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SOME INTEGRAL INEQUALITIES FOR m-CONVEX FUNCTIONS VIA CAPUTO k-FRACTIONAL DERIVATIVES

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Abstract. In this paper we will give Fejér Hadamard inequalities which are actually the generalization of Hadamard inequalities for m-convex functions via Caputo k-fractional derivatives. With the help of these results we will find out results for convex functions. Also we will derive several other Hadamard-type inequalities and deduce results for Caputo fractional derivatives.

1. Introduction

The concept of fractional calculus was introduced in 1965 when the question of semi-derivative was raised. Since 19th century its theory developed that it laid its importance for the number of applied disciplines including fractional geometry, fractional differential equations and fractional dynamics. Fractional calculus has got the attention of many mathematicians and no discipline of modern science is untouched by it. It's applications are very wide in the present century and has got success because of new fractional-order models as they are more accurate than integer-orderones. The beauty of the subject is that fractional derivatives (and integrals) are not a local (or point) quantities.

Fractional calculus is therefore an excellent tool for describing the memory and hereditary properties of various materials and processes [1].

Many branches of mathematics have developed on the basis of inequalities. Integral inequalities are very important in various classes of equations and almost hundreds of publications have been done on it. For studying existence, uniqueness and other properties of fractional differential equations fractional inequalities are very helpful.

Convex functions play an important role in different fields of Mathematics, Science and Engineering. A class of functions related to convex function is m-convex function introduced by Toader in [24].

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Definition 1. *A function $f:[a,b]\to\mathbb{R}$ is said to be convex if*

$$f\left(\lambda x+(1-\lambda)y\right)\leq \lambda f(x)+(1-\lambda)f(y),$$

holds for all $x,y\in[a,b]$ and $\lambda\in[0,1]$.

The function f is said to be concave if reverse of above inequality holds.

Definition 2. *A function $f:[0,b]\to\mathbb{R}$ such that $b>0$ is said to be m-convex, if*

$$f\left(tx+m(1-t)y\right)\leq tf(x)+m(1-t)f(y)$$

holds for all $x,y\in[0,b]$ and $t,m\in[0,1]$.

If we take $m=1$, then it will again become convex function defined on $[0,b]$.
If we take $m=0$, then it will give us the definition of star-shaped function defined as

Definition 3. *A function $f:[0,b]\to\mathbb{R}$ is called starshaped if*

$$f(tx)\leq tf(x) \text{ for all } t\in[0,b] \text{ and } x\in[0,b],$$

where $K_m(b)$ denote the set of m-convex functions on $[0,b]$ for which $f(0)<0$, which gives

$$K_1(b)\subset K_m(b)\subset K_0(b),$$

whenever $m\in(0,1)$. It should be noted that the class $K_1(b)$ are only convex functions $f:[0,b]\to\mathbb{R}$ for which $f(0)\leq 0$ (see [6]).

Example 1. *[18] A function $f:[0,\infty)\to\mathbb{R}$ given by*

$$f(x)=\frac{1}{12}\left(4x^3-15x^2+18x-5\right)$$

is $\frac{16}{17}-$convex function but it is not convex function.

A lot of work has been done on the Hadamard and the Fejér-Hadamard inequalities and their different extensions and generalizations have been found (see, [1–5, 9, 12, 13, 16, 22, 25] and references therein).

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function such that $a<b$. Then the following double inequality*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}, \tag{1.1}$$

is well known in literature as the Hadamard inequality.

In [11] Fejér gave the following generalization of Hadamard inequality which is well known in literature as the Fejér-Hadamard inequality.

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a convex function and $g : [a, b] \to \mathbb{R}$ is a non-negative, integrable and symmetric to $\frac{a+b}{2}$. Then the following inequality holds*

(1.2)
$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx.$$

In recent years many researchers have given importance to the Hadamard and the Fejér-Hadamard inequalities via fractional calculus and a lot of papers have been published in this regard (see, [13, 17, 21, 22] and references therein). In the following we define Caputo fractional derivatives [15].

Definition 4. *Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$. The Caputo fractional derivatives of order α are defined as follows:*

$$({}^C D^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, x > a$$

and

$$({}^C D^{\alpha}_{b-} f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, x < b$$

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative of order n exists, then Caputo fractional derivative $({}^C D^{\alpha}_{a+} f)(x)$ coincides with $f^{(n)}(x)$. In particular we have

$$({}^C D^{0}_{a+} f)(x) = ({}^C D^{0}_{b-} f)(x) = f(x)$$

where $n = 1$ and $\alpha = 0$.

In the following we define Caputo k-fractional derivatives.

Definition 5. *Let $\alpha > 0, k \ge 1$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$. The Caputo k-fractional derivatives of order α are defined as follows:*

$$({}^C D^{\alpha,k}_{a+} f)(x) = \frac{1}{k\Gamma_k(n-\frac{\alpha}{k})}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}}dt, x > a$$

and

$$({}^C D^{\alpha,k}_{b-} f)(x) = \frac{(-1)^n}{k\Gamma_k(n-\frac{\alpha}{k})}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}}dt, x < b$$

where $\Gamma_k(\alpha)$ *is the k-Gamma function defined as*

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{\frac{-t^k}{k}} dt,$$

also

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha)$$

If $\alpha = n \in \{1, 2, 3, ...\}$ *and usual derivative of order* n *exists, then Caputo k-fractional derivative* $({}^C D^{\alpha,k}_{a+} f)(x)$ *coincides with* $f^{(n)}(x)$*.*

In particular we have

$$({}^C D^{0,1}_{a+} f)(x) = ({}^C D^{0,1}_{b-} f)(x) = f(x)$$

where $n, k = 1$ and $\alpha = 0$. For $k = 1$, Caputo k-fractional derivatives give the definition of Caputo fractional derivatives.
In this paper, in Section 2 we give Fejér-Hadamard type inequalities for m-convex functions via Caputo k-fractional derivatives. In Section 3 we give some generalization of Hadamard-type inequalities for m-convex functions via Caputo k-fractional derivatives. We deduce results for m- convex function via Caputo fractional derivatives and for convex function via Caputo k-fractional derivatives as special cases of our results.
In the whole paper $C^n[a, b]$ denotes the space of n-times differentiable functions such that $f^{(n)}$ are continuous on $[a, b]$.

2. Main results

In order to prove our results we need the following lemma.

