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HERMITE-HADAMARD TYPE INEQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF h-CONVEX FUNCTIONS

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ABSTRACT. In this paper we establish some Hermite-Hadamard type inequalities for the Generalized Riemann-Liouville fractional integrals $I_{a+,g}^{\alpha}f$ and $I_{b-,g}^{\alpha}f$, g, where g is a strictly increasing function on (a,b), having a continuous derivative on (a,b) and under the assumption that the composite function $f\circ g^{-1}$ is h-convex on $(g(a), g(b))$. Some applications for Hadamard fractional integrals and s-Godunova-Levin type convex functions are also provided.

1. INTRODUCTION

The following integral inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2},$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality*.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [15], the recent survey paper [12] and the references therein.

Let (a,b) with $-\infty\leq a<b\leq\infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\operatorname{Re}(\alpha)>0$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Following [21, p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ by

$$I_{a+,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{g'(t)\,f(t)\,dt}{[g(x)-g(t)]^{1-\alpha}},\ a<x\leq b \tag{1.1}$$

and

$$I_{b-,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{g'(t)\,f(t)\,dt}{[g(t)-g(x)]^{1-\alpha}},\ a\leq x<b. \tag{1.2}$$

For $g(t)=t$ we have the classical *Riemann-Liouville fractional integrals*

$$J_{a+}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(t)\,dt}{(x-t)^{1-\alpha}},\ a<x\leq b \tag{1.3}$$

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and

$$J_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha}},\ a \le x < b, \tag{1.4}$$

while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [21, p. 111]

$$H_{a+}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0 \le a < x \le b \tag{1.5}$$

and

$$H_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0 \le a < x < b. \tag{1.6}$$

One can consider the function $g(t) = -t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha}f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}t^{\alpha+1}},\ 0 \le a < x \le b \tag{1.7}$$

and

$$R_{b-}^{\alpha}f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha}t^{\alpha+1}},\ 0 \le a < x < b. \tag{1.8}$$

Also, for $g(t) = t^p$, $p > 0$, we have the *p-Riemann-Liouville fractional integrals*

$$J_{a+,p}^{\alpha}f(x) := \frac{p}{\Gamma(\alpha)}\int_a^x \frac{t^{p-1}f(t)\,dt}{(x^p-t^p)^{1-\alpha}},\ 0 \le a < x \le b \tag{1.9}$$

and

$$J_{b-,p}^{\alpha}f(x) := \frac{p}{\Gamma(\alpha)}\int_x^b \frac{t^{p-1}f(t)\,dt}{(t^p-x^p)^{1-\alpha}},\ 0 \le a \le x < b. \tag{1.10}$$

For some recent papers on Hermite-Hadamard inequalities for the Riemann-Liouville fractional integrals see [1], [5], [6], [18]-[37] and the references therein.

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* a, $b \in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a,b) = A(a,b) := \frac{a+b}{2}$, the *arithmetic mean.* If $I = (0,\infty)$ and $g(t) = \ln t$, then $M_g(a,b) = G(a,b) := \sqrt{ab}$, the *geometric mean.* If $I = (0,\infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a,b) = H(a,b) := \frac{2ab}{a+b}$, the *harmonic mean.* If $I = (0,\infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a,b) = M_p(a,b) := \left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p.*

In the recent paper [13] we obtained the following result:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f \circ g^{-1}$*

is convex on $(g(a), g(b))$, *then for any* $x \in (a,b)$ *we have the inequalities*

$$\begin{aligned}(1.11)\qquad & f\circ g^{-1}\left(\frac{\alpha}{\alpha+1}\frac{g(a)+g(b)}{2}+\frac{1}{\alpha+1}g(x)\right)\\ &\leq \alpha\int_0^1 s^{\alpha-1} f\circ g^{-1}\left(s\frac{g(a)+g(b)}{2}+(1-s)g(x)\right)ds\\ &\leq \frac{1}{2}\Gamma(\alpha+1)\left[\frac{I^{\alpha}_{a+,g}f(x)}{(g(x)-g(a))^{\alpha}}+\frac{I^{\alpha}_{b-,g}f(x)}{(g(b)-g(x))^{\alpha}}\right]\\ &\leq \frac{\alpha}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{1}{\alpha}f(x)\right].\end{aligned}$$

In particular, we have the inequalities

$$\begin{aligned}(1.12)\qquad f(M_g(a,b)) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[I^{\alpha}_{a+,g}f(M_g(a,b))+I^{\alpha}_{b-,g}f(M_g(a,b))\right]\\ &\leq \frac{\alpha}{\alpha+1}\left[\frac{f(a)+f(b)}{2}+\frac{1}{\alpha}f(M_g(a,b))\right]\leq\frac{f(a)+f(b)}{2}.\end{aligned}$$

We also obtained the following complementary results [14]:

Theorem 2. *With the assumption of Theorem 1, we have for any* $x\in(a,b)$ *that*

$$\begin{aligned}(1.13)\qquad & f\circ g^{-1}\left(\frac{1}{\alpha+1}\frac{g(a)+g(b)}{2}+\frac{\alpha}{\alpha+1}g(x)\right)\\ &\leq \alpha\int_0^1 s^{\alpha-1} f\circ g^{-1}\left((1-s)\frac{g(a)+g(b)}{2}+sg(x)\right)ds\\ &\leq \frac{1}{2}\Gamma(\alpha+1)\left[\frac{I^{\alpha}_{x+,g}f(b)}{(g(b)-g(x))^{\alpha}}+\frac{I^{\alpha}_{x-,g}f(a)}{(g(x)-g(a))^{\alpha}}\right]\\ &\leq \frac{\alpha}{\alpha+1}\left[\frac{1}{\alpha}\frac{f(a)+f(b)}{2}+f(x)\right].\end{aligned}$$

In particular,

$$\begin{aligned}(1.14)\qquad f(M_g(a,b)) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[I^{\alpha}_{M_g(a,b)+,g}f(b)+I^{\alpha}_{M_g(a,b)-,g}f(a)\right]\\ &\leq \frac{\alpha}{\alpha+1}\left[\frac{1}{\alpha}\frac{f(a)+f(b)}{2}+f(M_g(a,b))\right]\leq\frac{f(a)+f(b)}{2}.\end{aligned}$$

Since $f\circ g^{-1}$ is convex, then

$$\begin{aligned}f\circ g^{-1}\left(\frac{\alpha}{\alpha+1}\frac{g(a)+g(b)}{2}+\frac{1}{\alpha+1}g(x)\right)+f\circ g^{-1}\left(\frac{1}{\alpha+1}\frac{g(a)+g(b)}{2}+\frac{\alpha}{\alpha+1}g(x)\right)\\ \geq 2f\circ g^{-1}\left[\frac{1}{2}\left(\frac{g(a)+g(b)}{2}+g(x)\right)\right]\end{aligned}$$

for any $x\in(a,b)$.

