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NEW OSTROWSKI TYPE FRACTIONAL INEQUALITIES FOR h-CONVEX FUNCTIONS VIA CAPUTO $k-$FRACTIONAL DERIVATIVE

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ABSTRACT. In this paper by using Caputo k-fractional derivatives we give some new fractional inequalities of Ostrowski type for h-convex functions. Also we deduce some results for $p-$funtions, convex functions and $s-$convex funtions in the second sense.

1. INTRODUCTION

The following inequality is known as Ostrowski inequality [13] (see also, [12, page 468]) which gives upper bound for approximation of integral average by the value $f(x)$ at point $x \in [a,b]$. It is provided by Ostrowski in 1938.

Theorem 1.1. *Let $f : I \to R$ where I is interval in $\mathbb{R}$ be a mapping differentiable in $I^\circ$the interior of I and $a, b \in I^\circ$, $a < b$. If $\left|f'(t)\right| \leq M$ for all $t \in [a,b]$, then we have*

$$\left|f(x) - \frac{1}{(b-a)}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M,\ x \in [a,b].$$

In numerical analysis many quadrature rules have been established to approximate the definite integrals. Ostrowski inequality provides the bounds of many numerical quadrature rules (see [2] and references there in). In recent decades Ostrowski inequality is studied in fractional calculus point of view by many mathematicians (see [5, 11, 10] and references their in).
We are interested to give Ostrowski type inequalities for mapping whose n^{th} derivative is $h-$convex via Caputo $k-$fractional derivatives.

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Definition 1.2. A function f is called convex function on the interval $[a,b]$ if for any two points $x, y \in [a,b]$ and any t where, $0 \leq t \leq 1$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Definition 1.3. [3] A non-negative function $f : I \to \mathbb{R}$ is said to be p-function, if for any two points $x, y \in I$ and $t \in [0,1]$

$$f\left(tx + (1-t)y\right) \leq f(x) + f(y).$$

s-convex functions in the second sense have been introduced by Hudzik and Maligranda in [8] as follows.

Definition 1.4. [8] A function $f : [0,\infty) \to \mathbb{R}$ is called $s-$convex in the second sense on the interval $[0,\infty)$ if for any two points $x, y \in [0,\infty)$ and any t where, $0 \leq t \leq 1$ and for some fixed $s \in (0,1]$

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y).$$

Definition 1.5. [14] Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a positive function. We say $f : I \to \mathbb{R}$ is a h-convex function, if f is non-negative and

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y) \tag{1}$$

for all $x, y \in I$ and $t \in (0,1)$. If above inequality is reversed, then f is called h-concave.

It is easy to see that
(i) If $h(t) = t$, then (1) gives non-negative convex functions.
(ii) If $h(t) = 1$, then (1) gives p-function.
(iii) If $h(t) = t^s$ where $s \in (0,1)$, then (1) gives s-convex function in the second sense.
In the following we give definitions of k-gamma and k-beta functions as well as their relationship.

Definition 1.6. [4] For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-gamma function is defined by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n nk^{\frac{x}{k}-1}}{(x)_{n,k}}.$$

Its integral representation is given by

$$\Gamma_k\left(\alpha\right) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \tag{2}$$

One can note that

$$\Gamma_k\left(\alpha + k\right) = \alpha \Gamma_k\left(\alpha\right).$$

For $k = 1$, (2) gives integral representation of gamma function.

Definition 1.7. [4] For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-beta function with two parameters x and y is defined as

$$\beta_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt. \tag{3}$$

For $k = 1$, (3) gives integral representation of beta function.

Theorem 1.8. [4] *Let $x, y > 0$, then for k-gamma and k-beta function following equality holds*

$$\beta_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \tag{4}$$

Definition 1.9. [9] Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in C^n[a, b]$ such that $f^{(n)}$ exists and are continuous on $[a, b]$. The Caputo fractional derivatives of order α are defined as follows:

$$^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, x > a \tag{5}$$

and

$$^{C}D^{\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, x < b \tag{6}$$

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative of order n exists, then Caputo fractional derivative $(^{C}D^{\alpha}_{a+}f)(x)$ coincides with $f^{(n)}(x)$. In particular we have

$$(^{C}D^{0}_{a+}f)(x) = (^{C}D^{0}_{b-}f)(x) = f(x) \tag{7}$$

where $n = 1$ and $\alpha = 0$.