Lemma 1. *Let* $f : [a, mb] \to \mathbb{R}$ *be a function such that* $f \in C^{n+1}[a, mb]$*, with* $a < mb$*. Also let* $g : [a, mb] \to \mathbb{R}$ *is continuous and integrable on* $[a, mb]$*, then the following equality for Caputo k-fractional derivatives* *holds*

$$\left(\int_a^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}} [f^{(n)}(a) + f^{(n)}(mb)] \tag{2.1}$$
$$- \left(n - \frac{\alpha}{k}\right) \int_a^{mb} \left(\int_a^t g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt$$
$$- \left(n - \frac{\alpha}{k}\right) \int_a^{mb} \left(\int_t^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt$$
$$= \int_a^{mb} \left(\int_a^t g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}} f^{(n+1)}(t)dt$$
$$- \int_a^{mb} \left(\int_t^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}} f^{(n+1)}(t)dt.$$

Proof. Since we have

$$\int_a^{mb} \left(\int_a^t g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}} f^{(n+1)}(t)dt$$
$$= \left(\int_a^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}} f^{(n)}(mb)$$
$$- \left(n - \frac{\alpha}{k}\right) \int_a^{mb} \left(\int_a^t g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt \tag{2.2}$$

and

$$\int_a^{mb} \left(\int_t^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}} f^{(n+1)}(t)dt$$
$$= -\left(\int_a^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}} f^{(n)}(a)$$
$$+ \left(n - \frac{\alpha}{k}\right) \int_a^{mb} \left(\int_t^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt. \tag{2.3}$$

Now subtracting (2.3) from (2.2) we have the required inequality (2.1). □

By using above lemma we will prove the following theorem.

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in C^{n+1}[a, b]$, with $a < b$ and let $g : [a, mb] \to \mathbb{R}$ is continuous on $[a, mb]$. If $|f^{(n+1)}|$ is m-convex on $[a, mb]$ with $a < mb$, then the following inequality with*

$\|g^{(n)}\|_\infty = sup|g^{(n)}(t)|$ for Caputo k-fractional derivatives holds

$$
\begin{aligned}
&\Bigg| \left(\int_a^{mb} g^{(n)}(s)ds \right)^{n-\frac{\alpha}{k}} [f^{(n)}(a) + f^{(n)}(mb)] \\
&- \left(n - \frac{\alpha}{k} \right) \int_a^{mb} \left(\int_a^t g^{(n)}(s)ds \right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t) f^{(n)}(t)dt \\
&- \left(n - \frac{\alpha}{k} \right) \int_a^{mb} \left(\int_t^{mb} g^{(n)}(s)ds \right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t) f^{(n)}(t)dt \Bigg| \\
&\leq \frac{(mb-a)^{n-\frac{\alpha}{k}+1} \|g^{(n)}\|_\infty^{n-\frac{\alpha}{k}}}{n-\frac{\alpha}{k}+1} [|f^{(n+1)}(a)| + m|f^{(n+1)}(b)|].
\end{aligned}
\tag{2.4}
$$

Proof. Using Lemma 1, we have

$$
\begin{aligned}
&\Bigg| \left(\int_a^{mb} g^{(n)}(s)ds \right)^{n-\frac{\alpha}{k}} [f^{(n)}(a) + f^{(n)}(mb)] \\
&- \left(n - \frac{\alpha}{k} \right) \int_a^{mb} \left(\int_a^t g^{(n)}(s)ds \right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t) f^{(n)}(t)dt \\
&- \left(n - \frac{\alpha}{k} \right) \int_a^{mb} \left(\int_t^{mb} g^{(n)}(s)ds \right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t) f^{(n)}(t)dt \Bigg| \\
&\leq \int_a^{mb} \left| \int_a^t g^{(n)}(s)ds \right|^{n-\frac{\alpha}{k}} |f^{(n+1)}(t)|dt \\
&+ \int_a^{mb} \left| \int_t^{mb} g^{(n)}(s)ds \right|^{n-\frac{\alpha}{k}} |f^{(n+1)}(t)|dt.
\end{aligned}
$$

Since $|g^{(n)}(t)| \leq \|g^{(n)}\|_\infty$, therefore

$$\begin{aligned}
&\left| \left(\int_a^{mb} g^{(n)}(s)ds \right)^{n-\frac{\alpha}{k}} [f^{(n)}(a) + f^{(n)}(mb)] \right. \\
&- \left(n - \frac{\alpha}{k} \right) \int_a^{mb} \left(\int_a^t g^{(n)}(s)ds \right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t) f^{(n)}(t)dt \\
&\left. - \left(n - \frac{\alpha}{k} \right) \int_a^{mb} \left(\int_t^{mb} g^{(n)}(s)ds \right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t) f^{(n)}(t)dt \right| \\
&\leq \|g^{(n)}\|_\infty^{n-\frac{\alpha}{k}} \left[\int_a^{mb} (t-a)^{n-\frac{\alpha}{k}} |f^{(n+1)}(t)|dt \right. \\
&\left. + \int_a^{mb} (mb-t)^{n-\frac{\alpha}{k}} |f^{(n+1)}(t)|dt \right] \\
&= \|g^{(n)}\|_\infty^{n-\frac{\alpha}{k}} \left[\int_a^{mb} (t-a)^{n-\frac{\alpha}{k}} \left| f^{(n+1)} \left(\frac{mb-t}{mb-a} a + m \frac{t-a}{mb-a} b \right) \right| dt \right. \\
&\left. + \int_a^{mb} (mb-t)^{n-\frac{\alpha}{k}} \left| f^{(n+1)} \left(\frac{mb-t}{mb-a} a + m \frac{t-a}{mb-a} b \right) \right| dt \right].
\end{aligned}$$

Since $|f^{(n+1)}|$ is m-convex, so we have

$$\begin{aligned}
&\left| \left(\int_a^{mb} g^{(n)}(s)ds \right)^{n-\frac{\alpha}{k}} [f^{(n)}(a) + f^{(n)}(mb)] \right. \\
&- \left(n - \frac{\alpha}{k} \right) \int_a^{mb} \left(\int_a^t g^{(n)}(s)ds \right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t) f^{(n)}(t)dt \\
&\left. - \left(n - \frac{\alpha}{k} \right) \int_a^{mb} \left(\int_t^{mb} g^{(n)}(s)ds \right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t) f^{(n)}(t)dt \right| \\
&\leq \|g^{(n)}\|_\infty^{n-\frac{\alpha}{k}} \left[\int_a^{mb} (t-a)^{n-\frac{\alpha}{k}} \left(\frac{mb-t}{mb-a} |f^{(n+1)}(a)| + m \frac{t-a}{mb-a} |f^{(n+1)}(b)| \right) dt \right. \\
&\left. + \int_a^{mb} (mb-t)^{n-\frac{\alpha}{k}} \left(\frac{mb-t}{mb-a} |f^{(n+1)}(a)| + m \frac{t-a}{mb-a} |f^{(n+1)}(b)| \right) dt \right],
\end{aligned}$$

which after a little calculation gives the required result. $\square$

Corollary 1. *If we put $k = 1$ we get the following inequality for m-convex function via Caputo fractional derivatives*

$$\begin{aligned}
&\Bigg|\left(\int_a^{mb} g^{(n)}(s)ds\right)^{n-\alpha}[f^{(n)}(a)+f^{(n)}(mb)] \\
&-(n-\alpha)\int_a^{mb}\left(\int_a^{t} g^{(n)}(s)ds\right)^{n-\alpha-1} g^{(n)}(t)f^{(n)}(t)dt \\
&-(n-\alpha)\int_a^{mb}\left(\int_t^{mb} g^{(n)}(s)ds\right)^{n-\alpha-1} g^{(n)}(t)f^{(n)}(t)dt\Bigg| \\
&\leq \frac{(mb-a)^{n-\alpha+1}\|g^{(n)}\|_\infty^{n-\alpha}}{n-\alpha+1}[|f^{(n+1)}(a)|+m|f^{(n+1)}(b)|].
\end{aligned}$$

