Received 09/06/17

Estimation Type Results Related to Fejér Inequality with Applications

[1]M. Rostamian Delavar* and and [2]S. S. Dragomir

[1]Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran

[2]Mathematics, College of Engineering & Science, Victoria University,

PO Box 14428, Melbourne City, MC 8001, Australia

E-mail: `m.rostamian@ub.ac.ir, sever.dragomir@vu.edu.au`

Abstract

This paper deals with some new theorems and inequalities about a famous kind of Fejér type integral inequalities which estimate the difference between right and middle parts in Fejér inequality with new bounds. Also as application higher moments of random variables and an error estimate for Trapezoid formula are given.

Keywords: Convex function, Fejér inequality, Trapezoid formula, Random variable.

2010 MSC: 26A51, 26D15, 52A01.

1 Introduction and Preliminaries

The Fejér integral inequality for convex functions has been proved in [2]:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then*

$$f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{1.1}$$

where $g : [a, b] \to \mathbb{R}^+ = [0, +\infty)$ is integrable and symmetric to $x = \frac{a+b}{2}$ $\Big(g(x) = g(a + b - x), \forall x \in [a, b]\Big)$.

An interesting problem in (1.1) is the estimation of difference for right-middle part and left-middle part of this inequality. The following theorem has been proved in [3], that estimates the difference between right and middle part in (1.1) using Hölder's inequality.

Theorem 1.2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $w : [a, b] \to \mathbb{R}^+$ be a differentiable mapping and symmetric to $\frac{a+b}{2}$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx - \frac{1}{b-a}\int_a^b f(x)w(x)dx\right| \le \frac{1}{2}\left[\int_0^1 \big(g(x)\big)^p dt\right]^{\frac{1}{p}}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

where

$$g(x) = \left|\int_{a+(b-a)t}^{b-(b-a)t} w(x)dx\right|$$

for $t \in [0, 1]$.

RGMIA Res. Rep. Coll. 20 (2017), Art. 80, 11 pp

The following two lemmas hold for symmetric functions as well and we use these results to obtain various inequalities in next section.

Lemma 1.3. *Suppose that $w : [a,b] \to \mathbb{R}$ is an integrable function on $[a,b]$ and symmetric to $\frac{a+b}{2}$. For any $t \in [0,1]$ with $0 \le t \le \frac{1}{2}$ we have*

$$\int_t^1 w\big(sa+(1-s)b\big)ds - \int_0^t w\big(sa+(1-s)b\big)ds = 2\int_t^{\frac{1}{2}} w\big(sa+(1-s)b\big)ds. \tag{1.2}$$

Proof. Using the change of variable $x = sa+(1-s)b$ for $0 \le t \le \frac{1}{2}$ we get

$$\int_t^1 w\big(sa+(1-s)b\big)ds - \int_0^t w\big(sa+(1-s)b\big)ds = \frac{1}{b-a}\left[\int_a^u w(x)dx - \int_u^b w(x)dx\right], \tag{1.3}$$

where $0 \le u \le \frac{a+b}{2}$.
Since w is symmetric to $\frac{a+b}{2}$ we have

$$\int_{\frac{a+b}{2}}^b w(x)dx = \int_a^{\frac{a+b}{2}} w(x)dx.$$

Then

$$\int_u^b w(x)dx = \int_u^{\frac{a+b}{2}} w(x)dx + \int_{\frac{a+b}{2}}^b w(x)dx = \int_u^{\frac{a+b}{2}} w(x)dx + \int_a^{\frac{a+b}{2}} w(x)dx.$$

Also

$$\int_a^{\frac{a+b}{2}} w(x)dx = \int_a^u w(x)dx + \int_u^{\frac{a+b}{2}} w(x)dx.$$

So

$$\frac{1}{b-a}\left[\int_a^u w(x)dx - \int_u^b w(x)dx\right] = \frac{-2}{b-a}\int_u^{\frac{a+b}{2}} w(x)dx = 2\int_t^{\frac{1}{2}} w\big(sa+(1-s)b\big)ds. \tag{1.4}$$

Using (1.4) in (1.3) we deduce (1.2). □

With the same argument used in the proof of Lemma 1.3 we can drive the following equality.