If we add the inequalities (1.11) and (1.13) and divide by 2 we get following more symmetric results

$$
\begin{aligned}
(1.15)\qquad & f\circ g^{-1}\left[\frac{1}{2}\left(\frac{g(a)+g(b)}{2}+g(x)\right)\right] \\
&\le \frac{1}{2}\left[f\circ g^{-1}\left(\frac{\alpha}{\alpha+1}\frac{g(a)+g(b)}{2}+\frac{1}{\alpha+1}g(x)\right)\right. \\
&\left.+f\circ g^{-1}\left(\frac{1}{\alpha+1}\frac{g(a)+g(b)}{2}+\frac{\alpha}{\alpha+1}g(x)\right)\right] \\
&\le \frac{1}{2}\alpha\left[\int_0^1 s^{\alpha-1}f\circ g^{-1}\left(s\frac{g(a)+g(b)}{2}+(1-s)\,g(x)\right)ds\right. \\
&\left.+\int_0^1 s^{\alpha-1}f\circ g^{-1}\left((1-s)\,\frac{g(a)+g(b)}{2}+sg(x)\right)ds\right] \\
&\le \frac{1}{4}\Gamma(\alpha+1)\left[\frac{I_{a+,g}^{\alpha}f(x)}{(g(x)-g(a))^{\alpha}}+\frac{I_{b-,g}^{\alpha}f(x)}{(g(b)-g(x))^{\alpha}}\right. \\
&\left.+\frac{I_{x+,g}^{\alpha}f(b)}{(g(b)-g(x))^{\alpha}}+\frac{I_{x-,g}^{\alpha}f(a)}{(g(x)-g(a))^{\alpha}}\right] \\
&\le \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f(x)\right]
\end{aligned}
$$

for any $x\in(a,b)$, and

$$
\begin{aligned}
(1.16)\qquad f(M_g(a,b)) &\le \frac{2^{\alpha-2}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))\right. \\
&\left.+I_{M_g(a,b)+,g}^{\alpha}f(b)+I_{M_g(a,b)-,g}^{\alpha}f(a)\right] \\
&\le \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f(M_g(a,b))\right]\le\frac{f(a)+f(b)}{2}.
\end{aligned}
$$

In this paper we establish some Hermite-Hadamard type inequalities for the Generalized Riemann-Liouville fractional integrals $I_{a+,g}^{\alpha}f$ and $I_{b-,g}^{\alpha}f$, under the assumption that the composite function $f\circ g^{-1}$ is h-convex on $(g(a),g(b))$. Some applications for Hadamard fractional integrals and s-Godunova-Levin type convex functions are also provided.

2. Some Identities

We have the following equalities of interest:

Theorem 3. *Let $f:[a,b]\to\mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then for any $x\in(a,b)$ we have the equalities*

$$
\begin{aligned}
(2.1)\quad & \frac{1}{2}\Gamma(\alpha)\left[\frac{I_{a+,g}^{\alpha}f(x)}{(g(x)-g(a))^{\alpha}}+\frac{I_{b-,g}^{\alpha}f(x)}{(g(b)-g(x))^{\alpha}}\right] \\
&=\frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\circ g^{-1}(sg(a)+(1-s)\,g(x))+f\circ g^{-1}((1-s)\,g(x)+sg(b))\right]ds
\end{aligned}
$$

and

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{I^{\alpha}_{x+,g}f(b)}{(g(b)-g(x))^{\alpha}}+\frac{I^{\alpha}_{x-,g}f(a)}{(g(x)-g(a))^{\alpha}}\right] \tag{2.2}$$
$$=\frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\circ g^{-1}(sg(x)+(1-s)g(b))+f\circ g^{-1}((1-s)g(a)+sg(x))\right]ds.$$

We also have

$$\frac{\Gamma(\alpha)}{(g(b)-g(a))^{\alpha}}\left[\frac{I^{\alpha}_{a+,g}f(b)+I^{\alpha}_{b-,g}f(a)}{2}\right] \tag{2.3}$$
$$=\frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\circ g^{-1}(sg(a)+(1-s)g(b))+f\circ g^{-1}((1-s)g(a)+sg(b))\right]ds.$$

Proof. Using the change of variable $u=g(t)$, then we have $du=g'(t)\,dt$, $t=g^{-1}(u)$ and

$$I^{\alpha}_{a+,g}f(x)=\frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x)}\frac{f\circ g^{-1}(u)\,du}{[g(x)-u]^{1-\alpha}}$$

for $a<x\le b$ and

$$I^{\alpha}_{b-,g}f(x)=\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)}\frac{f\circ g^{-1}(u)\,du}{[u-g(x)]^{1-\alpha}}$$

for $a\le x<b$.

Further, if we change the variable $u=(1-s)g(a)+sg(x)$, with $s\in[0,1]$, then for $a<x\le b$ we have

$$\begin{aligned} I^{\alpha}_{a+,g}f(x) \tag{2.4} &=\frac{1}{\Gamma(\alpha)}(g(x)-g(a))^{\alpha}\int_0^1(1-s)^{\alpha-1}f\circ g^{-1}((1-s)g(a)+sg(x))\,ds\\ &=\frac{1}{\Gamma(\alpha)}(g(x)-g(a))^{\alpha}\int_0^1 s^{\alpha-1}f\circ g^{-1}(sg(a)+(1-s)g(x))\,ds\end{aligned}$$

where for the last equality we replaced s by $1-s$.

If we change the variable $u=(1-s)g(x)+sg(b)$, with $s\in[0,1]$, then for $a\le x<b$ we also have

$$I^{\alpha}_{b-,g}f(x)=\frac{1}{\Gamma(\alpha)}(g(b)-g(x))^{\alpha}\int_0^1 s^{\alpha-1}f\circ g^{-1}((1-s)g(x)+sg(b))\,ds. \tag{2.5}$$

By using (2.4) and (2.5) we get (2.1).

Now, if we replace x with b in (2.4) and a with x, then we get

$$I^{\alpha}_{x+,g}f(b)=\frac{1}{\Gamma(\alpha)}(g(b)-g(x))^{\alpha}\int_0^1 s^{\alpha-1}f\circ g^{-1}(sg(x)+(1-s)g(b))\,ds.$$

Also, if we replace x with a and b with x in (2.5), then we get

$$I^{\alpha}_{x-,g}f(a)=\frac{1}{\Gamma(\alpha)}(g(x)-g(a))^{\alpha}\int_0^1 s^{\alpha-1}f\circ g^{-1}((1-s)g(a)+sg(x))\,ds. \tag{2.6}$$

By using these two equalities, we get (2.2).

Now, if we take $x=b$ in (2.4), then we have

$$I_{a+,g}^{\alpha}f\left(b\right)=\frac{1}{\Gamma\left(\alpha\right)}\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}\int_{0}^{1}s^{\alpha-1}f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(b\right)\right)ds$$

and if we take $x=a$ in (2.5), then we also have

$$I_{b-,g}^{\alpha}f(a)=\frac{1}{\Gamma\left(\alpha\right)}\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}\int_{0}^{1}s^{\alpha-1}f\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(b\right)\right)ds.$$

If we add these equalities, we then get (2.3). □

Corollary 1. *With the assumptions of Theorem 3, we have*

$$\begin{aligned}(2.7)\quad &\frac{2^{\alpha-1}\Gamma\left(\alpha\right)}{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}\left[I_{a+,g}^{\alpha}f(M_{g}\left(a,b\right))+I_{b-,g}^{\alpha}f(M_{g}\left(a,b\right))\right]\\ &=\frac{1}{2}\int_{0}^{1}s^{\alpha-1}\left[f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(f(M_{g}\left(a,b\right)\right)\right)\right.\\ &\qquad\left.+f\circ g^{-1}\left(\left(1-s\right)g\left(f(M_{g}\left(a,b\right))+sg\left(b\right)\right)\right)\right]ds\end{aligned}$$

and

$$\begin{aligned}(2.8)\quad &\frac{2^{\alpha-1}\Gamma\left(\alpha\right)}{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}\left[I_{M_{g}(a,b)+,g}^{\alpha}f(b)+I_{M_{g}(a,b)-,g}^{\alpha}f(a)\right]\\ &=\frac{1}{2}\int_{0}^{1}s^{\alpha-1}\left[f\circ g^{-1}\left(sg\left(M_{g}\left(a,b\right)\right)+\left(1-s\right)g\left(b\right)\right)\right.\\ &\qquad\left.+f\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(M_{g}\left(a,b\right)\right)\right)\right]ds.\end{aligned}$$