In the following we define Caputo k-fractional derivatives.

Definition 1.10. [6] Let $\alpha > 0, k \geq 1$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha]+1$, $f \in C^n[a, b]$ such that $f^{(n)}$ exists and are continuous on $[a, b]$. Then Caputo k-fractional derivatives of order α are defined as follows:

$$^{C}D^{\alpha,k}_{a+}f(x) = \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}}dt, x > a \tag{8}$$

and

$$^{C}D^{\alpha,k}_{b-}f(x) = \frac{(-1)^n}{k\Gamma_k(n-\frac{\alpha}{k})}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}}dt, x < b \tag{9}$$

We organize the paper in such a way that in the following section we prove some Ostrowski type inequalities for mappings whose n^{th} times derivative is h-convex via Caputo k-fractional derivatives.
Through out the paper $C^n[a,b]$ denotes the space of $n-$times differentiable functions such that $f^{(n)}$ are continuous on $[a,b]$.

2. Ostrowski type Caputo k-fractional inequalities for mappings whoes n^{th} times derivative is h-convex

In this section we present some Ostrowski type inequalities for h-convex functions via Caputo k-fractional integrals. The following lemma is very useful to obtain our results.

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f\in C^{n+1}[a,b]$, $a<b$. Then we have the following equality for Caputo $k-$fractional derivatives*

$$\begin{aligned}
&\left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a}\right]f^{(n)}(x)-\frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a}\times\\
&\left[\ {}^{c}D^{\alpha,k}_{x^-}f(a)+(-1)^n\ {}^{c}D^{\alpha,k}_{x^+}f(b)\right]\\
&=\frac{(x-a)^{n-\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{n-\frac{\alpha}{k}}f^{(n+1)}(tx+(1-t)a)dt\\
&-\frac{(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{n-\frac{\alpha}{k}}f^{(n+1)}(tx+(1-t)b)dt. \qquad (10)
\end{aligned}$$

Proof. It is easy to see that

$$\begin{aligned}
&\int_0^1 t^{n-\frac{\alpha}{k}}f^{(n+1)}(tx+(1-t)a)dt\\
&=\frac{t^{n-\frac{\alpha}{k}}f^{(n)}(tx+(1-t)a)}{x-a}\bigg|_0^1-\frac{n-\frac{\alpha}{k}}{x-a}\int_0^1 t^{n-\frac{\alpha}{k}-1}f^{(n)}(tx+(1-t)a)dt\\
&=\frac{f^{(n)}(x)}{x-a}-\frac{n-\frac{\alpha}{k}}{x-a}\int_a^x\left(\frac{y-a}{x-a}\right)^{n-\frac{\alpha}{k}-1}\frac{f^{(n)}(y)}{x-a}dy \qquad (11)\\
&=\frac{f^{(n)}(x)}{x-a}-\frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{(x-a)^{n-\frac{\alpha}{k}+1}}\ {}^{c}D^{\alpha,k}_{x^-}f(a)
\end{aligned}$$

and

$$
\begin{aligned}
&\int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)}(tx+(1-t)b)dt \\
&= \frac{t^{n-\frac{\alpha}{k}} f^{(n)}(tx+(1-t)b)}{x-b}\Bigg|_0^1 - \frac{n-\frac{\alpha}{k}}{x-b}\int_0^1 t^{n-\frac{\alpha}{k}-1} f^{(n)}(tx+(1-t)b)dt \\
&= \frac{f^{(n)}(x)}{x-b} - \frac{n-\frac{\alpha}{k}}{x-b}\int_x^b \left(\frac{y-b}{x-b}\right)^{n-\frac{\alpha}{k}-1} \frac{f^{(n)}(y)}{x-b}dy \\
&= \frac{-f^{(n)}(x)}{b-x} + \frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{(-1)^n(b-x)^{n-\frac{\alpha}{k}+1}}\ {}^cD^{\alpha,k}_{x^+} f(b).
\end{aligned} \tag{12}
$$

Multiplying (11) by $\frac{(x-a)^{n-\frac{\alpha}{k}+1}}{b-a}$ and (12) by $-\frac{(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}$, then adding resulting equations we get (10). □

Using above lemma we give the following Ostrowski fractional inequality.