Corollary 2. *If we put $m = 1$ we get the following inequality for convex function via Caputo k-fractional derivatives*

$$\begin{aligned}
&\Bigg|\left(\int_a^{b} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}}[f^{(n)}(a)+f^{(n)}(b)] \\
&-\left(n-\frac{\alpha}{k}\right)\int_a^{b}\left(\int_a^{t} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt \\
&-\left(n-\frac{\alpha}{k}\right)\int_a^{b}\left(\int_t^{b} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt\Bigg| \\
&\leq \frac{(b-a)^{n-\frac{\alpha}{k}+1}\|g^{(n)}\|_\infty^{n-\frac{\alpha}{k}}}{n-\frac{\alpha}{k}+1}[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|].
\end{aligned}$$

Using Lemma 1 and Hölder's inequality we prove the following result.

Theorem 4. *Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in C^{n+1}[a, b]$ with $a < b$. Also let $g : [a, mb] \to \mathbb{R}$ is continuous on $[a, mb]$. If $|f^{(n+1)}|^q$ is m-convex on $[a, mb]$, $q > 1$, with $a < mb$, then the following* *inequality with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\|g^{(n)}\|_{\infty}=sup|g^{(n)}(t)|$ *for Caputo k-fractional derivatives holds*

$$
\begin{aligned}
(2.5)\quad &\left|\left(\int_a^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}}[f^{(n)}(a)+f^{(n)}(mb)]\right.\\
&-\left(n-\frac{\alpha}{k}\right)\int_a^{mb}\left(\int_a^{t} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1}g^{(n)}(t)f^{(n)}(t)dt\\
&\left.-\left(n-\frac{\alpha}{k}\right)\int_a^{mb}\left(\int_t^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1}g^{(n)}(t)f^{(n)}(t)dt\right|\\
&\leq\frac{2(mb-a)^{n-\frac{\alpha}{k}+1}\|g^{(n)}\|_{\infty}^{n-\frac{\alpha}{k}}}{(n-\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\left[\frac{|f^{(n+1)}(a)|^q+m|f^{(n+1)}(b)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. Using the result obtained from Lemma 1 we have

$$
\begin{aligned}
&\left|\left(\int_a^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}}[f^{(n)}(a)+f^{(n)}(mb)]\right.\\
&-\left(n-\frac{\alpha}{k}\right)\int_a^{mb}\left(\int_a^{t} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1}g^{(n)}(t)f^{(n)}(t)dt\\
&\left.-\left(n-\frac{\alpha}{k}\right)\int_a^{mb}\left(\int_t^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1}g^{(n)}(t)f^{(n)}(t)dt\right|\\
&\leq\int_a^{mb}\left|\int_a^{t} g^{(n)}(s)ds\right|^{n-\frac{\alpha}{k}}|f^{(n+1)}(t)|dt\\
&+\int_a^{mb}\left|\int_t^{mb} g^{(n)}(s)ds\right|^{n-\frac{\alpha}{k}}|f^{(n+1)}(t)|dt.
\end{aligned}
$$

By applying Hölder's inequality on right hand side of the above inequality, we have

$$\begin{aligned}
&\Bigg|\left(\int_a^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}}\left[f^{(n)}(a)+f^{(n)}(mb)\right] \\
&-\left(n-\frac{\alpha}{k}\right)\int_a^{mb}\left(\int_a^{t} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt \\
&-\left(n-\frac{\alpha}{k}\right)\int_a^{mb}\left(\int_t^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt\Bigg| \\
&\leq \left(\int_a^{mb}\left|\int_a^{t} g^{(n)}(s)ds\right|^{np-\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}}\left(\int_a^{mb}|f^{(n+1)}(t)|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_a^{mb}\left|\int_t^{mb} g^{(n)}(s)ds\right|^{np-\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}}\left(\int_a^{mb}|f^{(n+1)}(t)|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

As $g^{(n)}(t) \leq \|g^{(n)}\|_\infty$, so we get

$$\begin{aligned}
&\Bigg|\left(\int_a^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}}\left[f^{(n)}(a)+f^{(n)}(mb)\right] \\
&-\left(n-\frac{\alpha}{k}\right)\int_a^{mb}\left(\int_a^{t} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt \\
&-\left(n-\frac{\alpha}{k}\right)\int_a^{mb}\left(\int_t^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt\Bigg| \\
&\leq \|g^{(n)}\|_\infty^{n-\frac{\alpha}{k}}\left[\left(\int_a^{mb}|t-a|^{np-\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}}+\left(\int_a^{mb}|mb-t|^{np-\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}}\right] \\
&\left(\int_a^{mb}|f^{(n+1)}(t)|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

From m-convexity of $|f^{(n+1)}|^q$ we have

$$\begin{aligned}
&\Bigg|\left(\int_a^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}}\left[f^{(n)}(a)+f^{(n)}(mb)\right]\\
&-\left(n-\frac{\alpha}{k}\right)\int_a^{mb}\left(\int_a^t g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt\\
&-\left(n-\frac{\alpha}{k}\right)\int_a^{mb}\left(\int_t^{mb} g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt\Bigg|\\
&\leq \|g^{(n)}\|_\infty^{n-\frac{\alpha}{k}}\left[\left(\int_a^{mb}|t-a|^{np-\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}}+\left(\int_a^{mb}|mb-t|^{np-\frac{\alpha p}{k}}dt\right)^{\frac{1}{p}}\right]\\
&\left(\int_a^b\left(\frac{mb-t}{mb-a}|f^{(n+1)}(a)|^q+m\frac{t-a}{mb-a}|f^{(n+1)}(b)|^q\right)dt\right)^{\frac{1}{q}}.
\end{aligned}$$

From which one can have inequality (2.5). □

Corollary 3. *If we put $k = 1$ in the above theorem we have the following Hadamard-type inequality for Caputo fractional derivatives*

$$\begin{aligned}
&\Bigg|\left(\int_a^{mb} g^{(n)}(s)ds\right)^{n-\alpha}\left[f^{(n)}(a)+f^{(n)}(mb)\right]\\
&-(n-\alpha)\int_a^{mb}\left(\int_a^t g^{(n)}(s)ds\right)^{n-\alpha-1} g^{(n)}(t)f^{(n)}(t)dt\\
&-(n-\alpha)\int_a^{mb}\left(\int_t^{mb} g^{(n)}(s)ds\right)^{n-\alpha-1} g^{(n)}(t)f^{(n)}(t)dt\Bigg|\\
&\leq \frac{2(mb-a)^{n-\alpha+1}\|g^{(n)}\|_\infty^{n-\alpha}}{(np-\alpha p+1)^{\frac{1}{p}}}\left[\frac{|f^{(n+1)}(a)|^q+m|f^{(n+1)}(b)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