Lemma 1.4. *Suppose that $w : [a,b] \to \mathbb{R}$ is an integrable function on $[a,b]$ and symmetric to $\frac{a+b}{2}$. For $t \in [0,1]$ with $\frac{1}{2} \le t \le 1$ we have*

$$\int_0^t w\big(sa+(1-s)b\big)ds - \int_t^1 w\big(sa+(1-s)b\big)ds = 2\int_{\frac{1}{2}}^t w\big(sa+(1-s)b\big)ds.$$

From Lemma 1.3 and Lemma 1.4, if w is a symmetric nonnegative function, we can obtain two integral inequalities that are useful for our further consideration.

Corollary 1.5. *Suppose that $w : [a,b] \to \mathbb{R}^+$ is an integrable function on $[a,b]$ and symmetric to $\frac{a+b}{2}$. Then*

$$\int_t^1 w\big(sa+(1-s)b\big)ds - \int_0^t w\big(sa+(1-s)b\big)ds \ge 0, \qquad 0 \le t \le \frac{1}{2},$$

and

$$\int_0^t w\big(sa+(1-s)b\big)ds - \int_t^1 w\big(sa+(1-s)b\big)ds \ge 0, \qquad \frac{1}{2} \le t \le 1.$$

The following identity has been obtained in [3] and will be used in the proof of theorems.

Lemma 1.6. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $w : [a, b] \to \mathbb{R}^+$ be a differentiable mapping. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{1}{b-a}\Big(\frac{f(a)+f(b)}{2}\int_a^b w(x)dx - \int_a^b f(x)w(x)dx\Big)$$
$$= \frac{b-a}{2}\int_0^1 p(t)f'\big(ta+(1-t)b\big)dt,$$

where

$$p(t) = \int_t^1 w\big(sa+(1-s)b\big)ds - \int_0^t w\big(sa+(1-s)b\big)ds.$$

2 Main Results

For the first result of this section, by using Corollary 1.5 and Lemma 1.6, we estimate the difference between the right and middle terms in (1.1) with a simple and new face, without need of using Hölder's inequality in the proof.

Theorem 2.1. *Suppose that $f : I \to \mathbb{R}$ is a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $w : [a, b] \to \mathbb{R}^+$ is a differentiable mapping symmetric to $\frac{a+b}{2}$. If $|f'|$ is a convex mapping on $[a, b]$, Then*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)dx - \int_a^b f(x)w(x)dx\right| \le \big[|f'(a)|+|f'(b)|\big]\int_{\frac{a+b}{2}}^b w(x)(b-x)dx. \tag{2.1}$$

Proof. From Lemma 1.6, Corollary 1.5 and convexity of $|f'|$ we have

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)dx - \int_a^b f(x)w(x)dx\right|$$
$$= \frac{(b-a)^2}{2}\left|\int_0^1\left[\int_t^1 w\big(sa+(1-s)b\big)ds - \int_0^t w\big(sa+(1-s)b\big)ds\right]f'(ta+(1-t)b)dt\right|$$
$$\le \frac{(b-a)^2}{2}\left\{\int_0^{\frac{1}{2}}\left|\int_t^1 w\big(sa+(1-s)b\big)ds - \int_0^t w\big(sa+(1-s)b\big)ds\right||f'|\big(ta+(1-t)b\big)dt\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left|\int_t^1 w\big(sa+(1-s)b\big)ds - \int_0^t w\big(sa+(1-s)b\big)ds\right||f'|\big(ta+(1-t)b\big)dt\right\}$$
$$\le \frac{(b-a)^2}{2}\left\{\int_0^{\frac{1}{2}}\left(\int_t^1 w\big(sa+(1-s)b\big)ds - \int_0^t w\big(sa+(1-s)b\big)ds\right)\big(t|f'(a)|+(1-t)|f'(b)|\big)dt\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left(\int_0^t w\big(sa+(1-s)b\big)ds - \int_t^1 w\big(sa+(1-s)b\big)ds\right)\big(t|f'(a)|+(1-t)|f'(b)|\big)\right\} = I.$$