Theorem 2.2. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a function such that $f \in C^{n+1}[a,b]$, $0 \le a < b$. If $\left|f^{(n+1)}\right|$ is h-convex on $[a,b]$ and $\left|f^{(n+1)}(x)\right| \le M$, $x \in [a,b]$, then the following inequality for Caputo k-fractional derivatives holds*

$$
\begin{aligned}
&\Bigg|\left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a}\right] f^{(n)}(x) - \frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a}\times \\
&\left[\ {}^cD^{\alpha,k}_{x^-} f(a) + (-1)^n\ {}^cD^{\alpha,k}_{x^+} f(b)\right]\Bigg| \\
&\le \frac{M(b-a)}{2}\int_0^1 t^{n-\frac{\alpha}{k}}\left[h(t)+h(1-t)\right]dt; x \in [a,b].
\end{aligned} \tag{13}
$$

Proof. Using Lemma 2.1, h-convexity of $\left|f^{(n+1)}\right|$, and upper bound of $\left|f^{(n+1)}(x)\right|$ we have

$$
\begin{aligned}
&\Bigg|\left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a}\right] f^{(n)}(x)-\frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a}\times \\
&\left[{}^{c}D^{\alpha,k}_{x^-}f(a)+(-1)^n\,{}^{c}D^{\alpha,k}_{x^+}f(b)\right]\Bigg| \\
&\leq \frac{(x-a)^{n-\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}(tx+(1-t)a)\right|dt \\
&+\frac{(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}(tx+(1-t)b)\right|dt \\
&\leq \frac{(x-a)^{n-\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{n-\frac{\alpha}{k}}\left[h(t)\left|f^{(n+1)}(x)\right|+h(1-t)\left|f^{(n+1)}(a)\right|\right]dt \\
&+\frac{(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{n-\frac{\alpha}{k}}\left[h(t)\left|f^{(n+1)}(x)\right|+h(1-t)\left|f^{(n+1)}(b)\right|\right]dt \\
&\leq \frac{M(x-a)^{n-\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{n-\frac{\alpha}{k}}\left[h(t)+h(1-t)\right]dt \\
&+\frac{M(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{n-\frac{\alpha}{k}}\left[h(t)+h(1-t)\right]dt \\
&= M\left[\frac{(x-a)^{n-\frac{\alpha}{k}+1}+(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\right]\int_0^1 t^{n-\frac{\alpha}{k}}\left[h(t)+h(1-t)\right]dt.
\end{aligned}
$$

This completes the proof. □

Now we give some special cases of Theorem 2.2.
The following result for Caputo fractional derivatives holds.

Corollary 2.3. *In Theorem 2.2, if we take $k=1$, then (13) becomes the following inequality*

$$
\begin{aligned}
&\Bigg|\left[\frac{(x-a)^{n-\alpha}+(b-x)^{n-\alpha}}{b-a}\right] f^{(n)}(x)-\frac{(n-\alpha)\Gamma_k(n-\alpha)}{b-a}\times \\
&\left[{}^{c}D^{\alpha}_{x^-}f(a)+(-1)^n\,{}^{c}D^{\alpha,k}_{x^+}f(b)\right]\Bigg|
\end{aligned}
$$

(14)

$$
\leq M\left[\frac{(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}}{b-a}\right]\int_0^1 t^{n-\alpha}\left[h(t)+h(1-t)\right]dt.
$$