We remark that, if we take into above equalities $g\left(t\right)=t$, then we get the following results for the classical Riemann-Liouville fractional integrals J_{a+}^{α} and J_{b-}^{α}, see for instance [11]

$$\begin{aligned}(2.9)\quad &\frac{1}{2}\Gamma\left(\alpha\right)\left[\frac{J_{a+}^{\alpha}f(x)}{\left(x-a\right)^{\alpha}}+\frac{J_{b-}^{\alpha}f(x)}{\left(b-x\right)^{\alpha}}\right]\\ &=\frac{1}{2}\int_{0}^{1}s^{\alpha-1}\left[f\left(sa+\left(1-s\right)x\right)+f\left(\left(1-s\right)x+sb\right)\right]ds\end{aligned}$$

and

$$\begin{aligned}(2.10)\quad &\frac{1}{2}\Gamma\left(\alpha\right)\left[\frac{J_{x+}f(b)}{\left(b-x\right)^{\alpha}}+\frac{J_{x-}^{\alpha}f(a)}{\left(x-a\right)^{\alpha}}\right]\\ &=\frac{1}{2}\int_{0}^{1}s^{\alpha-1}\left[f\left(sx+\left(1-s\right)b\right)+f\left(\left(1-s\right)a+sx\right)\right]ds.\end{aligned}$$

We also have

$$\begin{aligned}(2.11)\quad &\frac{\Gamma\left(\alpha\right)}{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}\left[\frac{J_{a+}^{\alpha}f\left(b\right)+J_{b-}^{\alpha}f(a)}{2}\right]\\ &=\frac{1}{2}\int_{0}^{1}s^{\alpha-1}\left[f\left(sa+\left(1-s\right)b\right)+f\left(\left(1-s\right)a+sb\right)\right]ds.\end{aligned}$$

In particular, we have the mid-point equalities

$$\frac{2^{\alpha-1}}{(b-a)^{\alpha}}\Gamma(\alpha)\left[J_{a+}^{\alpha}f(\frac{a+b}{2})+J_{b-}^{\alpha}f(\frac{a+b}{2})\right] = \frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\left(sa+(1-s)\frac{a+b}{2}\right)+f\left((1-s)\frac{a+b}{2}+sb\right)\right]ds \tag{2.12}$$

and

$$\frac{2^{\alpha-1}}{(b-a)^{\alpha}}\Gamma(\alpha)\left[J_{\frac{a+b}{2}+}f(b)+J_{\frac{a+b}{2}-}^{\alpha}f(a)\right] = \frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\left(s\frac{a+b}{2}+(1-s)b\right)+f\left((1-s)a+s\frac{a+b}{2}\right)\right]ds. \tag{2.13}$$

If we take $f(t)=\ln t$, $t\in[a,b]\subset(0,\infty)$ into above general equalities, then we get the following identities for the Hadamard fractional integrals

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{H_{a+}^{\alpha}f(x)}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}}+\frac{H_{b-}^{\alpha}f(x)}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}\right] = \frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\left(a^s x^{1-s}\right)+f\left(x^{1-s}b^s\right)\right]ds \tag{2.14}$$

and

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{H_{x+}^{\alpha}f(b)}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}}+\frac{H_{x-}^{\alpha}f(a)}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}\right] = \frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\left(a^{1-s}x^{s}\right)+f\left(x^{s}b^{1-s}\right)\right]ds \tag{2.15}$$

for any $x\in(a,b)$.

We also have

$$\frac{\Gamma(\alpha)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}\left[\frac{H_{a+}^{\alpha}f(b)+H_{b-}^{\alpha}f(a)}{2}\right] = \frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\left(a^{1-s}b^{s}\right)+f\left(a^{s}b^{1-s}\right)\right]ds. \tag{2.16}$$

In particular, we have

$$\frac{2^{\alpha-1}\Gamma(\alpha)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}\left[H_{a+}^{\alpha}f(G(a,b))+H_{b-}^{\alpha}f(G(a,b))\right] = \frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\left(a^{s}G^{1-s}(a,b)\right)+f\left(G^{1-s}(a,b)\,b^{s}\right)\right]ds \tag{2.17}$$

and

$$\frac{2^{\alpha-1}\Gamma(\alpha)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}\left[H_{G(a,b)+}^{\alpha}f(b)+H_{G(a,b)-}^{\alpha}f(a)\right] = \frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\left(a^{1-s}G^{s}(a,b)\right)+f\left(G^{s}(a,b)\,b^{1-s}\right)\right]ds, \tag{2.18}$$

where $G(a,b):=\sqrt{ab}$ is the geometric mean of a, $b>0$.

If we take $g(t)=-t^{-1}$, $t\in(a,b)\subset(0,\infty)$, then $g^{-1}(t)=-t^{-1}$ and by the above equalities we have for the Harmonic fractional integrals

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{a^{\alpha}x^{\alpha}R_{a+}^{\alpha}f(x)}{(x-a)^{\alpha}}+\frac{x^{\alpha}b^{\alpha}R_{b-}^{\alpha}f(x)}{(b-x)^{\alpha}}\right] = \frac{1}{2}\int_{0}^{1}s^{\alpha-1}\left[f\left(\frac{ax}{(1-s)a+sx}\right)+f\left(\frac{xb}{(1-s)b+sx}\right)\right]ds \tag{2.19}$$

and

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{x^{\alpha}b^{\alpha}R_{x+}^{\alpha}f(b)}{(b-x)^{\alpha}}+\frac{a^{\alpha}x^{\alpha}R_{x-}^{\alpha}f(a)}{(x-a)^{\alpha}}\right] = \frac{1}{2}\int_{0}^{1}s^{\alpha-1}\left[f\left(\frac{ax}{(1-s)x+sa}\right)+f\left(\frac{xb}{(1-s)x+sb}\right)\right]ds \tag{2.20}$$

for any $x\in(a,b)$.

We also have

$$\frac{a^{\alpha}b^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[\frac{R_{a+}^{\alpha}f(b)+R_{b-}^{\alpha}f(a)}{2}\right] = \frac{1}{2}\int_{0}^{1}s^{\alpha-1}\left[f\left(\frac{ab}{(1-s)b+sa}\right)+f\left(\frac{ab}{(1-s)a+sb}\right)\right]ds. \tag{2.21}$$

In particular, we have

$$\frac{2^{\alpha-1}a^{\alpha}b^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[R_{a+}^{\alpha}f(H(a,b))+R_{b-}^{\alpha}f(H(a,b))\right] = \frac{1}{2}\int_{0}^{1}s^{\alpha-1}\left[f\left(\frac{H(a,b)\,b}{(1-s)b+sH(a,b)}\right)+f\left(\frac{aH(a,b)}{(1-s)a+sH(a,b)}\right)\right]ds \tag{2.22}$$

and

$$\frac{2^{\alpha-1}a^{\alpha}b^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[R_{H(a,b)+}^{\alpha}f(b)+R_{H(a,b)-}^{\alpha}f(a)\right] = \frac{1}{2}\int_{0}^{1}s^{\alpha-1}\left[f\left(\frac{H(a,b)\,b}{sb+(1-s)H(a,b)}\right)+f\left(\frac{aH(a,b)}{sa+(1-s)H(a,b)}\right)\right]ds, \tag{2.23}$$

where $H(a,b):=\frac{2ab}{a+b}$ is the harmonic mean of $a,\ b>0$.