If we change the order of integration in I, then

$$I=\frac{(b-a)^2}{2}\Bigg\{\int_0^{\frac{1}{2}}\int_0^{s} w\big(sa+(1-s)b\big)\big(t|f'(a)|+(1-t)|f'(b)|\big)dtds$$

$$+\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} w\big(sa+(1-s)b\big)\big(t|f'(a)|+(1-t)|f'(b)|\big)dtds$$

$$-\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w\big(sa+(1-s)b\big)\big(t|f'(a)|+(1-t)|f'(b)|\big)dtds$$

$$+\int_{\frac{1}{2}}^{1}\int_s^{1} w\big(sa+(1-s)b\big)\big(t|f'(a)|+(1-t)|f'(b)|\big)dtds$$

$$+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} w\big(sa+(1-s)b\big)\big(t|f'(a)|+(1-t)|f'(b)|\big)dtds$$

$$-\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w\big(sa+(1-s)b\big)\big(t|f'(a)|+(1-t)|f'(b)|\big)dtds\Bigg\}.$$

Calculating all inner integrals in I we get

$$I=\frac{(b-a)^2}{2}\Bigg\{\int_0^{\frac{1}{2}} w\big(sa+(1-s)b\big)\Big(\frac{1}{2}s^2|f'(a)|+(s-\frac{1}{2}s^2)|f'(b)|\Big)ds$$

$$+\int_{\frac{1}{2}}^{1} w\big(sa+(1-s)b\big)\Big(\frac{1}{8}|f'(a)|+\frac{3}{8}|f'(b)|\Big)ds$$

$$-\int_0^{\frac{1}{2}} w\big(sa+(1-s)b\big)\Big((\frac{1}{8}-\frac{1}{2}s^2)|f'(a)|+(\frac{3}{8}-s+\frac{1}{2}s^2)|f'(b)|\Big)ds$$

$$+\int_{\frac{1}{2}}^{1} w\big(sa+(1-s)b\big)\Big((\frac{1}{2}-\frac{1}{2}s^2)|f'(a)|+(\frac{1}{2}-s+\frac{1}{2}s^2)|f'(b)|\Big)ds$$

$$+\int_0^{\frac{1}{2}} w\big(sa+(1-s)b\big)\Big(\frac{3}{8}|f'(a)|+\frac{1}{8}|f'(b)|\Big)ds$$

$$-\int_{\frac{1}{2}}^{1} w\big(sa+(1-s)b\big)\Big((\frac{1}{2}s^2-\frac{1}{8})|f'(a)|+(s-\frac{1}{2}s^2-\frac{3}{8})|f'(b)|\Big)ds\Bigg\}.$$

Simple form of I can be obtained as the following:

$$I=\frac{(b-a)^2}{2}\Bigg\{\int_0^{\frac{1}{2}} w\big(sa+(1-s)b\big)\Big((s^2+\frac{1}{4})|f'(a)|+(-s^2+2s-\frac{1}{4})|f'(b)|\Big)ds$$

$$+\int_{\frac{1}{2}}^{1} w\big(sa+(1-s)b\big)\Big((-s^2+\frac{3}{4})|f'(a)|+(s^2-2s+\frac{5}{4})|f'(b)|\Big)ds\Bigg\}.$$

If we use the change of variable $x=sa+(1-s)b$ in I, then

$$I=\frac{(b-a)}{2}\Bigg\{\int_{\frac{a+b}{2}}^{b} w(x)\Big(\Big[(\frac{x-b}{a-b})^2+\frac{1}{4}\Big]|f'(a)|+\Big[2(\frac{x-b}{a-b})-(\frac{x-b}{a-b})^2-\frac{1}{4}\Big]|f'(b)|\Big)dx$$

$$+\int_a^{\frac{a+b}{2}} w(x)\Big(\Big[\frac{3}{4}-(\frac{x-b}{a-b})^2\Big]|f'(a)|+\Big[(\frac{x-b}{a-b})^2-2(\frac{x-b}{a-b})+\frac{5}{4}\Big]|f'(b)|\Big)dx.\Bigg\}$$