Corollary 2.4. *In Theorem 2.2, if we take $h(t)=1$, which means that $\left|f^{(n+1)}\right|$ is p-function, then (13) becomes the following inequality*

$$\begin{aligned}
&\Bigg|\left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a}\right]f^{(n)}(x)-\frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a}\times\\
&\left[{}^{c}D^{\alpha,k}_{x^-}f(a)+(-1)^n\,{}^{c}D^{\alpha,k}_{x^+}f(b)\right]\Bigg|\\
&\leq\frac{2M}{n-\frac{\alpha}{k}+1}\left[\frac{(x-a)^{n-\frac{\alpha}{k}+1}+(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\right];x\in[a,b]. \qquad (15)
\end{aligned}$$

Corollary 2.5. *In Theorem 2.2, if we take $h(t)=t$, which means that $\left|f^{(n+1)}\right|$ is convex function, then (13) becomes the following inequality*

$$\begin{aligned}
&\Bigg|\left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a}\right]f^{(n)}(x)-\frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a}\times\\
&\left[{}^{c}D^{\alpha,k}_{x^-}f(a)+(-1)^n\,{}^{c}D^{\alpha,k}_{x^+}f(b)\right]\Bigg|\\
&\leq\frac{M}{n-\frac{\alpha}{k}+1}\left[\frac{(x-a)^{n-\frac{\alpha}{k}+1}+(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\right];x\in[a,b]. \qquad (16)
\end{aligned}$$

Corollary 2.6. *In Theorem 2.2, if we take $h(t)=t^s$, which means that $\left|f^{(n+1)}\right|$ is s-function in the second sense, then (13) becomes the following inequality*

$$\begin{aligned}
&\Bigg|\left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a}\right]f^{(n)}(x)-\frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a}\times\\
&\left[{}^{c}D^{\alpha,k}_{x^-}f(a)+(-1)^n\,{}^{c}D^{\alpha,k}_{x^+}f(b)\right]\Bigg|\\
&\leq M\left[\frac{(x-a)^{n-\frac{\alpha}{k}+1}+(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\right]\Bigg[\frac{1}{n-\frac{\alpha}{k}+s+1}\\
&+\frac{k\Gamma_k(nk-\alpha+k)\Gamma_k(sk+k)}{\Gamma_k(nk-\alpha+sk+2k)}\Bigg];x\in[a,b]. \qquad (17)
\end{aligned}$$

Theorem 2.7. *Let $f:I\subseteq[0,\infty)\to\mathbb{R}$ be a function such that $f\in C^{n+1}[a,b]$, $0\leq a<b$. If $\left|f^{(n+1)}\right|^q, q>1$, is h-convex on $[a,b]$ and $\left|f^{(n+1)}(x)\right|\leq M$, $x\in[a,b]$, then the following inequality for Caputo* *k-fractional integrals holds*

$$\left|\left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a}\right]f^{(n)}(x)-\frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a}\times\right.$$
$$\left.\left[{}^{c}D^{\alpha,k}_{x^-}f(a)+(-1)^n\;{}^{c}D^{\alpha,k}_{x^+}f(b)\right]\right|$$
$$\leq M\left[\frac{(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}}{b-a}\right]\left(\frac{1}{p(n-\frac{\alpha}{k})+1}\right)^{\frac{1}{p}}\times$$
$$\left(\int_0^1[h(t)+h(1-t)]\,dt\right)^{\frac{1}{q}};x\in[a,b], \tag{18}$$

with $\alpha,k>0$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1 and Holder's inequality we have

$$\left|\left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a}\right]f^{(n)}(x)-\frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a}\times\right.$$
$$\left.\left[{}^{c}D^{\alpha,k}_{x^-}f(a)+(-1)^n\;{}^{c}D^{\alpha,k}_{x^+}f(b)\right]\right|$$
$$\leq\frac{(x-a)^{n-\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}(tx+(1-t)a)\right|dt$$
$$+\frac{(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}(tx+(1-t)b)\right|dt$$
$$\leq\frac{(x-a)^{n-\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{p(n-\frac{\alpha}{k})}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n+1)}(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{p(n-\frac{\alpha}{k})}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n+1)}(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f^{(n+1)}\right|^q$ is h-convex and $\left|f^{(n+1)}(x)\right| \leq M$, $x \in [a,b]$, there for we have

