(b)}{2} + \frac{1}{\alpha} f(G(a,b))\right] \leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{3.4}$$

provided $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ is GA-convex.

We also have the integral inequalities

$$
\begin{aligned}
(3.5)\qquad & \frac{1}{\alpha} f\left([G(a,b)]^{\frac{\alpha}{\alpha+1}}[I(a,b)]^{\frac{1}{\alpha+1}}\right) \\
& \leq \frac{1}{\alpha}\frac{1}{b-a}\int_a^b f\left([G(a,b)]^{\frac{\alpha}{\alpha+1}} x^{\frac{1}{\alpha+1}}\right)dx \\
& \leq \frac{1}{b-a}\int_a^b \left(\int_0^1 s^{\alpha-1} f\left([G(a,b)]^s x^{1-s}\right)ds\right)dx \\
& \leq \frac{1}{2}\Gamma(\alpha)\frac{1}{b-a}\int_a^b \left[\frac{H_{a+}^{\alpha} f(x)}{(\ln x-\ln a)^{\alpha}}+\frac{H_{b-}^{\alpha} f(x)}{(\ln b-\ln x)^{\alpha}}\right]dx \\
& \leq \frac{1}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{1}{\alpha}\frac{1}{b-a}\int_a^b f(x)\,dx\right].
\end{aligned}
$$

since $\ln I(a,b)=\frac{1}{b-a}\int_a^b \ln x dx$ and the *identric mean* $I(a,b)$ is defined by $I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ for $0<a<b$.

4. Application for Exponential Fractional Integral

Consider the *"Exponential fractional integrals"*

$$E_{a+}^{\alpha} f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x \frac{\exp(t)\,f(t)\,dt}{[\exp(x)-\exp(t)]^{1-\alpha}},\ a<x\leq b$$

and

$$E_{b-}^{\alpha} f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b \frac{\exp(t)\,f(t)\,dt}{[\exp(t)-\exp(x)]^{1-\alpha}},\ a\leq x<b.$$

Let $f:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]\subset\mathbb{R}$. If $f\circ\ln$ is convex on $(\exp(a),\exp(b))$, then for any $x\in(a,b)$ we have by Theorem 1 the inequalities

$$
\begin{aligned}
(4.1)\qquad & \frac{1}{\alpha} f\left[\ln\left(\frac{\alpha}{\alpha+1}\frac{\exp a+\exp b}{2}+\frac{1}{\alpha+1}\exp x\right)\right] \\
& \leq \int_0^1 s^{\alpha-1} f\left[\ln\left(s\frac{\exp a+\exp b}{2}+(1-s)\exp x\right)\right]ds \\
& \leq \frac{1}{2}\Gamma(\alpha)\left[\frac{E_{a+}^{\alpha} f(x)}{(\exp x-\exp a)^{\alpha}}+\frac{E_{b-}^{\alpha} f(x)}{(\exp b-\exp x)^{\alpha}}\right] \\
& \leq \frac{1}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{1}{\alpha} f(x)\right].
\end{aligned}
$$

From (2.3) we have

$$
\begin{aligned}
(4.2)\qquad & f(LME(a,b)) \\
& \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\exp b-\exp a)^{\alpha}}\left[E_{a+}^{\alpha} f(LME(a,b))+E_{b-}^{\alpha} f(LME(a,b))\right] \\
& \leq \frac{\alpha}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{1}{\alpha} f(LME(a,b))\right]\leq\frac{f(a)+f(b)}{2},
\end{aligned}
$$

where $LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right)$, provided that $f\circ\ln$ is convex on $(\exp(a),\exp(b))$.

Finally, by (2.5) we have

$$\begin{aligned}
&\frac{1}{\alpha}f\left[\ln\left(\frac{\alpha}{\alpha+1}\frac{\exp a+\exp b}{2}+\frac{1}{\alpha+1}\frac{\exp b-\exp a}{b-a}\right)\right] \\
&\leq \frac{1}{\alpha}\frac{1}{b-a}\int_a^b f\left[\ln\left(\frac{\alpha}{\alpha+1}\frac{\exp a+\exp b}{2}+\frac{1}{\alpha+1}\exp x\right)\right]dx \\
&\leq \frac{1}{b-a}\int_a^b\left(\int_0^1 s^{\alpha-1}f\left[\ln\left(s\frac{\exp a+\exp b}{2}+(1-s)\exp x\right)\right]ds\right)dx \\
&\leq \frac{1}{2}\Gamma(\alpha)\frac{1}{b-a}\int_a^b\left[\frac{E_{a+}^{\alpha}f(x)}{(g(x)-g(a))^{\alpha}}+\frac{E_{b-}^{\alpha}f(x)}{(g(b)-g(x))^{\alpha}}\right]dx \\
&\leq \frac{1}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{1}{\alpha}\frac{1}{b-a}\int_a^b f(x)\,dx\right],
\end{aligned} \tag{4.3}$$

provided that $f\circ\ln$ is convex on $(\exp(a),\exp(b))$.

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