On the other hand since w is symmetric to $\frac{a+b}{2}$ then

$$\int_a^{\frac{a+b}{2}} w(x)\Big(\Big[\frac{3}{4}-(\frac{x-b}{a-b})^2\Big]|f'(a)|+\Big[(\frac{x-b}{a-b})^2-2(\frac{x-b}{a-b})+\frac{5}{4}\Big]|f'(b)|\Big)dx$$
$$=\int_{\frac{a+b}{2}}^b w(x)\Big(\Big[\frac{3}{4}-(\frac{a-x}{a-b})^2\Big]|f'(a)|+\Big[(\frac{a-x}{a-b})^2-2(\frac{a-x}{a-b})+\frac{5}{4}\Big]|f'(b)|\Big)dx.$$

So

$$I=\frac{(b-a)}{2}\Big\{\int_{\frac{a+b}{2}}^b w(x)\Big(\Big[1+(\frac{x-b}{a-b})^2-(\frac{a-x}{a-b})^2\Big]|f'(a)|$$
$$+\Big[1-2(\frac{a-x}{a-b})-(\frac{x-b}{a-b})^2+(\frac{a-x}{a-b})^2+2(\frac{x-b}{a-b})\Big]|f'(b)|\Big)dx\Big\}$$
$$=\frac{(b-a)}{2}\Big\{\int_{\frac{a+b}{2}}^b w(x)\Big(2\Big[\frac{x-b}{a-b}\Big]|f'(a)|+2\Big[\frac{x-b}{a-b}\Big]|f'(b)|\Big)dx$$
$$=\big(|f'(a)|+|f'(b)|\big)\int_{\frac{a+b}{2}}^b w(x)(b-x)dx.$$

□

Corollary 2.2. (Theorem 2.2 in [1]) If in Theorem 2.1 we consider $w\equiv 1$, then

$$\Big|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)dx\Big|\le\frac{(b-a)^2}{8}\big(|f'(a)|+|f'(b)|\big).$$

If the derivative of the function is bounded from below and above, then we can drive an estimation type result related to Fejér inequality.

Theorem 2.3. *Suppose that $f:I\to\mathbb{R}$ is a differentiable mapping on I°, $a,b\in I^\circ$ with $a<b$ and $w:[a,b]\to\mathbb{R}^+$ is a differentiable mapping . Assume that f' is integrable on $[a,b]$ and there exist constants $m<M$ such that*

$$-\infty<m\le f'(x)\le M<\infty\qquad for\ all\ x\in[a,b].$$

Then

$$\Big|\frac{f(a)+f(b)}{2(b-a)}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx-\frac{m+M}{4}\int_0^1 p(t)dt\Big|$$
$$\le\frac{(M-m)(b-a)}{4}\int_0^1|p(t)|dt, \tag{2.2}$$

where $p(t)$ is defined in Lemma 1.6.

Proof. From Lemma 1.6 we have

$$\frac{1}{b-a}\Big(\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\int_a^b f(x)w(x)dx\Big)$$
$$=\frac{b-a}{2}\int_0^1 p(t)\Big[f'\big(ta+(1-t)b\big)-\frac{m+M}{2}+\frac{m+M}{2}\Big]dt$$
$$=\frac{(m+M)(b-a)}{4}\int_0^1 p(t)dt+\frac{b-a}{2}\int_0^1 p(t)\Big[f'\big(ta+(1-t)b\big)-\frac{m+M}{2}\Big]dt.$$

So

$$I = \frac{1}{b-a}\left(\frac{f(a)+f(b)}{2}\int_a^b w(x)dx - \int_a^b f(x)w(x)dx\right) - \frac{(m+M)(b-a)}{4}\int_0^1 p(t)dt$$
$$= \frac{b-a}{2}\int_0^1 p(t)\Big[f'\big(ta+(1-t)b\big) - \frac{m+M}{2}\Big]dt.$$