Corollary 4. *If we put $m = 1$ in above theorem we get the following inequality for convex function via Caputo k-fractional derivatives*

$$\begin{aligned}
&\Bigg|\left(\int_a^b g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}}[f^{(n)}(a)+f^{(n)}(b)] \\
&-\left(n-\frac{\alpha}{k}\right)\int_a^b\left(\int_a^t g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt \\
&-\left(n-\frac{\alpha}{k}\right)\int_a^b\left(\int_t^b g^{(n)}(s)ds\right)^{n-\frac{\alpha}{k}-1} g^{(n)}(t)f^{(n)}(t)dt\Bigg| \\
&\leq \frac{2(b-a)^{n-\frac{\alpha}{k}+1}\|g^{(n)}\|_\infty^{n-\frac{\alpha}{k}}}{(n-\frac{\alpha p}{k}+1)^{\frac{1}{p}}}\left[\frac{|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

3. Generalized Hadamard-type integral inequalities for m-convex functions via Caputo k-fractional derivatives

In this section we will give some Hadamard-type inequalities for differentiable m-convex functions via Caputo k-fractional derivatives. Also we deduce some results of [20–22]. Firstly we give the following lemma which will be used further for our results.

Lemma 2. *Let $f : [a, mb] \to \mathbb{R}$ be a function such that $f \in C^{n+1}[a, mb]$ with $a < mb$. Then the following equality for Caputo k-fractional derivatives holds*

(3.1)

$$\begin{aligned}
&-\frac{f^{(n)}\left(m\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)} \\
&+\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}}\times\Big[({}^C D^{\alpha,k}_{(m\lambda b+(1-\lambda)a)^+}f)(\lambda a+m(1-\lambda)b) \\
&+(-1)^n({}^C D^{\alpha,k}_{(\lambda a+m(1-\lambda)b)^-}f)(m\lambda b+(1-\lambda)a)\Big] \\
&=\int_0^1[(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}] \\
&f^{(n+1)}\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)dt,
\end{aligned}$$

where $\lambda \in [0, 1] \setminus \{\frac{1}{2}\}$.

Proof. Since

$$\int_0^1 [(1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}]$$

$$f^{(n+1)}\left(t\left(\lambda a + m(1-\lambda)b\right) + m(1-t)\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right) dt$$

(3.2)

$$= \int_0^1 (1-t)^{n-\frac{\alpha}{k}} f^{(n+1)}\left(t\left(\lambda a + m(1-\lambda)b\right) + m(1-t)\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right) dt$$

$$- \int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)}\left(t\left(\lambda a + m(1-\lambda)b\right) + m(1-t)\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right) dt.$$

Consider

$$\int_0^1 (1-t)^{n-\frac{\alpha}{k}} f^{(n+1)}\left(t\left(\lambda a + m(1-\lambda)b\right) + m(1-t)\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right) dt$$

$$= -\frac{f^{(n)}\left(m\lambda b + (1-\lambda)a\right)}{(1-2\lambda)(mb-a)}$$

$$+ \frac{n-\frac{\alpha}{k}}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}} \int_{m\lambda b+(1-\lambda)a}^{\lambda a+m(1-\lambda)b} \left(\lambda a + m(1-\lambda)b - x\right)^{n-\frac{\alpha}{k}-1} f^{(n)}(x)dx$$

$$= -\frac{f^{(n)}(m\lambda b + (1-\lambda)a)}{(1-2\lambda)(mb-a)}$$

$$+ \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}} ({}^C D^{\alpha,k}_{(m\lambda b+(1-\lambda)a)^+} f)(\lambda a + m(1-\lambda)b).$$

Similarly, we have

$$\int_0^1 t^{n-\frac{\alpha}{k}} f^{(n+1)}\left(t\left(\lambda a + m(1-\lambda)b\right) + m(1-t)\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right) dt$$

$$= \frac{f^{(n)}\left(\lambda a + m(1-\lambda)b\right)}{(1-2\lambda)(mb-a)}$$

$$- \frac{n-\frac{\alpha}{k}}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}} \int_{m\lambda b+(1-\lambda)a}^{\lambda a+m(1-\lambda)b} \left(x - m\lambda b + (1-\lambda)a\right)^{n-\frac{\alpha}{k}-1} f^{(n)}(x)dx$$

$$= \frac{f^{(n)}(\lambda a + m(1-\lambda)b)}{(1-2\lambda)(mb-a)}$$

$$- \frac{k\Gamma_k(n-\frac{\alpha}{k}+k)\left[(-1)^n({}^C D^{\alpha,k}_{(\lambda a+m(1-\lambda)b)^-} f)(m\lambda b + (1-\lambda)a)\right]}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}}.$$

Now putting the above values in (3.2), we get the required inequality (3.1). □

Using Lemma 2 and power mean inequality we give the following Hadamard-type inequality for m-convex functions.

Theorem 5. *Let $f : [a, mb] \to \mathbb{R}$ be a function such that $f \in C^{(n+1)}[a, mb]$ with $0 \leq a < mb$. If $\left|f^{(n+1)}\right|^q$, $q \geq 1$ is m-convex on $[a, mb]$, then the following inequality for Caputo k-fractional derivatives holds*

(3.3)

$$\Bigg|\frac{f^{(n)}(m\lambda b + (1-\lambda)a) + f^{(n)}(\lambda a + m(1-\lambda)b)}{(1-2\lambda)(mb-a)}$$
$$-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}} \times \Big[({}^{C}D^{\alpha,k}_{(m\lambda b+(1-\lambda)a)^+}f)(\lambda a + m(1-\lambda)b)$$
$$+(-1)^n({}^{C}D^{\alpha,k}_{(\lambda a+m(1-\lambda)b)^-}f)(m\lambda b + (1-\lambda)a)\Big]\Bigg|$$
$$\leq \frac{2}{n-\frac{\alpha}{k}+1}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)$$
$$\left[\frac{\left|f^{(n+1)}(\lambda a + m(1-\lambda)b)\right|^q + m\left|f^{(n+1)}(\lambda b + (1-\lambda)\frac{a}{m})\right|^q}{2}\right]^{\frac{1}{q}}$$

where $\lambda \in [0,1] \setminus \{\frac{1}{2}\}$.