$$
\begin{aligned}
&\left| \left[\frac{(x-a)^{n-\frac{\alpha}{k}} + (b-x)^{n-\frac{\alpha}{k}}}{b-a} \right] f^{(n)}(x) - \frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a} \times \right. \\
&\left. \left[{}^{c}D^{\alpha,k}_{x^-} f(a) + (-1)^n \, {}^{c}D^{\alpha,k}_{x^+} f(b) \right] \right| \\
&\leq \frac{(x-a)^{n-\frac{\alpha}{k}+1}}{b-a} \left(\int_0^1 t^{p(n-\frac{\alpha}{k})} dt \right)^{\frac{1}{p}} \times \\
&\left(\int_0^1 \left[h(t) \left|f^{(n+1)}(x)\right|^q + h(1-t) \left|f^{(n+1)}(a)\right|^q \right] dt \right)^{\frac{1}{q}} \\
&+ \frac{(b-x)^{n-\frac{\alpha}{k}+1}}{b-a} \left(\int_0^1 t^{p(n-\frac{\alpha}{k})} dt \right)^{\frac{1}{p}} \times \\
&\left(\int_0^1 \left[h(t) \left|f^{(n+1)}(x)\right|^q + h(1-t) \left|f^{(n+1)}(b)\right|^q \right] dt \right)^{\frac{1}{q}} \\
&\leq \frac{M(x-a)^{n-\frac{\alpha}{k}+1}}{b-a} \left(\frac{1}{p(n-\frac{\alpha}{k})+1} \right)^{\frac{1}{p}} \left(\int_0^1 \left[h(t) + h(1-t) \right] dt \right)^{\frac{1}{q}} \\
&+ \frac{M(b-x)^{n-\frac{\alpha}{k}+1}}{b-a} \left(\frac{1}{p(n-\frac{\alpha}{k})+1} \right)^{\frac{1}{p}} \left(\int_0^1 \left[h(t) + h(1-t) \right] dt \right)^{\frac{1}{q}} \\
&= M \left[\frac{(x-a)^{n-\frac{\alpha}{k}+1} + (b-x)^{n-\frac{\alpha}{k}+1}}{b-a} \right] \left(\frac{1}{p(n-\frac{\alpha}{k})+1} \right)^{\frac{1}{p}} \times \\
&\left(\int_0^1 \left[h(t) + h(1-t) \right] dt \right)^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof. $\square$

Now we give some special cases of Theorem 2.7.
The following result for Caputo fractional derivatives holds.

Corollary 2.8. *In Theorem 2.7, if we take $k = 1$, then (18) becomes the following inequality*

$$\left| \left[\frac{(x-a)^{n-\alpha} + (b-x)^{n-\alpha}}{b-a} \right] f^{(n)}(x) - \frac{(n-\alpha)\Gamma_k(n-\alpha)}{b-a} \times \right.$$
$$\left. \left[{}^{c}D^{\alpha}_{x^-} f(a) + (-1)^n \, {}^{c}D^{\alpha,k}_{x^+} f(b) \right] \right|$$
$$\leq M \left[\frac{(x-a)^{n-\alpha+1} + (b-x)^{n-\alpha+1}}{b-a} \right] \left(\frac{1}{p(n-\alpha)+1} \right)^{\frac{1}{p}} \times$$
$$\left(\int_0^1 [h(t) + h(1-t)] \, dt \right)^{\frac{1}{q}}. \tag{19}$$

Corollary 2.9. *In Theorem 2.7, if we take $h(t) = 1$, which means that $\left| f^{(n+1)} \right|$ is p-function, then (18) becomes the following inequality*

$$\left| \left[\frac{(x-a)^{n-\frac{\alpha}{k}} + (b-x)^{n-\frac{\alpha}{k}}}{b-a} \right] f^{(n)}(x) - \frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a} \times \right.$$
$$\left. \left[{}^{c}D^{\alpha,k}_{x^-} f(a) + (-1)^n \, {}^{c}D^{\alpha,k}_{x^+} f(b) \right] \right|$$
$$\leq M \left[\frac{(x-a)^{n-\frac{\alpha}{k}+1} + (b-x)^{n-\frac{\alpha}{k}+1}}{b-a} \right] \left(\frac{1}{p(n-\frac{\alpha}{k})+1} \right)^{\frac{1}{p}} (2)^{\frac{1}{q}}, \tag{20}$$

with $x \in [a,b]$ and $\alpha, k > 0$.