Finally, if we take the power function $g(t)=t^{p}$, $t\in(a,b)\subset(0,\infty)$, then by Theorem 3 we have

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{a+,p}^{\alpha}f(x)}{(x^{p}-a^{p})^{\alpha}}+\frac{J_{b-,p}^{\alpha}f(x)}{(b^{p}-x^{p})^{\alpha}}\right] = \frac{1}{2}\int_{0}^{1}s^{\alpha-1}\left(f\left[(sa^{p}+(1-s)x^{p})^{1/p}\right]+f\left[((1-s)x^{p}+sb^{p})^{1/p}\right]\right)ds \tag{2.24}$$

and

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{J_{x+,p}^{\alpha}f(b)}{(b^{p}-x^{p})^{\alpha}}+\frac{J_{x-,p}^{\alpha}f(a)}{(x^{p}-a^{p})^{\alpha}}\right] = \frac{1}{2}\int_{0}^{1}s^{\alpha-1}\left(f\left[(sx^{p}+(1-s)b^{p})^{1/p}\right]+f\left[((1-s)a^{p}+sx^{p})^{1/p}\right]\right)ds \tag{2.25}$$

for any $x\in(a,b)$.

We also have

$$\frac{\Gamma(\alpha)}{(b^p-a^p)^\alpha}\left[\frac{J^\alpha_{a+,p}f(b)+J^\alpha_{b-,p}f(a)}{2}\right] = \frac{1}{2}\int_0^1 s^{\alpha-1}\left[\left(f\left[(sa^p+(1-s)b^p)^{1/p}\right]+f\left[((1-s)a^p+sb^p)^{1/p}\right]\right)\right]ds. \tag{2.26}$$

In particular, by replacing x with $M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the power mean with exponent p, we have

$$\begin{aligned}\frac{2^{\alpha-1}\Gamma(\alpha)}{(b^p-a^p)^\alpha}&\left[J^\alpha_{a+,p}f(M_p(a,b))+J^\alpha_{b-,p}f(M_p(a,b))\right]\\ &=\frac{1}{2}\int_0^1 s^{\alpha-1}\left(f\left[\left(sa^p+(1-s)\left(\frac{a^p+b^p}{2}\right)\right)^{1/p}\right]\right.\\ &\qquad\left.+f\left[\left((1-s)\left(\frac{a^p+b^p}{2}\right)+sb^p\right)^{1/p}\right]\right)ds\\ &=\frac{1}{2}\int_0^1 s^{\alpha-1}\left(f\left[\left(\frac{(1+s)a^p+(1-s)b^p}{2}\right)^{1/p}\right]\right.\\ &\qquad\left.+f\left[\left(\frac{(1-s)a^p+(1+s)b^p}{2}\right)^{1/p}\right]\right)ds\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}\frac{2^{\alpha-1}\Gamma(\alpha)}{(b^p-a^p)^\alpha}&\left[J^\alpha_{M_p(a,b)+,p}f(b)+J^\alpha_{M_p(a,b)-,p}f(a)\right]\\ &=\frac{1}{2}\int_0^1 s^{\alpha-1}\left(f\left[\left(s\left(\frac{a^p+b^p}{2}\right)+(1-s)b^p\right)^{1/p}\right]\right.\\ &\qquad\left.+f\left[\left((1-s)a^p+s\left(\frac{a^p+b^p}{2}\right)\right)^{1/p}\right]\right)ds\\ =\frac{1}{2}\int_0^1 s^{\alpha-1}&\left(f\left[\left(\frac{sa^p+(2-s)b^p}{2}\right)^{1/p}\right]+f\left[\left(\frac{(2-s)a^p+sb^p}{2}\right)^{1/p}\right]\right)ds.\end{aligned} \tag{2.28}$$

3. Inequalities for h-Convexity

Assume that I and J are intervals in $\mathbb{R}$, $(0,1)\subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 1 ([34]). *Let $h:J\to[0,\infty)$ with h not identical to 0. We say that $f:I\to[0,\infty)$ is an h-convex function if for all $x,y\in I$ we have*

$$f(tx+(1-t)y)\le h(t)f(x)+h(1-t)f(y) \tag{3.1}$$

for all $t\in(0,1)$.

For some results concerning this class of functions see [34], [3], [19], [28] and [33].

We can give the following examples of h-convex functions.

Definition 2 ([17]). *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \le \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{3.2}$$

Definition 3 ([16]). *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \le f(x) + f(y). \tag{3.3}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions.*

We have introduced the class of functions [10] that generalize both Godunova-Levin type functions and P type functions.

Definition 4. *We say that the function $f : I \to [0,\infty)$ is of s-Godunova-Levin type, with $s \in [0,1]$, if*

$$f(tx + (1-t)y) \le \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{3.4}$$

for all $t \in (0,1)$ and $x, y \in I$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(I)$ the class of s-Godunova-Levin functions defined on I, then we obviously have

$$P(I) = Q_0(I) \subseteq Q_{s_1}(I) \subseteq Q_{s_2}(I) \subseteq Q_1(I) = Q(I)$$

for $0 \le s_1 \le s_2 \le 1$.

Definition 5 ([4]). *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

for all x, $y \in [0,\infty)$ and $t \in [0,1]$.

We have:

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f \circ g^{-1}$ is h-convex on $(g(a), g(b))$, then for any $x \in (a,b)$ we have the inequalities*

$$\begin{aligned} &\frac{1}{h\left(\frac{1}{2}\right)} \int_0^1 s^{\alpha-1} f \circ g^{-1} \left(s \frac{g(a) + g(b)}{2} + (1-s) g(x) \right) ds \\ &\le \frac{1}{2} \Gamma(\alpha) \left[\frac{I^{\alpha}_{a+,g} f(x)}{(g(x) - g(a))^{\alpha}} + \frac{I^{\alpha}_{b-,g} f(x)}{(g(b) - g(x))^{\alpha}} \right] \\ &\le \frac{f(a) + f(b)}{2} \int_0^1 s^{\alpha-1} h(s)\, ds + f(x) \int_0^1 s^{\alpha-1} h(1-s)\, ds. \end{aligned} \tag{3.5}$$

In particular, we have the inequalities

$$\begin{aligned}
(3.6)\qquad & \frac{1}{h\left(\frac{1}{2}\right)} f\left(M_g\left(a,b\right)\right) \\
& \leq \frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}\left[I_{a+,g}^{\alpha} f(M_g\left(a,b\right))+I_{b-,g}^{\alpha} f(M_g\left(a,b\right))\right] \\
& \leq \alpha\left[\frac{f\left(a\right)+f\left(b\right)}{2}\int_0^1 s^{\alpha-1}h\left(s\right)ds+f\left(M_g\left(a,b\right)\right)\int_0^1 s^{\alpha-1}h\left(1-s\right)ds\right].
\end{aligned}$$