Proof. Consider $q = 1$. Using Lemma 2 and m-convexity of $\left|f^{(n+1)}\right|^q$, we have

(3.4)

$$\left|\frac{f^{(n)}(m\lambda b + (1-\lambda)a) + f^{(n)}(\lambda a + m(1-\lambda)b)}{(1-2\lambda)(mb-a)}\right.$$

$$-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}} \times \left[({}^{C}D^{\alpha,k}_{(m\lambda b+(1-\lambda)a)^{+}}f)(\lambda a + m(1-\lambda)b)\right.$$

$$\left.\left. +(-1)^n({}^{C}D^{\alpha,k}_{(\lambda a+m(1-\lambda)b)^{-}}f)(m\lambda b + (1-\lambda)a)\right]\right|$$

(3.5)

$$\leq \int_0^1 \left|(1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}\right|$$

$$\left|f^{(n+1)}\left(t(\lambda a + m(1-\lambda)b) + m(1-t)\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right)\right| dt$$

$$\leq \int_0^1 \left|(1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}\right| \left[t\left|f^{(n+1)}(\lambda a + m(1-\lambda)b)\right|\right.$$

$$\left. +m(1-t)\left|f^{(n+1)}\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right|\right] dt$$

$$= \int_0^{\frac{1}{2}} \left[(1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}\right] \left[t\left|f^{(n+1)}(\lambda a + m(1-\lambda)b)\right|\right.$$

$$\left. +m(1-t)\left|f^{(n+1)}\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right|\right] dt$$

$$+ \int_{\frac{1}{2}}^1 \left[t^{n-\frac{\alpha}{k}} - (1-t)^{n-\frac{\alpha}{k}}\right] \left[t\left|f^{(n+1)}(\lambda a + m(1-\lambda)b)\right|\right.$$

(3.6)

$$\left. +m(1-t)\left|f^{(n+1)}\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right|\right] dt.$$

Since we have

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} \left[(1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}}\right] \left[t\left|f^{(n+1)}\left(\lambda a + m(1-\lambda)b\right)\right|\right.\\
&\left.+m(1-t)\left|f^{(n+1)}\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right|\right] dt\\
&= \left|f^{(n+1)}(\lambda a + m(1-\lambda)b)\right| \int_0^{\frac{1}{2}} \left[t(1-t)^{n-\frac{\alpha}{k}} - t^{n-\frac{\alpha}{k}+1}\right] dt\\
&+ m\left|f^{(n+1)}\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right| \int_0^{\frac{1}{2}} \left[(1-t)^{n-\frac{\alpha}{k}+1} - (1-t)t^{n-\frac{\alpha}{k}}\right] dt\\
&= \left|f^{(n+1)}(\lambda a + m(1-\lambda)b)\right| \left[\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)} - \frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right]\\
&+ m\left|f^{(n+1)}\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right| \left[\frac{1}{n-\frac{\alpha}{k}+2} - \frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right].
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_{\frac{1}{2}}^{1} \left[t^{n-\frac{\alpha}{k}} - (1-t)^{n-\frac{\alpha}{k}}\right]\\
&\left[t\left|f^{(n+1)}\left(\lambda a + m(1-\lambda)b\right)\right| + m(1-t)\left|f^{(n+1)}\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right|\right] dt\\
&= \left|f^{(n+1)}(\lambda a + m(1-\lambda)b)\right| \left[\int_{\frac{1}{2}}^{1} t^{n-\frac{\alpha}{k}+1} dt - \int_{\frac{1}{2}}^{1} t(1-t)^{n-\frac{\alpha}{k}} dt\right]\\
&+ m\left|f^{(n+1)}\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right| \left[\int_{\frac{1}{2}}^{1} t^{n-\frac{\alpha}{k}}(1-t) dt - \int_{\frac{1}{2}}^{1} (1-t)^{n-\frac{\alpha}{k}+1} dt\right]\\
&= \left|f^{(n+1)}(\lambda a + m(1-\lambda)b)\right| \left[\frac{1}{n-\frac{\alpha}{k}+2} - \frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right]\\
&+ m\left|f^{(n+1)}\left(\lambda b + (1-\lambda)\frac{a}{m}\right)\right| \left[\frac{1}{(n-\frac{\alpha}{k}+1)(n-\frac{\alpha}{k}+2)} - \frac{(\frac{1}{2})^{n-\frac{\alpha}{k}+1}}{n-\frac{\alpha}{k}+1}\right].
\end{aligned}
$$

Putting above values in (3.4), we get

$$\left|\frac{f^{(n)}\left(m\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}\right.$$
$$-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}}\times\left[({}^{C}D^{\alpha,k}_{(m\lambda b+(1-\lambda)a)^{+}}f)(\lambda a+m(1-\lambda)b)\right.$$
$$\left.\left.+(-1)^{n}({}^{C}D^{\alpha,k}_{(\lambda a+m(1-\lambda)b)^{-}}f)(m\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq\frac{2}{n-\frac{\alpha}{k}+1}\left[1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right]$$
$$\left[\frac{\left|f^{(n+1)}(\lambda a+m(1-\lambda)b)\right|+m\left|f^{(n+1)}(\lambda b+(1-\lambda)\frac{a}{m})\right|}{2}\right].$$

Now consider $q>1$. Then by using Lemma 2 and power mean inequality, we get

$$\int_0^1\left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|$$
$$\left|f^{(n+1)}\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|dt$$
$$\leq\left(\int_0^1\left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|dt\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1\left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|\right.$$
$$\left.\left|f^{(n+1)}\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}.$$

Since $\left|f^{(n+1)}\right|^q$ is m-convex, so we have

$$\left|\frac{f^{(n)}\left(m\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}\right.$$
$$-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}}\times\left[({}^C D^{\alpha,k}_{(m\lambda b+(1-\lambda)a)^+}f)(\lambda a+m(1-\lambda)b)\right.$$
$$\left.\left.+(-1)^n({}^C D^{\alpha,k}_{(\lambda a+m(1-\lambda)b)^-}f)(m\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq\left(\int_0^1\left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|dt\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1\left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|\right.$$
$$\left.\left|f^{(n+1)}\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\left(\int_0^{\frac{1}{2}}\left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|dt+\int_{\frac{1}{2}}^1\left|t^{n-\frac{\alpha}{k}}-(1-t)^{n-\frac{\alpha}{k}}\right|dt\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1\left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|\left|f^{(n+1)}\left(t\left(\lambda a+m(1-\lambda)b\right)\right|^q\right.$$
$$\left.+m(1-t)\left|f^{(n+1)}\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\left[\frac{2}{n-\frac{\alpha}{k}+1}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)\right]^{1-\frac{1}{q}}\left[\frac{2}{n-\frac{\alpha}{k}+1}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)\right]^{\frac{1}{q}}$$
$$\left[\frac{\left|f^{(n+1)}(\lambda a+m(1-\lambda)b)\right|^q+m\left|f^{(n+1)}\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}$$
$$\leq\frac{2}{n-\frac{\alpha}{k}+1}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)$$
$$\left[\frac{\left|f^{(n+1)}(\lambda a+m(1-\lambda)b)\right|^q+m\left|f^{(n+1)}\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}},$$