Corollary 2.10. *In Theorem 2.7, if we take $h(t) = t$, which means that $\left| f^{(n+1)} \right|$ is convex function, then (18) becomes the following inequality*

$$\left| \left[\frac{(x-a)^{n-\frac{\alpha}{k}} + (b-x)^{n-\frac{\alpha}{k}}}{b-a} \right] f^{(n)}(x) - \frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a} \times \right.$$
$$\left. \left[{}^{c}D^{\alpha,k}_{x^-} f(a) + (-1)^n \, {}^{c}D^{\alpha,k}_{x^+} f(b) \right] \right|$$
$$\leq M \left[\frac{(x-a)^{n-\frac{\alpha}{k}+1} + (b-x)^{n-\frac{\alpha}{k}+1}}{b-a} \right] \left(\frac{1}{p(n-\frac{\alpha}{k})+1} \right)^{\frac{1}{p}}; x \in [a,b]. \tag{21}$$

Corollary 2.11. *In Theorem 2.7, if we take $h(t) = t^s$, which means that $\left| f^{(n+1)} \right|$ is s-function in the second sense, then (18) becomes the* *following inequality*

$$\left|\left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a}\right]f^{(n)}(x)-\frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a}\times\right.$$
$$\left.\left[{}^{c}D^{\alpha,k}_{x^-}f(a)+(-1)^n\ {}^{c}D^{\alpha,k}_{x^+}f(b)\right]\right|$$

(22)

$$\leq M\left[\frac{(x-a)^{n-\frac{\alpha}{k}+1}+(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\right]\left(\frac{1}{p(n-\frac{\alpha}{k})+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}.$$

Theorem 2.12. *Let $f : I \subseteq [0,\infty) \to \mathbb{R}$ be a function such that $f \in C^{n+1}[a,b]$, $0 \leq a < b$. If $\left|f^{(n+1)}\right|^q, q > 1$ is h-convex on $[a,b]$ $q \geq 1$, and $\left|f^{(n+1)}(x)\right| \leq M$, $x \in [a,b]$, then the following inequality for Caputo k-fractional integrals holds*

$$\left|\left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a}\right]f^{(n)}(x)-\frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a}\times\right.$$
$$\left.\left[{}^{c}D^{\alpha,k}_{x^-}f(a)+(-1)^n\ {}^{c}D^{\alpha,k}_{x^+}f(b)\right]\right|$$
$$\leq M\left[\frac{(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}}{b-a}\right]\left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}\times$$
$$\left(\int_0^1 t^{n-\frac{\alpha}{k}}\left[h(t)+h(1-t)\right]dt\right)^{\frac{1}{q}};x\in[a,b]. \tag{23}$$

Proof. Using Lemma 2.1 and power mean inequality we have

$$\Bigg| \left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a}\right] f^{(n)}(x) - \frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a}\times$$

$$\left[{}^{c}D^{\alpha,k}_{x^-}f(a)+(-1)^n\,{}^{c}D^{\alpha,k}_{x^+}f(b)\right]\Bigg|$$

$$\leq \frac{(x-a)^{n-\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}(tx+(1-t)a)\right|dt$$

$$+\frac{(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}(tx+(1-t)b)\right|dt$$

$$\leq \frac{(x-a)^{n-\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{n-\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\left(\int_0^1 t^{n-\frac{\alpha}{k}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-\frac{\alpha}{k}}\left|f^{(n+1)}(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f^{(n+1)}\right|^q$ is h-convex and $\left|f^{(n+1)}(x)\right| \leq M$, $x \in [a,b]$, there for we have

$$\left| \left[\frac{(x-a)^{n-\frac{\alpha}{k}} + (b-x)^{n-\frac{\alpha}{k}}}{b-a}\right] f^{(n)}(x) - \frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a} \times \right.$$