Therefore

$$|I| \le \frac{b-a}{2}\int_0^1 |p(t)|\Big|f'\big(ta+(1-t)b\big) - \frac{m+M}{2}\Big|dt \le \frac{(M-m)(b-a)}{4}\int_0^1 |p(t)|dt,$$

since from inequality $m \le f'\big(ta+(1-t)b\big) \le M$ we have

$$m - \frac{m+M}{2} \le f'\big(ta+(1-t)b\big) - \frac{m+M}{2} \le M - \frac{m+M}{2},$$

which implies that

$$\Big|f'\big(ta+(1-t)b\big) - \frac{m+M}{2}\Big| \le \frac{M-m}{2}.$$

□

Remark 2.4. If in Theorem 2.3 we assume that w is symmetric to $\frac{a+b}{2}$, then from Lemma 1.3 and Lemma 1.4 we have

$$\int_0^1 |p(t)|dt = 2\int_0^1 \Big|\int_t^{\frac{1}{2}} w\big(sa+(1-s)b\big)ds\Big|dt \le 2\int_0^1 \big|t-\frac{1}{2}\big| \sup_{s\in[t,\frac{1}{2}]} \big|w\big(sa+(1-s)b\big)\big|dt \le \frac{1}{2}||w||_\infty.$$

Then

$$\left|\frac{f(a)+f(b)}{2(b-a)}\int_a^b w(x)dx - \frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{m+M}{4}\int_0^1 p(t)dt\right|$$
$$\le \frac{(M-m)(b-a)}{8}||w||_\infty.$$

Also using Hölder's inequality we have

$$\int_0^1 |p(t)|dt \le 2\int_0^1 \big|t-\frac{1}{2}\big|^{\frac{1}{p}}\left(\int_t^{\frac{1}{2}} |w\big(sa+(1-s)b\big)|^q ds\right)^{\frac{1}{q}} dt \le 2||w||_q \int_0^1 \big|t-\frac{1}{2}\big|^{\frac{1}{p}} dt,$$

which implies that

$$\left|\frac{f(a)+f(b)}{2(b-a)}\int_a^b w(x)dx - \frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{m+M}{4}\int_0^1 p(t)dt\right|$$
$$\le \frac{(M-m)(b-a)}{2}||w||_q \int_0^1 \big|t-\frac{1}{2}\big|^{\frac{1}{p}} dt.$$

Corollary 2.5. *In Theorem 2.3 if $w \equiv 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{m(1+a-b)+M(1+b-a)}{8}.$$

Proof. If we consider $w \equiv 1$, then the relations $||w||_\infty = 1$ and $\int_0^1 |p(t)|dt \le \frac{1}{2}$ imply that

$$\frac{1}{b-a}\left|\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)dx\right| \le \left|\frac{m+M}{4}\int_0^1 p(t)dt\right| + \frac{(M-m)(b-a)}{8}$$
$$\le \frac{m+M}{8} + \frac{(M-m)(b-a)}{8} = \frac{m(1+a-b)+M(1+b-a)}{8}.$$

□

Estimation for difference between the right and middle terms of (1.1) when the derivative of the function satisfies a Lipschitz condition is our next aim.

Theorem 2.6. *Suppose that $f : I \to \mathbb{R}$ is a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and $w : [a, b] \to \mathbb{R}^+$ is a differentiable mapping. Assume that f' is integrable on $[a, b]$ and satisfies a Lipschitz condition for some $K > 0$. Then*

$$\left|\frac{f(a)+f(b)}{2(b-a)}\int_a^b w(x)dx - \frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{2}f'\Big(\frac{a+b}{2}\Big)\int_0^1 p(t)dt\right| \tag{2.3}$$
$$\le K\frac{(b-a)}{2}\int_0^1 |t-\frac{1}{2}||p(t)|dt,$$

where $p(t)$ is defined in Lemma 1.6.