which is the required inequality. $\square$

Corollary 5. *In Theorem 5, if we put $k = 1$ we get the following inequality for Caputo fractional derivatives*

$$\begin{aligned}
&\Bigg|\frac{f^{(n)}\left(m\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}\\
&-\frac{\Gamma(n-\alpha+1)}{(1-2\lambda)^{n-\alpha+1}(mb-a)^{n-\alpha+1}}\times\Big[({}^{C}D^{\alpha}_{(m\lambda b+(1-\lambda)a)^{+}}f)(\lambda a+m(1-\lambda)b)\\
&+(-1)^{n}({}^{C}D^{\alpha}_{(\lambda a+m(1-\lambda)b)^{-}}f)(m\lambda b+(1-\lambda)a)\Big]\Bigg|\\
&\leq\frac{2}{n-\alpha+1}\left(1-\frac{1}{2^{n-\alpha}}\right)\\
&\left[\frac{\left|f^{(n+1)}(\lambda a+m(1-\lambda)b)\right|^{q}+m\left|f^{(n+1)}(\lambda b+(1-\lambda)\frac{a}{m})\right|^{q}}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

Corollary 6. *In above theorem if we put $m = 1$ then we get the following result for convex functions via Caputo k-fractional derivatives*

$$\begin{aligned}
&\Bigg|\frac{f^{(n)}\left(\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)}\\
&-\frac{k\Gamma_{k}(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(b-a)^{n-\frac{\alpha}{k}+1}}\times\Big[({}^{C}D^{\alpha,k}_{(\lambda b+(1-\lambda)a)^{+}}f)(\lambda a+(1-\lambda)b)\\
&+(-1)^{n}({}^{C}D^{\alpha,k}_{(\lambda a+(1-\lambda)b)^{-}}f)(\lambda b+(1-\lambda)a)\Big]\Bigg|\\
&\leq\frac{2}{n-\frac{\alpha}{k}+1}\left(1-\frac{1}{2^{n-\frac{\alpha}{k}}}\right)\\
&\left[\frac{\left|f^{(n+1)}(\lambda a+(1-\lambda)b)\right|^{q}+\left|f^{(n+1)}(\lambda b+(1-\lambda)a)\right|^{q}}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

Using Lemma 2 and Hölder inequality we give the following results.

Theorem 6. *Let $f : [a, mb] \to \mathbb{R}$ be a function such that $f \in C^{(n+1)}[a, mb]$ with $0 \leq a < mb$. If $\left|f^{(n+1)}\right|^{q}$ is m-convex on $[a, mb]$ for some fixed $q \geq 1$, then the following inequality for Caputo k-fractional derivatives* *holds*

(3.7)

$$\left|\frac{f^{(n)}\left(m\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}\right.$$
$$-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}}\times\Big[({}^{C}D^{\alpha,k}_{(m\lambda b+(1-\lambda)a)^+}f)(\lambda a+m(1-\lambda)b)$$
$$\left.+(-1)^n({}^{C}D^{\alpha,k}_{(\lambda a+m(1-\lambda)b)^-}f)(m\lambda b+(1-\lambda)a)\Big]\right|$$
$$\leq\left[\frac{2}{np-\frac{\alpha p}{k}+1}\left(1-\frac{1}{2^{np-\frac{\alpha p}{k}}}\right)\right]^{\frac{1}{p}}$$
$$\left[\frac{\left|f^{(n+1)}(\lambda a+m(1-\lambda)b)\right|^q+m\left|f^{(n+1)}(\lambda b+(1-\lambda)\frac{a}{m})\right|^q}{2}\right]^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$.

Proof. If we use Lemma 2, Hölder inequality and m-convexity of $|f(n+1)|^q$ respectively, we get

$$\left|\frac{f^{(n)}\left(m\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}\right.$$
$$-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}}\times\left[({}^{C}D^{\alpha,k}_{(m\lambda b+(1-\lambda)a)^+}f)(\lambda a+m(1-\lambda)b)\right.$$
$$\left.\left.+(-1)^n({}^{C}D^{\alpha,k}_{(\lambda a+m(1-\lambda)b)^-}f)(m\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq\int_0^1\left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|$$
$$\left|f^{(n+1)}\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|dt$$
$$\leq\left(\int_0^1\left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}$$
$$\left(\int_0^1\left|f^{(n+1)}\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\left(\int_0^{\frac{1}{2}}\left[(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right]^p dt+\int_{\frac{1}{2}}^1\left[t^{n-\frac{\alpha}{k}}-(1-t)^{n-\frac{\alpha}{k}}\right]^p dt\right)^{\frac{1}{p}}$$
$$\left(\int_0^1\left(t\left|f^{(n+1)}\left(\lambda a+m(1-\lambda)b\right)\right|^q\right.\right.$$
$$\left.\left.+m(1-t)\left|f^{(n+1)}\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

Using result

$$(A-B)^q\leq A^q-B^q,\quad A\geq B\geq 0,$$

we have

$$\begin{aligned}
&\Bigg|\frac{f^{(n)}\left(m\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}\\
&-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}}\times\Big[({}^CD^{\alpha,k}_{(m\lambda b+(1-\lambda)a)^+}f)(\lambda a+m(1-\lambda)b)\\
&+(-1)^n({}^CD^{\alpha,k}_{(\lambda a+m(1-\lambda)b)^-}f)(m\lambda b+(1-\lambda)a)\Big]\Bigg|\\
&\leq\left(\int_0^{\frac{1}{2}}\left[(1-t)^{np-\frac{\alpha p}{k}}-t^{np-\frac{\alpha p}{k}}\right]dt+\int_{\frac{1}{2}}^1\left[t^{np-\frac{\alpha p}{k}}-(1-t)^{np-\frac{\alpha p}{k}}\right]dt\right)^{\frac{1}{p}}\\
&\left(\frac{\left|f^{(n+1)}(\lambda a+m(1-\lambda)b)\right|^q+m\left|f^{(n+1)}(\lambda b+(1-\lambda)\frac{a}{m})\right|^q}{2}\right)^{\frac{1}{q}}\\
&\leq\left(\frac{2}{np-\frac{\alpha p}{k}+1}\left(1-\frac{1}{2^{np-\frac{\alpha p}{k}}}\right)\right)^{\frac{1}{p}}\\
&\left(\frac{\left|f^{(n+1)}(\lambda a+m(1-\lambda)b)\right|^q+m\left|f^{(n+1)}(\lambda b+(1-\lambda)\frac{a}{m})\right|^q}{2}\right)^{\frac{1}{q}},
\end{aligned}$$

which is the required inequality. □

Corollary 7. *In Theorem 6, if we put $k=1$ we get the following result for Caputo fractional derivatives*

$$\begin{aligned}
&\Bigg|\frac{f^{(n)}\left(m\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}\\
&-\frac{\Gamma(n-\alpha+1)}{(1-2\lambda)^{n-\alpha+1}(mb-a)^{n-\alpha+1}}\times\Big[({}^CD^{\alpha}_{(m\lambda b+(1-\lambda)a)^+}f)(\lambda a+m(1-\lambda)b)\\
&+(-1)^n({}^CD^{\alpha}_{(\lambda a+m(1-\lambda)b)^-}f)(m\lambda b+(1-\lambda)a)\Big]\Bigg|\\
&\leq\left[\frac{2}{np-\alpha p+1}\left(1-\frac{1}{2^{np-\alpha p}}\right)\right]^{\frac{1}{p}}\\
&\left[\frac{\left|f^{(n+1)}(\lambda a+m(1-\lambda)b)\right|^q+m\left|f^{(n+1)}(\lambda b+(1-\lambda)\frac{a}{m})\right|^q}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