$$\left. \left[{}^cD^{\alpha,k}_{x^-} f(a) + (-1)^n \ {}^cD^{\alpha,k}_{x^+} f(b)\right] \right|$$

$$\leq \frac{(x-a)^{n-\frac{\alpha}{k}+1}}{b-a} \left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \times$$

$$\left(\int_0^1 t^{n-\frac{\alpha}{k}} \left[h(t)\left|f^{(n+1)}(x)\right|^q + h(1-t)\left|f^{(n+1)}(a)\right|^q\right] dt\right)^{\frac{1}{q}}$$

$$+ \frac{(b-x)^{n-\frac{\alpha}{k}+1}}{b-a} \left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \times$$

$$\left(\int_0^1 t^{n-\frac{\alpha}{k}} \left[h(t)\left|f^{(n+1)}(x)\right|^q + h(1-t)\left|f^{(n+1)}(b)\right|^q\right] dt\right)^{\frac{1}{q}}$$

$$\leq \frac{M(x-a)^{n-\frac{\alpha}{k}+1}}{b-a} \left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \left(\int_0^1 t^{n-\frac{\alpha}{k}} \left[h(t) + h(1-t)\right] dt\right)^{\frac{1}{q}}$$

$$+ \frac{M(b-x)^{n-\frac{\alpha}{k}+1}}{b-a} \left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \left(\int_0^1 t^{n-\frac{\alpha}{k}} \left[h(t) + h(1-t)\right] dt\right)^{\frac{1}{q}}$$

$$= M\left[\frac{(x-a)^{n-\alpha+1} + (b-x)^{n-\alpha+1}}{b-a}\right] \left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \times$$

$$\left(\int_0^1 t^{n-\frac{\alpha}{k}} \left[h(t) + h(1-t)\right] dt\right)^{\frac{1}{q}}.$$

This completes the proof. □

Now we give some special cases of Theorem 2.12.
The following result for Caputo fractional derivatives holds.

Corollary 2.13. *In Theorem* 2.12*, if we take $k=1$, then (23) becomes the following inequality*

$$\left| \left[\frac{(x-a)^{n-\alpha}+(b-x)^{n-\alpha}}{b-a} \right] f^{(n)}(x) - \frac{(n-\alpha)\Gamma_k(n-\alpha)}{b-a} \times \right.$$
$$\left. \left[{}^cD^{\alpha}_{x^-} f(a) + (-1)^n \, {}^cD^{\alpha,k}_{x^+} f(b) \right] \right|$$
$$\leq M \left[\frac{(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}}{b-a} \right] \left(\frac{1}{n-\alpha+1} \right)^{1-\frac{1}{q}} \times$$
$$\left(\int_0^1 t^{n-\alpha} \left[h(t) + h(1-t) \right] dt \right)^{\frac{1}{q}}. \tag{24}$$

Corollary 2.14. *In Theorem* 2.12*, if we take $h(t)=1$, which means that $\left|f^{(n+1)}\right|$ is p-function, then (23) becomes the following inequality*

$$\left| \left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a} \right] f^{(n)}(x) - \frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a} \times \right.$$
$$\left. \left[{}^cD^{\alpha,k}_{x^-} f(a) + (-1)^n \, {}^cD^{\alpha,k}_{x^+} f(b) \right] \right|$$
$$\leq M \left[\frac{(x-a)^{n-\frac{\alpha}{k}+1}+(b-x)^{n-\frac{\alpha}{k}+1}}{b-a} \right] \left(\frac{1}{n-\frac{\alpha}{k}+1} \right)^{1-\frac{1}{q}} \left(\frac{2}{n-\frac{\alpha}{k}+1} \right)^{\frac{1}{q}} \tag{25}$$

