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ON SOME EQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF ABSOLUTELY CONTINUOUS FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper we establish some equalities of interest for the generalized Riemann-Liouville fractional integrals $I_{a+,g}^{\alpha}f$ and $I_{b-,g}^{\alpha}f$ for the absolutely continuous functions f and for g being a strictly increasing function on (a,b) and having a continuous derivative g' on (a,b). As applications, some inequalities in the case when the function $\left|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right|$ is convex on $(g(a), g(b))$ are provided. Examples for the Hadamard fractional integrals are also given.

1. Introduction

Let (a,b) with $-\infty \le a < b \le \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\operatorname{Re}(\alpha) > 0$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Following [15, p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ by

$$f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t)\, dt}{[g(x)-g(t)]^{1-\alpha}},\ a < x \le b \tag{1.1}$$

and

$$I_{b-,g}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t)\, dt}{[g(t)-g(x)]^{1-\alpha}},\ a \le x < b. \tag{1.2}$$

For $g(t) = t$ we have the classical *Riemann-Liouville fractional integrals*

$$J_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)\, dt}{(x-t)^{1-\alpha}},\ a < x \le b \tag{1.3}$$

and

$$J_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)\, dt}{(t-x)^{1-\alpha}},\ a \le x < b, \tag{1.4}$$

while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [15, p. 111]

$$H_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t},\ 0 \le a < x \le b \tag{1.5}$$

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and

$$H_{b-}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0\leq a<x<b. \tag{1.6}$$

One can consider the function $g(t)=-t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_{a}^{x}\frac{f(t)\,dt}{(x-t)^{1-\alpha}t^{\alpha+1}},\ 0\leq a<x\leq b \tag{1.7}$$

and

$$R_{b-}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_{x}^{b}\frac{f(t)\,dt}{(t-x)^{1-\alpha}t^{\alpha+1}},\ 0\leq a<x<b. \tag{1.8}$$

Also, for $g(t)=t^{p}$, $p>0$, we have the *p-Riemann-Liouville fractional integrals*

$$J_{a+,p}^{\alpha}f(x):=\frac{p}{\Gamma(\alpha)}\int_{a}^{x}\frac{t^{p-1}f(t)\,dt}{(x^{p}-t^{p})^{1-\alpha}},\ 0\leq a<x\leq b \tag{1.9}$$

and

$$J_{b-,p}^{\alpha}f(x):=\frac{p}{\Gamma(\alpha)}\int_{x}^{b}\frac{t^{p-1}f(t)\,dt}{(t^{p}-x^{p})^{1-\alpha}},\ 0\leq a\leq x<b. \tag{1.10}$$

For some recent papers on various equalities and inequalities for the Riemann-Liouville fractional integrals see [1], [3], [4], [13]-[28] and the references therein.

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a,\ b\in I$ as

$$M_{g}(a,b):=g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_{g}(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_{g}(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_{g}(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^{p}$, $p\neq0$, then $M_{g}(a,b)=M_{p}(a,b):=\left(\frac{a^{p}+b^{p}}{2}\right)^{1/p}$, the *power mean with exponent p*.

In this paper we establish some equalities of interest for the generalized Riemann-Liouville fractional integrals $I_{a+,g}^{\alpha}f$ and $I_{b-,g}^{\alpha}f$ for the absolutely continuous functions f and for g being a strictly increasing function on (a,b) and having a continuous derivative g' on (a,b). As applications, some inequalities in the case when the function $\left|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right|$ is convex on $(g(a),g(b))$ are provided. Examples for the Hadamard fractional integrals are also given.