Proof. From Lemma 1.6

$$\frac{1}{b-a}\Big(\frac{f(a)+f(b)}{2}\int_a^b w(x)dx - \int_a^b f(x)w(x)dx\Big)$$
$$= \frac{b-a}{2}\int_0^1 p(t)\Big[f'\big(ta+(1-t)b\big) - f'\Big(\frac{a+b}{2}\Big) + f'\Big(\frac{a+b}{2}\Big)\Big]dt$$
$$= \frac{b-a}{2}\int_0^1 p(t)\Big[f'\big(ta+(1-t)b\big) - f'\Big(\frac{a+b}{2}\Big)\Big]dt + \frac{b-a}{2}f'\Big(\frac{a+b}{2}\Big)\int_0^1 p(t)dt.$$

Then

$$\frac{1}{b-a}\Big(\frac{f(a)+f(b)}{2}\int_a^b w(x)dx - \int_a^b f(x)w(x)dx\Big) - \frac{b-a}{2}f'\Big(\frac{a+b}{2}\Big)\int_0^1 p(t)dt$$
$$= \frac{b-a}{2}\int_0^1 p(t)\Big[f'\big(ta+(1-t)b\big) - f'\Big(\frac{a+b}{2}\Big)\Big]dt.$$

Since f' satisfies a Lipschitz condition for some $K > 0$, then

$$\Big|f'\big(ta+(1-t)b\big) - f'\Big(\frac{a+b}{2}\Big)\Big| \le K\Big|ta+(1-t)b - \frac{a+b}{2}\Big| = K\big|t-\frac{1}{2}\big|(b-a).$$

Hence

$$\left|\frac{f(a)+f(b)}{2(b-a)}\int_a^b w(x)dx - \frac{1}{b-a}\int_a^b f(x)w(x)dx - \frac{1}{2}f'\Big(\frac{a+b}{2}\Big)\int_0^1 p(t)dt\right|$$
$$\le K\frac{(b-a)}{2}\int_0^1 |t-\frac{1}{2}||p(t)|dt.$$

□

Remark 2.7. In Theorem 2.6 assume that w is symmetric to $\frac{a+b}{2}$. With the same argument as Remark 2.4, using Lemma 1.3 and Lemma 1.4 we get

$$\begin{aligned}&\Big|\frac{f(a)+f(b)}{2(b-a)}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx-\frac{1}{2}f'\Big(\frac{a+b}{2}\Big)\int_0^1 p(t)dt\Big|\\&\leq K(b-a)\int_0^1\int_t^{\frac{1}{2}}|t-\frac{1}{2}||w\big(sa+(1-s)b\big)|dsdt.\end{aligned}$$

Also we have

$$\begin{aligned}&\Big|\frac{f(a)+f(b)}{2(b-a)}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx-\frac{1}{2}f'\Big(\frac{a+b}{2}\Big)\int_0^1 p(t)dt\Big|\\&\leq 2K(b-a)||w||_\infty\int_0^1(t-\frac{1}{2})^2dt=\frac{K(b-a)}{6}||w||_\infty,\end{aligned}$$

which implies that

$$\Big|\frac{f(a)+f(b)}{2(b-a)}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx\Big|\leq\Big(\frac{K}{6}+\frac{1}{4}\Big|f'\Big(\frac{a+b}{2}\Big)\Big|\Big)||w||_\infty(b-a).$$

Corollary 2.8. *In Theorem 2.6 if $w\equiv 1$, then*

$$\Big|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\Big|\leq\Big(\frac{K}{6}+\frac{1}{4}\Big|f'\Big(\frac{a+b}{2}\Big)\Big|\Big)(b-a).$$

3 Application

3.1 Random Variable

Suppose that for $0<a<b$, $w:[a,b]\to[0,+\infty)$ is a continuous probability density function which is symmetric about $\frac{a+b}{2}$. Also for $r\in\mathbb{R}$, suppose that the r-moment

$$E_r(X)=\int_a^b x^r w(x)dx$$

is finite.