Corollary 2.15. *In Theorem* 2.12*, if we take $h(t)=t$, which means that $\left|f^{(n+1)}\right|$ is convex function, then (23) becomes the following inequality*

$$\left| \left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a} \right] f^{(n)}(x) - \frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a} \times \right.$$
$$\left. \left[{}^cD^{\alpha,k}_{x^-} f(a) + (-1)^n \, {}^cD^{\alpha,k}_{x^+} f(b) \right] \right|$$
$$\leq M \left[\frac{(x-a)^{n-\frac{\alpha}{k}+1}+(b-x)^{n-\frac{\alpha}{k}+1}}{b-a} \right] \left(\frac{1}{n-\frac{\alpha}{k}+1} \right) \tag{26}$$

Corollary 2.16. *In Theorem* 2.12*, if we take $h(t)=t^s$, which means that $\left|f^{(n+1)}\right|$ is s-function in the second sense, then (23) becomes the* *following inequality*

$$\left|\left[\frac{(x-a)^{n-\frac{\alpha}{k}}+(b-x)^{n-\frac{\alpha}{k}}}{b-a}\right]f^{(n)}(x)-\frac{(nk-\alpha)\Gamma_k(n-\frac{\alpha}{k})}{b-a}\times\right.$$

$$\left.\left[{}^{c}D^{\alpha,k}_{x^-}f(a)+(-1)^n\ {}^{c}D^{\alpha,k}_{x^+}f(b)\right]\right|$$

$$\leq M\left[\frac{(x-a)^{n-\frac{\alpha}{k}+1}+(b-x)^{n-\frac{\alpha}{k}+1}}{b-a}\right]\left(\frac{1}{n-\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{1}{n-\frac{\alpha}{k}+s+1}\right.$$

(27)

$$\left.+\frac{k\Gamma_k(nk-\alpha+k)\Gamma_k(sk+k)}{\Gamma_k(nk-\alpha+sk+2k)}\right]^{\frac{1}{q}};x\in[a,b].$$

References

[1] M. Alomari, M. Darus, S. S. Dragomir and P. Corne, *Ostrowski type inequalities for the functions whose derivative are s-convex in second sense*, Appl. Math. Lett., 23(9) (2010), 1071–1076.

[2] S. S. Dragomir, *Ostrowski-type inequalities for Lebesgue integral: A survey of recent results*, Aust. J. Math. Anal. Appl., 14(1) (2017), 1–287.

[3] S. S. Dragomir, J. Pečarič and L. E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math, 21 (1995), 335-341.

[4] R. Diaza and E. Pariglan, *On hypergeometric function and k-pochemer*, Divulgaciones matematicas, 15(2) (2007), 179–192.

[5] G. Farid, *Some new Ostrowski type inequalities via fractional integrals*, Int. J. Anal. App., 14(1) (2017), 64-68.

[6] G. Farid, A. Javed, A. U. Rehman , *On Hadamard inequalities for $n-$times differentiable functions which are relative convex via Caputo $k-$fractional derivatives*, Nonlinear Anal. Forum, to appear.

[7] E. K. Godunova and V. I. Levin, *Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions*, Moskov. Gos. Ped. Inst. Moscow, 166 (1985), 138-142.

[8] H. Hudzik and L. Maligranda, *Some remarks on s-convex functions*, Aequationes Mathematicae, 48 (1994), 100111.

[9] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of fractional differential Equations*, North-Holland Math. Stud. **204**, Elsevier, New York-London, (2006).

[10] W. Liu, *Ostrowski type fractional integral inequalities for MT-convex function*, Miskole Mathematical Notes, 16(1) (2015), 249–256.

[11] M. Matloka, *Ostrowski type inequalities for functions whose derivatives are h-convex via fractional integrals*, Journal of Scientific Research and Reports 3(12) (2014), 1633-1641.

[12] D. S. Mitrinovic, J. E. Pecaric, and A. M. Fink, *Inequalities involving functions and their integrals and derivatives*, ser. Mathematics and its applications(East European series). Kluwer Academic Publisher Group, Dordrecht, 53 (1991).

[13] A. Ostrowski, *Uber die Absolutabweichung einer differentiierbaren funktion von ihrem integralmittelwert*, Comment. Math. Helv., 10(1) (1938), 226–227.
[14] S. Varosanec, *On h-convexity*, J. Math. Anal. Appl. 326(1) (2007), 303–311.

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