Proof. First of all, we observe that if a function $F:(c,d)\to\mathbb{R}$ is h-convex on the interval (c,d) then for any $u,v\in(c,d)$ and $s\in(0,1)$ we have

$$F\left(\left(1-s\right)u+sv\right)\leq h\left(1-s\right)F\left(u\right)+h\left(s\right)F\left(v\right)$$

and

$$\frac{1}{h\left(\frac{1}{2}\right)}F\left(\frac{u+v}{2}\right)\leq F\left(u\right)+F\left(v\right). \tag{3.7}$$

Since $f\circ g^{-1}$ is h-convex on $\left(g\left(a\right),g\left(b\right)\right)$, then we have

$$\begin{aligned}
& \frac{1}{2}\left[f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(x\right)\right)+f\circ g^{-1}\left(\left(1-s\right)g\left(x\right)+sg\left(b\right)\right)\right] \\
& \leq \frac{1}{2}\left[h\left(s\right)f\circ g^{-1}\left(g\left(a\right)\right)+h\left(1-s\right)f\circ g^{-1}\left(g\left(x\right)\right)\right. \\
& \left.+h\left(1-s\right)f\circ g^{-1}\left(g\left(x\right)\right)+h\left(s\right)f\circ g^{-1}\left(g\left(b\right)\right)\right] \\
& = \frac{f\left(a\right)+f\left(b\right)}{2}h\left(s\right)+h\left(1-s\right)f\left(x\right)
\end{aligned}$$

for any $x\in(a,b)$ and $s\in(0,1)$.

If we multiply this inequality by $s^{\alpha-1}$ and integrate over s, then we get

$$\begin{aligned}
& \frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(x\right)\right)+f\circ g^{-1}\left(\left(1-s\right)g\left(x\right)+sg\left(b\right)\right)\right]ds \\
& \leq \frac{f\left(a\right)+f\left(b\right)}{2}\int_0^1 s^{\alpha-1}h\left(s\right)ds+f\left(x\right)\int_0^1 s^{\alpha-1}h\left(1-s\right)ds
\end{aligned}$$

for any $x\in(a,b)$.

By using this inequality and the representation (2.1) we get the last part of (3.5).

We also have by (3.7) that

$$\begin{aligned}
& \frac{1}{h\left(\frac{1}{2}\right)}f\circ g^{-1}\left(\frac{sg\left(a\right)+\left(1-s\right)g\left(x\right)+\left(1-s\right)g\left(x\right)+sg\left(b\right)}{2}\right) \\
& \leq \frac{1}{2}\left[f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(x\right)\right)+f\circ g^{-1}\left(\left(1-s\right)g\left(x\right)+sg\left(b\right)\right)\right],
\end{aligned}$$

namely

$$\begin{aligned}
& \frac{1}{h\left(\frac{1}{2}\right)}f\circ g^{-1}\left(s\frac{g\left(a\right)+g\left(b\right)}{2}+\left(1-s\right)g\left(x\right)\right) \\
& \leq \frac{1}{2}\left[f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(x\right)\right)+f\circ g^{-1}\left(\left(1-s\right)g\left(x\right)+sg\left(b\right)\right)\right],
\end{aligned}$$

for any $x\in(a,b)$.

If we multiply this inequality by $s^{\alpha-1}$, integrate over s and use the representation (2.1), then we get the first part of (3.5).

Now, if we take

$$x=M_{g}\left(a,b\right)=g^{-1}\left(\frac{g\left(a\right)+g\left(b\right)}{2}\right)$$

then we have

$$\begin{aligned}
&\int_{0}^{1}s^{\alpha-1}f\circ g^{-1}\left(s\frac{g\left(a\right)+g\left(b\right)}{2}+\left(1-s\right)g\left(x\right)\right)ds\\
&=\int_{0}^{1}s^{\alpha-1}f\circ g^{-1}\left(s\frac{g\left(a\right)+g\left(b\right)}{2}+\left(1-s\right)\frac{g\left(a\right)+g\left(b\right)}{2}\right)ds\\
&=f\circ g^{-1}\left(\frac{g\left(a\right)+g\left(b\right)}{2}\right)\int_{0}^{1}s^{\alpha-1}ds=\frac{1}{\alpha}f\circ g^{-1}\left(\frac{g\left(a\right)+g\left(b\right)}{2}\right)\\
&=\frac{1}{\alpha}f\circ g^{-1}\left(M_{g}\left(a,b\right)\right).
\end{aligned}$$

Also

$$\begin{aligned}
&\frac{1}{2}\Gamma\left(\alpha\right)\left[\frac{I_{a+,g}^{\alpha}f(M_{g}\left(a,b\right))}{\left(g\left(M_{g}\left(a,b\right)\right)-g\left(a\right)\right)^{\alpha}}+\frac{I_{b-,g}^{\alpha}f(M_{g}\left(a,b\right))}{\left(g\left(b\right)-g\left(M_{g}\left(a,b\right)\right)\right)^{\alpha}}\right]\\
&=\frac{1}{2}\Gamma\left(\alpha\right)\left[\frac{I_{a+,g}^{\alpha}f(M_{g}\left(a,b\right))}{\left(\frac{g(a)+g(b)}{2}-g\left(a\right)\right)^{\alpha}}+\frac{I_{b-,g}^{\alpha}f(M_{g}\left(a,b\right))}{\left(g\left(b\right)-\frac{g(a)+g(b)}{2}\right)^{\alpha}}\right]\\
&=\frac{2^{\alpha-1}}{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}\Gamma\left(\alpha\right)\left[I_{a+,g}^{\alpha}f(M_{g}\left(a,b\right))+I_{b-,g}^{\alpha}f(M_{g}\left(a,b\right))\right].
\end{aligned}$$

By using (3.5) for $x=M_{g}\left(a,b\right)$ we get the desired result (3.6). □

Remark 1. *If we take the integral mean in (3.5), then we get*

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)}\frac{1}{b-a}\int_{a}^{b}\left(\int_{0}^{1}s^{\alpha-1}f\circ g^{-1}\left(s\frac{g\left(a\right)+g\left(b\right)}{2}+\left(1-s\right)g\left(x\right)\right)ds\right)dx \quad (3.8)\\
&\leq\frac{1}{2}\Gamma\left(\alpha\right)\frac{1}{b-a}\int_{a}^{b}\left[\frac{I_{a+,g}^{\alpha}f(x)}{\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}}+\frac{I_{b-,g}^{\alpha}f(x)}{\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}}\right]dx\\
&\leq\frac{f\left(a\right)+f\left(b\right)}{2}\int_{0}^{1}s^{\alpha-1}h\left(s\right)ds+\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\int_{0}^{1}s^{\alpha-1}h\left(1-s\right)ds.
\end{aligned}$$

Theorem 5. *Assume that f and g are as in Theorem 4, then for any $x\in(a,b)$ we have the inequalities*