Corollary 8. *By putting $m = 1$ in above theorem we get the following inequality for convex function via Caputo k-fractional derivatives*

$$\begin{aligned}
&\Bigg|\frac{f^{(n)}\left(\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)}\\
&-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(b-a)^{n-\frac{\alpha}{k}+1}}\times\Big[({}^{C}D^{\alpha,k}_{(\lambda b+(1-\lambda)a)^{+}}f)(\lambda a+(1-\lambda)b)\\
&+(-1)^{n}({}^{C}D^{\alpha,k}_{(\lambda a+(1-\lambda)b)^{-}}f)(\lambda b+(1-\lambda)a)\Big]\Bigg|\\
&\leq\left[\frac{2}{np-\frac{\alpha p}{k}+1}\left(1-\frac{1}{2^{np-\frac{\alpha p}{k}}}\right)\right]^{\frac{1}{p}}\\
&\left[\frac{\left|f^{(n+1)}(\lambda a+(1-\lambda)b)\right|^{q}+\left|f^{(n+1)}(\lambda b+(1-\lambda)a)\right|^{q}}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

Theorem 7. *Let $f:[a,mb]\to\mathbb{R}$ be a function such that $f\in C^{(n+1)}[a,mb]$ with $0\leq a<mb$. If $\left|f^{(n+1)}\right|^{q}$ is m-convex on $[a,mb]$ for some fixed $q\geq 1$, then the following inequality for Caputo k-fractional derivatives holds*

(3.8)

$$\begin{aligned}
&\Bigg|\frac{f^{(n)}\left(m\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}\\
&-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}}\times\Big[({}^{C}D^{\alpha,k}_{(m\lambda b+(1-\lambda)a)^{+}}f)(\lambda a+m(1-\lambda)b)\\
&+(-1)^{n}({}^{C}D^{\alpha,k}_{(\lambda a+m(1-\lambda)b)^{-}}f)(m\lambda b+(1-\lambda)a)\Big]\Bigg|\\
&\leq\left[\frac{2}{nq-\frac{\alpha q}{k}+1}\left(1-\frac{1}{2^{nq-\frac{\alpha q}{k}}}\right)\right]^{\frac{1}{q}}\\
&\left[\frac{\left|f^{(n+1)}(\lambda a+m(1-\lambda)b)\right|^{q}+m\left|f^{(n+1)}(\lambda b+(1-\lambda)\frac{a}{m})\right|^{q}}{2}\right]^{\frac{1}{q}}
\end{aligned}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$.

Proof. If we use Lemma 2, Hölder inequality and m-convexity of $|f(n+1)|^q$ respectively, we have

$$
\begin{aligned}
&\Bigg|\frac{f^{(n)}\left(m\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)} \\
&-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}}\times\Big[({}^C D^{\alpha,k}_{(m\lambda b+(1-\lambda)a)^+}f)(\lambda a+m(1-\lambda)b) \\
&+(-1)^n({}^C D^{\alpha,k}_{(\lambda a+m(1-\lambda)b)^-}f)(m\lambda b+(1-\lambda)a)\Big]\Bigg| \\
&\leq \int_0^1 \left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right| \\
&\left|f^{(n+1)}\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|dt \\
&\leq \left(\int_0^1 1^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|^q\right. \\
&\left.\left|f^{(n+1)}\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^{\frac{1}{2}} \left|(1-t)^{n-\frac{\alpha}{k}}-t^{n-\frac{\alpha}{k}}\right|^q\right. \\
&\left|f^{(n+1)}\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^q dt \\
&+\int_{\frac{1}{2}}^1 \left|t^{n-\frac{\alpha}{k}}-(1-t)^{n-\frac{\alpha}{k}}\right|^q \\
&\left.\left|f^{(n+1)}\left(t\left(\lambda a+m(1-\lambda)b\right)+m(1-t)\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}
$$

Using result

$$(A-B)^q \leq A^q - B^q, \quad A \geq B \geq 0, q \geq 1,$$

we have

$$\Bigg|\frac{f^{(n)}\left(m\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}$$

$$-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(mb-a)^{n-\frac{\alpha}{k}+1}}\times\Big[({}^CD^{\alpha,k}_{(m\lambda b+(1-\lambda)a)^+}f)(\lambda a+m(1-\lambda)b)$$

$$+(-1)^n({}^CD^{\alpha,k}_{(\lambda a+m(1-\lambda)b)^-}f)(m\lambda b+(1-\lambda)a)\Big]\Bigg|$$

$$\leq\Bigg(\left|f^{(n+1)}\left(\lambda a+m(1-\lambda)b\right)\right|^q\int_0^{\frac{1}{2}}\left[(1-t)^{nq-\frac{\alpha q}{k}}t-t^{nq-\frac{\alpha q}{k}+1}\right]dt$$

$$+m\left|f^{(n+1)}\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|^q\int_0^{\frac{1}{2}}\left[(1-t)^{nq-\frac{\alpha q}{k}+1}-t^{nq-\frac{\alpha q}{k}}(1-t)\right]dt$$

$$+\left|f^{(n+1)}\left(\lambda a+m(1-\lambda)b\right)\right|^q\int_{\frac{1}{2}}^{1}\left[t^{nq-\frac{\alpha q}{k}+1}-(1-t)^{nq-\frac{\alpha q}{k}}t\right]dt$$

$$+m\left|f^{(n+1)}\left(\lambda b+(1-\lambda)\frac{a}{m}\right)\right|^q\int_{\frac{1}{2}}^{1}\left[t^{nq-\frac{\alpha q}{k}}(1-t)-(1-t)^{nq-\frac{\alpha q}{k}+1}\right]dt\Bigg)^{\frac{1}{q}}$$

$$\leq\left[\frac{2}{nq-\frac{\alpha q}{k}+1}\left(1-\frac{1}{2^{nq-\frac{\alpha q}{k}}}\right)\right]^{\frac{1}{q}}$$

$$\left[\frac{\left|f^{(n+1)}(\lambda a+m(1-\lambda)b)\right|^q+m\left|f^{(n+1)}(\lambda b+(1-\lambda)\frac{a}{m})\right|^q}{2}\right]^{\frac{1}{q}}.$$