(1) If we consider $f(x)=x^r$ for $r\geq 2$ and $x\in[a,b]$, then $|f'(x)|=rx^{r-1}$ which is a convex function and so from (2.1) in Theorem 2.1 we have

$$\Big|\frac{a^r+b^r}{2}-E_r(X)\Big|\leq\frac{r(b-a)}{4}\Big(a^{r-1}+b^{r-1}\Big),$$

since

$$\begin{aligned}&\Big|\frac{a^r+b^r}{2}-E_r(X)\Big|\leq r\Big(a^{r-1}+b^{r-1}\Big)\int_{\frac{a+b}{2}}^b w(x)(b-x)dx\\&\leq r\Big(a^{r-1}+b^{r-1}\Big)\frac{b-a}{2}\int_{\frac{a+b}{2}}^b w(x)dx=r\Big(a^{r-1}+b^{r-1}\Big)\frac{b-a}{4},\end{aligned}$$

where from the fact that w is symmetric and $\int_a^b w(x)dx=1$, we have $\int_{\frac{a+b}{2}}^b w(x)dx=\frac{1}{2}$.
If $r=1$, $E(X)$ is the expectation of the random variable X and from above inequality we obtain the following

known bound

$$\left|\frac{a+b}{2} - E(X)\right| \leq \frac{b-a}{2}.$$

(2) If we consider $f(x) = x^r$ for $r \in \mathbb{R}$ and $x \in [a,b]$, then $m = ra^{r-1} \leq f'(x) = rx^{r-1} \leq rb^{r-1} = M$ and so from (2.2) in Theorem 2.3 we have

$$\frac{1}{b-a}\left|\frac{a^r+b^r}{2} - E_r(X) - \frac{r(a^{r-1}+b^{r-1})(b-a)}{4}\int_0^1 p(t)dt\right|$$
$$\leq \frac{r(b^{r-1}-a^{r-1})(b-a)}{4}\int_0^1 |p(t)|dt \leq \frac{r(b^{r-1}-a^{r-1})(b-a)}{4}.$$

It follows that

$$\frac{1}{b-a}\left|\frac{a^r+b^r}{2} - E_r(X)\right| \leq \frac{r(a^{r-1}-b^{r-1})(b-a)}{4} + \frac{r(a^{r-1}+b^{r-1})}{4}\int_0^1 |p(t)|dt$$
$$\leq \frac{r(b^{r-1}-a^{r-1})(b-a)}{4} + \frac{r(a^{r-1}+b^{r-1})}{4}.$$

Therefore

$$\left|\frac{a^r+b^r}{2} - E_r(X)\right| \leq \frac{r(a^{r-1}-b^{r-1})(b-a)^2}{4} + \frac{r(a^{r-1}+b^{r-1})(b-a)}{4}.$$

If we consider $r = 1$ in above inequality, then we have

$$\left|\frac{a+b}{2} - E(X)\right| \leq \frac{(b-a)}{2}. \tag{3.1}$$

(3) If we consider $f(x) = x^r$ for $r \in \mathbb{R}$ and $x \in [a,b]$ then

$$|p(t)| = \Big|\int_t^1 w\big(sa+(1-s)b\big)ds - \int_0^t w\big(sa+(1-s)b\big)ds\Big|$$
$$\leq \int_t^1 w\big(sa+(1-s)b\big)ds + \int_0^t w\big(sa+(1-s)b\big)ds = \int_0^1 w\big(sa+(1-s)b\big)ds = 1.$$

Also for f the Lipschitz constant $K = \sup_{x\in[a,b]} |f'(x)| = \sup_{x\in[a,b]} rx^{r-1}$ is equivalent to

$$K = \begin{cases} rb^{r-1}, & r \geq 1; \\ ra^{r-1}, & r < 1. \end{cases}$$

So from (2.3) in Theorem 2.6 we have

$$\left|\frac{a^r+b^r}{2} - E_r(X)\right| \leq K\frac{(b-a)^2}{2}\int_0^1 |t-\frac{1}{2}||p(t)|dt + f'\Big(\frac{a+b}{2}\Big)\frac{b-a}{2}\int_0^1 |p(t)|dt$$
$$\leq K\frac{(b-a)^2}{2}\int_0^1 |t-\frac{1}{2}|dt + f'\Big(\frac{a+b}{2}\Big)\frac{b-a}{2} = K\frac{(b-a)^2}{8} + f'\Big(\frac{a+b}{2}\Big)\frac{b-a}{2},$$