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)}\int_{0}^{1}s^{\alpha-1}f\circ g^{-1}\left(\left(1-s\right)\frac{g\left(a\right)+g\left(b\right)}{2}+sg\left(x\right)\right)ds \quad (3.9)\\
&\leq\frac{1}{2}\Gamma\left(\alpha\right)\left[\frac{I_{x+,g}^{\alpha}f(b)}{\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}}+\frac{I_{x-,g}^{\alpha}f(a)}{\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}}\right]\\
&\leq\frac{f\left(a\right)+f\left(b\right)}{2}\int_{0}^{1}s^{\alpha-1}h\left(1-s\right)ds+f\left(x\right)\int_{0}^{1}s^{\alpha-1}h\left(s\right)ds.
\end{aligned}$$

In particular, we have the inequalities

$$\begin{aligned}
(3.10)\quad & \frac{1}{h\left(\frac{1}{2}\right)} f\left(M_g\left(a,b\right)\right) \\
& \leq \frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}\left[I^{\alpha}_{M_g(a,b)+,g}f(b)+I^{\alpha}_{M_g(a,b)-,g}f(a)\right] \\
& \leq \alpha\left[\frac{f\left(a\right)+f\left(b\right)}{2}\int_0^1 s^{\alpha-1}h\left(1-s\right)ds+f\left(M_g\left(a,b\right)\right)\int_0^1 s^{\alpha-1}h\left(s\right)ds\right].
\end{aligned}$$

Proof. By the h-convexity of $f\circ g^{-1}$ we have, as in the proof of Theorem 4, that

$$\begin{aligned}
(3.11)\quad & \frac{1}{h\left(\frac{1}{2}\right)} f\circ g^{-1}\left(\left(1-s\right)\frac{g\left(a\right)+g\left(b\right)}{2}+sg\left(x\right)\right) \\
& \leq \left[f\circ g^{-1}\left(sg\left(x\right)+\left(1-s\right)g\left(b\right)\right)+f\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(x\right)\right)\right] \\
& \leq \frac{f\left(a\right)+f\left(b\right)}{2}h\left(1-s\right)+h\left(s\right)f\left(x\right),
\end{aligned}$$

for any $x\in(a,b)$ and $s\in(0,1)$.

If we multiply by $s^{\alpha-1}$, integrate over s and use the representation (2.2), then we get the desired result (3.9). □

Remark 2. *If we take the integral mean in (3.9), then we get*

$$\begin{aligned}
(3.12)\quad & \frac{1}{h\left(\frac{1}{2}\right)}\frac{1}{b-a}\int_a^b\left(\int_0^1 s^{\alpha-1}f\circ g^{-1}\left(\left(1-s\right)\frac{g\left(a\right)+g\left(b\right)}{2}+sg\left(x\right)\right)ds\right)dx \\
& \leq \frac{1}{2}\Gamma\left(\alpha\right)\frac{1}{b-a}\int_a^b\left[\frac{I^{\alpha}_{x+,g}f(b)}{\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}}+\frac{I^{\alpha}_{x-,g}f(a)}{\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}}\right]dx \\
& \leq \frac{f\left(a\right)+f\left(b\right)}{2}\int_0^1 s^{\alpha-1}h\left(1-s\right)ds+\frac{1}{b-a}\int_a^b f\left(x\right)dx\int_0^1 s^{\alpha-1}h\left(s\right)ds.
\end{aligned}$$

The following trapezoid type inequalities also hold.

Theorem 6. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}
(3.13)\quad \frac{1}{h\left(\frac{1}{2}\right)} f\left(M_g\left(a,b\right)\right) & \leq \frac{\Gamma\left(\alpha+1\right)}{\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}\left[\frac{I^{\alpha}_{a+,g}f\left(b\right)+I^{\alpha}_{b-,g}f(a)}{2}\right] \\
& \leq \frac{f\left(a\right)+f\left(b\right)}{2}\alpha\int_0^1 s^{\alpha-1}\left[h\left(s\right)+h\left(1-s\right)\right]ds.
\end{aligned}$$

Proof. By the h-convexity of $f\circ g^{-1}$ we have,

$$\begin{aligned}
& \frac{1}{h\left(\frac{1}{2}\right)} f\circ g^{-1}\left(\frac{g\left(a\right)+g\left(b\right)}{2}\right) \\
& \leq \frac{1}{2}\left[f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(b\right)\right)+f\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(b\right)\right)\right] \\
& \leq \frac{f\left(a\right)+f\left(b\right)}{2}\left[h\left(s\right)+h\left(1-s\right)\right]
\end{aligned}$$

for any $s\in(0,1)$.

If we multiply this inequality by $s^{\alpha-1}$, integrate over s and use the identity (2.3) we get the desired result (3.13). □

The above results provide some interesting inequalities if one takes particular examples for the function g that lead to various fractional integrals as exemplified in the introduction.

If f is h-convex on (a,b), then from (3.6), (3.10) and (3.13) we get the following inequalities for the classical Riemann-Liouville fractional integrals J_{a+}^{α} and J_{b+}^{α}

$$\begin{aligned}&\frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\\&\leq\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(b-a\right)^{\alpha}}\left[J_{a+}^{\alpha}f(\frac{a+b}{2})+J_{b-}^{\alpha}f(\frac{a+b}{2})\right]\\&\leq\alpha\left[\frac{f\left(a\right)+f\left(b\right)}{2}\int_{0}^{1}s^{\alpha-1}h\left(s\right)ds+f\left(\frac{a+b}{2}\right)\int_{0}^{1}s^{\alpha-1}h\left(1-s\right)ds\right],\end{aligned}\tag{3.14}$$

$$\begin{aligned}&\frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\\&\leq\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(b-a\right)^{\alpha}}\left[J_{\frac{a+b}{2}+}^{\alpha}f(b)+J_{\frac{a+b}{2}-}^{\alpha}f(a)\right]\\&\leq\alpha\left[\frac{f\left(a\right)+f\left(b\right)}{2}\int_{0}^{1}s^{\alpha-1}h\left(1-s\right)ds+f\left(\frac{a+b}{2}\right)\int_{0}^{1}s^{\alpha-1}h\left(s\right)ds\right]\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)&\leq\frac{\Gamma\left(\alpha+1\right)}{\left(b-a\right)^{\alpha}}\left[\frac{J_{a+}^{\alpha}f\left(b\right)+J_{b-}^{\alpha}f(a)}{2}\right]\\&\leq\frac{f\left(a\right)+f\left(b\right)}{2}\alpha\int_{0}^{1}s^{\alpha-1}\left[h\left(s\right)+h\left(1-s\right)\right]ds.\end{aligned}\tag{3.16}$$

If $f\circ\exp$ is h-convex on $(\ln a,\ln b)$, then from (3.6), (3.10) and (3.13) we get the following inequalities for the classical Hadamard fractional integrals H_{a+}^{α} and H_{b-}^{α}

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)}f\left(\sqrt{ab}\right)&\leq\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}\left[H_{a+}^{\alpha}f(\sqrt{ab})+H_{b-}^{\alpha}f(\sqrt{ab})\right]\\&\leq\alpha\left[\frac{f\left(a\right)+f\left(b\right)}{2}\int_{0}^{1}s^{\alpha-1}h\left(s\right)ds+f\left(\sqrt{ab}\right)\int_{0}^{1}s^{\alpha-1}h\left(1-s\right)ds\right],\end{aligned}\tag{3.17}$$

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)}f\left(\sqrt{ab}\right)&\leq\frac{2^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}\left[H_{\sqrt{ab}+}^{\alpha}f(b)+H_{\sqrt{ab}-}^{\alpha}f(a)\right]\\&\leq\alpha\left[\frac{f\left(a\right)+f\left(b\right)}{2}\int_{0}^{1}s^{\alpha-1}h\left(1-s\right)ds+f\left(\sqrt{ab}\right)\int_{0}^{1}s^{\alpha-1}h\left(s\right)ds\right]\end{aligned}\tag{3.18}$$

and

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)}f\left(\sqrt{ab}\right)&\leq\frac{\Gamma\left(\alpha+1\right)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}\left[\frac{H_{a+}^{\alpha}f\left(b\right)+I_{b-}^{\alpha}f(a)}{2}\right]\\&\leq\frac{f\left(a\right)+f\left(b\right)}{2}\alpha\int_{0}^{1}s^{\alpha-1}\left[h\left(s\right)+h\left(1-s\right)\right]ds.\end{aligned}\tag{3.19}$$

Similar results can be stated for Harmonic fractional integrals and p-Riemann-Liouville fractional integrals as defined in the introduction. The details are not presented here.