□

Corollary 9. *In Theorem 7, if we put $k=1$ we get the following result for Caputo fractional derivatives*

$$\Bigg|\frac{f^{(n)}\left(m\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+m(1-\lambda)b)}{(1-2\lambda)(mb-a)}$$

$$-\frac{\Gamma(n-\alpha+1)}{(1-2\lambda)^{n-\alpha+1}(mb-a)^{n-\alpha+1}}\times\Big[({}^CD^{\alpha}_{(m\lambda b+(1-\lambda)a)^+}f)(\lambda a+m(1-\lambda)b)$$

$$+(-1)^n({}^CD^{\alpha}_{(\lambda a+m(1-\lambda)b)^-}f)(m\lambda b+(1-\lambda)a)\Big]\Bigg|$$

$$\leq\left[\frac{2}{nq-\alpha q+1}\left(1-\frac{1}{2^{nq-\alpha q}}\right)\right]^{\frac{1}{q}}$$

$$\left[\frac{\left|f^{(n+1)}(\lambda a+m(1-\lambda)b)\right|^q+m\left|f^{(n+1)}(\lambda b+(1-\lambda)\frac{a}{m})\right|^q}{2}\right]^{\frac{1}{q}}$$

Corollary 10. *By taking $m = 1$ in above theorem we get the following result for convex function via Caputo k-fractional derivatives*

$$\Bigg|\frac{f^{(n)}\left(\lambda b+(1-\lambda)a\right)+f^{(n)}(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)}$$
$$-\frac{k\Gamma_k(n-\frac{\alpha}{k}+k)}{(1-2\lambda)^{n-\frac{\alpha}{k}+1}(b-a)^{n-\frac{\alpha}{k}+1}}\times\Big[\big({}^{C}D^{\alpha,k}_{(\lambda b+(1-\lambda)a)^{+}}f\big)(\lambda a+(1-\lambda)b)$$
$$+(-1)^{n}\big({}^{C}D^{\alpha,k}_{(\lambda a+(1-\lambda)b)^{-}}f\big)(\lambda b+(1-\lambda)a)\Big]\Bigg|$$
$$\leq\left[\frac{2}{nq-\frac{\alpha q}{k}+1}\left(1-\frac{1}{2^{nq-\frac{\alpha q}{k}}}\right)\right]^{\frac{1}{q}}$$
$$\left[\frac{\left|f^{(n+1)}(\lambda a+(1-\lambda)b)\right|^{q}+\left|f^{(n+1)}(\lambda b+(1-\lambda)a)\right|^{q}}{2}\right]^{\frac{1}{q}}$$

References

[1] A. Azócar, K. Nikodem and G. Roa, Fejér-type inequalities for strongly convex functions, *Ann. Math. Sil.*, 26 (2012), 43-54.

[2] A. G. Azpeitia, Convex functions and the Hadamard inequality, *Rev. Colomb. Mat.*, 28(1) (1994), 7-12.

[3] M. Alomari and M. Darus, On the Hadamard's inequality for log convex functions on coordinates, *J. Inequal. Appl.*, Article ID 283147 (2009), 13 pp.

[4] M. K. Bakula and J. Pečarić, Note on some Hadamard-type inequalities, *J. Ineq. Pure Appl. Math.*, 5(3) (2004).

[5] M. K. Bakula, M. E. Ozdemir and J. Pečarić, Hadamard type inequalities for m-convex and (α, m)-convex functions, *J. Ineq. Pure Appl. Math.*, 9(4) (2008), 12 pp.

[6] S. S. Dragomir, On some new inequalities of Hermite-Hadamard type for m–convex functions, *Tamkang J. Math.*, 33(1) (2002) 45–56.

[7] S. S. Dragomir and G. H. Toader, Some inequalities for m-convex functions, *Stud. Univ. Babeş-Bolyai, Math.*, 38(1) (1993), 21-28.

[8] S. S. Dragmoir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.*, 11(5) (1998), 91–95.

[9] S. S. Dragomir and C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, RGMIA Monographs, Victoria University, 2000. Math. Sic. Marh. Roum., **47**, (2004), 3–14.

[10] G. Farid, M. Marwan and A. U. Rehman, New mean value theorems and generalization of Hadamard inequality via coordinated

m-convex functions, *J. Inequal. Appl.* 2015, Article ID 283, (2015) 11pp.

[11] L. Fejér, Uberdie fourierreihen, II, Math. Naturwise. Anz Ungar. Akad., Wiss, 24, 369–390 (1906) (In Hungarian).

[12] P. M. Gill, C. E. M. Pearce and J. Pečarić, Hadamard's inequality for r-convex functions, *J. Math. Anal. Appl.*, 215(2) (1997), 461–470.

[13] İ. İşcan, Hermite Hadamard Fejér type inequalities for convex functions via fractional integrals, *Stud. Univ. Babeş-Bolyai Math.*, 60(3) (2015), 355–366.

[14] İ. İşcan, New estimates on generalization of some integral inequalities for (α, m)-convex functions, *Contemp. Anal. Appl. Math.*, 1(2) (2013), 253–264.

[15] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, *North-Holland Math. Stud.*, 204, Elsevier, New York-London, 2006.

[16] U. S. Kirmaci, M. K. Bakula, M. E. Ozdemir and J. Pečarić, Hadamard-type inequalities for s-convex functions, *Appl. Math. Comput.*, 193 (2007), 26–35.

[17] S. Mubeen and G. M. Habibullah, k-fractional integrals and applications, *Int. J. Contemp. Math. Sci.*, 7 (2012), 89–94 .

[18] P.T. Mocanu, I. Serb and G. Toader, Real star-convex functions, *Studia Univ. Babes-Bolyai, Math.*, 42(3) (1997) 65–80.

[19] M. E. Ozdemir, M. Avci and E. Set, On some inequalities of HermiteHadamard type via m-convexity, *Appl. Math. Lett.*, 23(9) (2010), 1065-1070.

[20] C. E. M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, *Appl. Math. Lett.*, 13(2) (2000), 51-55.

[21] M. Z. Sarikaya and H. Budak, Generalized Hermite–Hadamard type integral inequalities for fractional integrals, *Filomat.*, 30(5) (2016), 1315–1326.

[22] M. Z. Sarikaya, E. Set, H. Yaldiz and N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *J. Math. Comput. Model.*, 57 (2013), 2403–2407.

[23] M. Z. Sarikaya and S. Erden, On the Hermite-Hadamard Féjer-type integral inequality for convex function, *turkish journal of analysis and number theory.*, 2(3) (2014), 85–89.

[24] G. H. Toader, Some generalizations of the convexity, *Proc. Colloq. Approx. Optim. Cluj-Naploca.*, (Romania) (1984), 329-338.

[25] J. Wang, C. Zhu and Y. Zhou, New generalized Hermite-Hadamard type inequalities and applications to special means, *J. Inequal. Appl.*, 2013, Article ID 325, 15pp.

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