which implies that

$$\left|\frac{a^r+b^r}{2} - E_r(X)\right| \leq \begin{cases} \frac{r(b-a)}{2}\left[\frac{b^{r-1}(b-a)}{4} + \left(\frac{a+b}{2}\right)^{r-1}\right], & r \geq 1; \\ \\ \frac{r(b-a)}{2}\left[\frac{a^{r-1}(b-a)}{4} + \left(\frac{a+b}{2}\right)^{r-1}\right], & r < 1. \end{cases}$$

If we consider $r = 1$ in above inequality, then we have

$$\left|\frac{a+b}{2} - E(X)\right| \leq \frac{(b-a)^2}{8}.$$

This inequality is sometime better and other times worse than the above inequality (3.1), depending on the difference $b - a$.

3.2 Trapezoidal Formula

Consider the partition (P) of interval $[a, b]$ as $a = x_0 < x_1 < x_2 < ... < x_n = b$. The quadrature formula is

$$\int_a^b f(x)w(x)dx = T(f, w, P) + E(f, w, P),$$

where

$$T(f, w, P) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \int_{x_i}^{x_{i+1}} w(x)dx,$$

is the Trapezoidal form and $E(f, w, P)$ is the associated approximation error.

For each $i \in \{0, 1, ..., n-1\}$ consider interval $[x_i, x_{i+1}]$ of partition (P) of interval $[a, b]$. Suppose that all conditions of Theorem 2.1 are satisfied on $[x_i, x_{i+1}]$. Then

$$\begin{aligned}&\left|\frac{f(x_i) + f(x_{i+1})}{2} \int_{x_i}^{x_{i+1}} w(x)dx - \int_{x_i}^{x_{i+1}} f(x)w(x)dx\right| \\ &\leq \Big[|f'(x_i)| + |f'(x_{i+1})|\Big] \int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}} w(x)(x_{i+1} - x)dx.\end{aligned} \tag{3.2}$$

Now if all conditions of Theorem 2.1 are satisfied for the partition (P) on interval $[a, b]$ then using inequality (3.2), summing with respect to i from $i = 0$ to $i = n - 1$ and using triangle inequality we obtain

$$\begin{aligned}&\left|T(f, w, P) - \int_a^b f(x)w(x)dx\right| = \left|\sum_{i=0}^{n-1} \Big[\frac{f(x_i) + f(x_{i+1})}{2} \int_{x_i}^{x_{i+1}} w(x)dx - \int_{x_i}^{x_{i+1}} f(x)w(x)dx\Big]\right| \\ &\leq \sum_{i=0}^{n-1} \left|\frac{f(x_i) + f(x_{i+1})}{2} \int_{x_i}^{x_{i+1}} w(x)dx - \int_{x_i}^{x_{i+1}} f(x)w(x)dx\right| \\ &\leq \sum_{i=0}^{n-1} \Big[|f'(x_i)| + |f'(x_{i+1})|\Big] \int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}} w(x)(x_{i+1} - x)dx.\end{aligned}$$

So we get the error bound:

$$|E(f, w, P)| \leq \sum_{i=0}^{n-1} \Big[|f'(x_i)| + |f'(x_{i+1})|\Big] \int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}} w(x)(x_{i+1} - x)dx.$$

References

[1] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Applied Mathematics Letters, **11** (1998) 91–95.

[2] L. Fejér, *Über die fourierreihen*, II, Math. Naturwise. Anz Ungar. Akad. Wiss. **24** (1906) 369–390.

[3] M. Z. Sarikaya, *On new Hermite Hadamard Fejer type integral inequalities*, Stud. Univ. Babes-Bolyai Math. **57**(3) (2012) 377–386.