4. Some Examples

The above results provide various inequalities when $f\circ g^{-1}$ is h-convex on $(g(a),g(b))$ and h is one the examples provided at the beginning of the above paragraph.

If we assume that $f\circ g^{-1}$ is of s-Godunova-Levin type, with $s\in[0,1)$ on $(g(a),g(b))$, namely $h(t)=\frac{1}{t^s}$, then by (3.6) we have the following Hermite-Hadamard type inequalities

$$\begin{aligned}2^s f(M_g(a,b)) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))\right]\\ &\leq \frac{f(a)+f(b)}{2}\frac{\alpha}{\alpha-s}+f(M_g(a,b))\,\alpha B(\alpha,1-s)\end{aligned}\tag{4.1}$$

provided $\alpha>s\geq 0$ and $s\in[0,1)$, where $B(\cdot,\cdot)$ is Beta function, i.e.

$$B(\alpha,\beta):=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt,\ \ \alpha,\ \ \beta>0.$$

From (3.10) we get

$$\begin{aligned}2^s f(M_g(a,b)) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[I_{M_g(a,b)+,g}^{\alpha}f(b)+I_{M_g(a,b)-,g}^{\alpha}f(a)\right]\\ &\leq \frac{f(a)+f(b)}{2}\alpha B(\alpha,1-s)+f(M_g(a,b))\frac{\alpha}{\alpha-s}\end{aligned}\tag{4.2}$$

while from (3.13) we get

$$\begin{aligned}2^s f(M_g(a,b)) &\leq \frac{\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[\frac{I_{a+,g}^{\alpha}f(b)+I_{b-,g}^{\alpha}f(a)}{2}\right]\\ &\leq \left[\alpha B(\alpha,1-s)+\frac{\alpha}{\alpha-s}\right]\frac{f(a)+f(b)}{2}.\end{aligned}\tag{4.3}$$

If $f\circ g^{-1}$ is of P-type, namely $s=0$, then we have the simpler inequalities

$$\begin{aligned}f(M_g(a,b)) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))\right]\\ &\leq \frac{f(a)+f(b)}{2}+f(M_g(a,b))\end{aligned}\tag{4.4}$$

$$\begin{aligned}f(M_g(a,b)) &\leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[I_{M_g(a,b)+,g}^{\alpha}f(b)+I_{M_g(a,b)-,g}^{\alpha}f(a)\right]\\ &\leq \frac{f(a)+f(b)}{2}+f(M_g(a,b))\end{aligned}\tag{4.5}$$

and

$$f(M_g(a,b))\leq \frac{\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[\frac{I_{a+,g}^{\alpha}f(b)+I_{b-,g}^{\alpha}f(a)}{2}\right]\leq f(a)+f(b).\tag{4.6}$$

More particular, if we assume that f is of s-Godunova-Levin type, with $s\in[0,1)$ on (a,b), then from (4.1)-(4.3) we get the following inequalities for the classical

Riemann-Liouville fractional integrals

$$2^{s} f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+,}^{\alpha} f(\frac{a+b}{2})+J_{b-}^{\alpha} f(\frac{a+b}{2})\right] \tag{4.7}$$
$$\leq \frac{f(a)+f(b)}{2} \frac{\alpha}{\alpha-s}+f\left(\frac{a+b}{2}\right) \alpha B(\alpha, 1-s)$$

$$2^{s} f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}+}^{\alpha} f(b)+J_{\frac{a+b}{2}-}^{\alpha} f(a)\right] \tag{4.8}$$
$$\leq \frac{f(a)+f(b)}{2} \alpha B(\alpha, 1-s)+f\left(\frac{a+b}{2}\right) \frac{\alpha}{\alpha-s}$$

and

$$2^{s} f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\frac{J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)}{2}\right] \tag{4.9}$$
$$\leq\left[\alpha B(\alpha, 1-s)+\frac{\alpha}{\alpha-s}\right] \frac{f(a)+f(b)}{2},$$

provided $\alpha>s \geq 0$ and $s \in[0,1)$.

Also, if f is of P-type, namely $s=0$, then we have the simpler inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+,}^{\alpha} f(\frac{a+b}{2})+J_{b-}^{\alpha} f(\frac{a+b}{2})\right] \tag{4.10}$$
$$\leq \frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right),$$

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1} \Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}+}^{\alpha} f(b)+J_{\frac{a+b}{2}-}^{\alpha} f(a)\right] \tag{4.11}$$
$$\leq \frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)$$

and

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\frac{J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)}{2}\right] \leq f(a)+f(b). \tag{4.12}$$

Moreover, if $f \circ \exp$ is of s-Godunova-Levin type, with $s \in[0,1)$ on $(\ln a, \ln b)$, then from (4.1)-(4.3) we get the following inequalities for the classical Hadamard fractional integrals

$$2^{s} f(\sqrt{a b}) \leq \frac{2^{\alpha-1} \Gamma(\alpha+1)}{\left[\ln \left(\frac{b}{a}\right)\right]^{\alpha}}\left[H_{a+}^{\alpha} f(\sqrt{a b})+H_{b-}^{\alpha} f(\sqrt{a b})\right] \tag{4.13}$$
$$\leq \frac{f(a)+f(b)}{2} \frac{\alpha}{\alpha-s}+f(\sqrt{a b}) \alpha B(\alpha, 1-s),$$

$$2^{s} f(\sqrt{a b}) \leq \frac{2^{\alpha-1} \Gamma(\alpha+1)}{\left[\ln \left(\frac{b}{a}\right)\right]^{\alpha}}\left[H_{\sqrt{a b}+}^{\alpha} f(b)+H_{\sqrt{a b}-}^{\alpha} f(a)\right] \tag{4.14}$$
$$\leq \frac{f(a)+f(b)}{2} \alpha B(\alpha, 1-s)+f(\sqrt{a b}) \frac{\alpha}{\alpha-s}$$

and

$$2^{s} f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}} \left[\frac{H_{a+}^{\alpha} f(b) + H_{b-}^{\alpha} f(a)}{2}\right] \leq \left[\alpha B(\alpha, 1-s) + \frac{\alpha}{\alpha - s}\right] \frac{f(a)+f(b)}{2}, \tag{4.15}$$

provided $\alpha > s \geq 0$ and $s \in [0,1)$.

Also, if $f \circ \exp$ is of P-type, namely $s = 0$, then we have the simpler inequalities

$$f\left(\sqrt{ab}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}} \left[H_{a+}^{\alpha} f(\sqrt{ab}) + H_{b-}^{\alpha} f(\sqrt{ab})\right] \leq \frac{f(a)+f(b)}{2} + f\left(\sqrt{ab}\right), \tag{4.16}$$

$$f\left(\sqrt{ab}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}} \left[H_{\sqrt{ab}+}^{\alpha} f(b) + H_{\sqrt{ab}-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} + f\left(\sqrt{ab}\right) \tag{4.17}$$

and

$$f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\alpha+1)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}} \left[\frac{H_{a+}^{\alpha} f(b) + H_{b-}^{\alpha} f(a)}{2}\right] \leq f(a) + f(b). \tag{4.18}$$

Similar results can be stated for Harmonic fractional integrals and p-Riemann-Liouville fractional integrals as defined in the introduction. The details are not presented here.

References

[1] P. Agarwal, M. Jleli and M. Tomar, Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals, *Journal of Inequalities and Applications* **2017** (2017) :55.

[2] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, *J. Math. Anal. Appl.* **335** (2007) 1294–1308.

[3] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities. *Comput. Math. Appl.* **58** (2009), no. 9, 1869–1877.

[4] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. (German) *Publ. Inst. Math.* (Beograd) (N.S.) **23(37)** (1978), 13–20.

[5] Z. Dahmani, On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, *Ann. Funct. Anal.* **1**(1) (2010), 51-58.

[6] J. Deng and J. Wang, Fractional Hermite-Hadamard inequalities for $(\alpha; m)$-logarithmically convex functions. *J. Inequal. Appl.* **2013**, Article ID 364 (2013)

[7] S. S. Dragomir, Inequalities of Hermite-Hadamard type for GA-convex functions, RGMIA *Res. Rep. Coll.* **18** (2015), Art. 30. [Online `http://rgmia.org/papers/v18/v18a30.pdf`].

[8] S. S. Dragomir, Some new inequalities of Hermite-Hadamard type for GA-convex functions, RGMIA *Res. Rep. Coll.* **18** (2015), Art. 33. [Online `http://rgmia.org/papers/v18/v18a33.pdf`].

[9] S. S. Dragomir, Inequalities of Jensen type for GA-convex functions, RGMIA *Res. Rep. Coll.* **18** (2015), Art. 35. [Online `http://rgmia.org/papers/v18/v18a35.pdf`].

[10] S. S. Dragomir, Symmetrized convexity and Hermite-Hadamard type inequalities, *J. Math. Inequal.*, Volume **10** (2016), No. 4, 901–918.

[11] S. S. Dragomir, Some inequalities of Hermite-Hadamard type for convex functions and Riemann-Liouville fractional integrals, RGMIA *Res. Rep. Coll.* **20** (2017), Art. 40. [Online `http://rgmia.org/papers/v20/v20a40.pdf`].

[12] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results, *Australian J. Math. Anal. Appl.*, Volume **14**, Issue 1, Article 1, pp. 1-287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[13] S. S. Dragomir, Hermite-Hadamard type inequalities of first kind for generalized Riemann-Liouville fractional integrals, RGMIA *Res. Rep. Coll.* **20** (2017), Art. 76. [Online http://rgmia.org/papers/v20/v20a76.pdf].

[14] S. S. Dragomir, Hermite-Hadamard type inequalities of second kind for generalized Riemann-Liouville fractional integrals, RGMIA *Res. Rep. Coll.* **20** (2017), Art. 79. [Online http://rgmia.org/papers/v20/v20a79.pdf].

[15] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000.[Online http://rgmia.org/monographs/hermite_hadamard.html].

[16] S. S. Dragomir, J. Pečarić and L. Persson, Some inequalities of Hadamard type. *Soochow J. Math.* **21** (1995), no. 3, 335–341.

[17] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (Russian) *Numerical mathematics and mathematical physics* (Russian), 138–142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985

[18] S.-R. Hwang, K.-L. Tseng and K.-C. Hsu, New inequalities for fractional integrals and their applications, *Turk. J. Math.* **40** (2016): 471-486.

[19] M. A. Latif, On some inequalities for h-convex functions. *Int. J. Math. Anal.* (Ruse) **4** (2010), no. 29-32, 1473–1482.

[20] A. Kashuri and R. Liko, Hermite-Hadamard type fractional integral inequalities for the generalized $(1;m)$-preinvex functions, *Nonlinear Analysis and Differential Equations*, Vol. **4**, 2016, no. 8, 353 - 367.

[21] A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006. xvi+523 pp. ISBN: 978-0-444-51832-3; 0-444-51832-0

[22] M. Kunt and I. Iscan, Hermite-Hadamard-Fejer type inequalities for p-convex functions via fractional integrals, *Communication in Mathematical Modeling and Applications*, **2** (2017), No. 1, 1-15.

[23] M. A. Khan, Y. Khurshid and T. Ali, Hermite-Hadamard inequality for fractional integrals via η-convex functions, *Acta Math. Univ. Comenianae* Vol. **LXXXVI**, 1 (2017), pp. 153-164.

[24] W. Liu and W. Wen, Some generalizations of different type of integral inequalities for MT-convex functions, *Filomat* **30**:2 (2016), 333–342

[25] M. E. Özdemir and Ç. Yildiz, The Hadamard's inequality for quasi-convex functions via fractional integrals, *Annals of the University of Craiova, Mathematics and Computer Science Series*, Volume **40** (2), 2013, Pages 167-173.

[26] S. Qaisar, M. Iqbal, S. Hussain, S. I. Butt and M. A. Meraj, New inequalities on Hermite-Hadamard utilizing fractional integrals, *Kragujevac J. Math.*, **42** (1) (2018), Pages 15–27.

[27] M. Z. Sarikaya, H. Filiz and M. E. Kiris, On some generalized integral inequalities for Riemann-Liouville fractional integrals, *Filomat*, **29**:6 (2015), 1307–1314.

[28] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions. *Acta Math. Univ. Comenian.* (N.S.) **79** (2010), no. 2, 265–272.

[29] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Mathematical and Computer Modelling*, **57** (2013), pp.2403–2407, doi:10.1016/j.mcm.2011.12.048.

[30] M. Z. Sarikaya and H. Yildirim, On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals, *Miskolc Mathematical Notes*, **17** (2016), No. 2, pp. 1049–1059.

[31] D.-P. Shi, B.-Y. Xi and F. Qi, Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals of (α, m)-convex functions, *Fractional Differential Calculus*, Volume **4**, Number 1 (2014), 31–43

[32] M. Tunç, On new inequalities for h-convex functions via Riemann-Liouville fractional integration, *Filomat* **27**:4 (2013), 559–565.

[33] M. Tunç, Ostrowski-type inequalities via h-convex functions with applications to special means. *J. Inequal. Appl.* **2013**, 2013:326.

[34] S. Varošanec, On h-convexity. *J. Math. Anal. Appl.* **326** (2007), no. 1, 303–311.

[35] J.-R. Wang, X. Lia and Y. Zhou, Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals via s-convex functions and applications to special Means, *Filomat* **30**:5 (2016), 1143–1150.

[36] X.-M. Zhang, Y.-M. Chu and X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its application, *Journal of Inequalities and Applications*, Volume **2010**, Article ID 507560, 11 pages.

[37] Y. Zhang and J. Wang, On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals, *J. Inequal. Appl.* **2013**, Article ID 220 (2013)

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