2. General Identities

We have the following equalities of interest, see also [11]:

Lemma 1. *Let $f:[a,b]\rightarrow\mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then for* *any* $x \in (a, b)$ *we have the equalities*

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{I^{\alpha}_{a+,g}f(x)}{(g(x)-g(a))^{\alpha}}+\frac{I^{\alpha}_{b-,g}f(x)}{(g(b)-g(x))^{\alpha}}\right] \tag{2.1}$$
$$=\frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\circ g^{-1}(sg(a)+(1-s)g(x))+f\circ g^{-1}((1-s)g(x)+sg(b))\right]ds$$

and

$$\frac{1}{2}\Gamma(\alpha)\left[\frac{I^{\alpha}_{x+,g}f(b)}{(g(b)-g(x))^{\alpha}}+\frac{I^{\alpha}_{x-,g}f(a)}{(g(x)-g(a))^{\alpha}}\right] \tag{2.2}$$
$$=\frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\circ g^{-1}(sg(x)+(1-s)g(b))+f\circ g^{-1}((1-s)g(a)+sg(x))\right]ds.$$

We also have

$$\frac{\Gamma(\alpha)}{(g(b)-g(a))^{\alpha}}\left[\frac{I^{\alpha}_{a+,g}f(b)+I^{\alpha}_{b-,g}f(a)}{2}\right] \tag{2.3}$$
$$=\frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\circ g^{-1}(sg(a)+(1-s)g(b))+f\circ g^{-1}((1-s)g(a)+sg(b))\right]ds.$$

Proof. For the sake of completeness, we give here a short proof. Using the change of variable $u=g(t)$, then we have $du=g'(t)\,dt$, $t=g^{-1}(u)$ and

$$I^{\alpha}_{a+,g}f(x)=\frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x)}\frac{f\circ g^{-1}(u)\,du}{[g(x)-u]^{1-\alpha}}$$

for $a<x\le b$ and

$$I^{\alpha}_{b-,g}f(x)=\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)}\frac{f\circ g^{-1}(u)\,du}{[u-g(x)]^{1-\alpha}}$$

for $a\le x<b$.

Further, if we change the variable $u=(1-s)g(a)+sg(x)$, with $s\in[0,1]$, then for $a<x\le b$ we have

$$\begin{aligned} I^{\alpha}_{a+,g}f(x) &= \frac{1}{\Gamma(\alpha)}(g(x)-g(a))^{\alpha}\int_0^1(1-s)^{\alpha-1}f\circ g^{-1}((1-s)g(a)+sg(x))\,ds \\ &= \frac{1}{\Gamma(\alpha)}(g(x)-g(a))^{\alpha}\int_0^1 s^{\alpha-1}f\circ g^{-1}(sg(a)+(1-s)g(x))\,ds \end{aligned} \tag{2.4}$$

where for the last equality we replaced s by $1-s$.

If we change the variable $u=(1-s)g(x)+sg(b)$, with $s\in[0,1]$, then for $a\le x<b$ we also have

$$I^{\alpha}_{b-,g}f(x)=\frac{1}{\Gamma(\alpha)}(g(b)-g(x))^{\alpha}\int_0^1 s^{\alpha-1}f\circ g^{-1}((1-s)g(x)+sg(b))\,ds. \tag{2.5}$$

By using (2.4) and (2.5) we get (2.1).

Now, if we replace x with b in (2.4) and a with x, then we get

$$I^{\alpha}_{x+,g}f(b)=\frac{1}{\Gamma(\alpha)}(g(b)-g(x))^{\alpha}\int_0^1 s^{\alpha-1}f\circ g^{-1}(sg(x)+(1-s)g(b))\,ds.$$

Also, if we replace x with a and b with x in (2.5), then we get

$$I_{x-,g}^{\alpha}f(a)=\frac{1}{\Gamma(\alpha)}\left(g(x)-g(a)\right)^{\alpha}\int_0^1 s^{\alpha-1}f\circ g^{-1}\left((1-s)g(a)+sg(x)\right)ds. \tag{2.6}$$

By using these two equalities, we get (2.2).

Now, if we take $x=b$ in (2.4), then we have

$$I_{a+,g}^{\alpha}f(b)=\frac{1}{\Gamma(\alpha)}\left(g(b)-g(a)\right)^{\alpha}\int_0^1 s^{\alpha-1}f\circ g^{-1}\left(sg(a)+(1-s)g(b)\right)ds$$

and if we take $x=a$ in (2.5), then we also have

$$I_{b-,g}^{\alpha}f(a)=\frac{1}{\Gamma(\alpha)}\left(g(b)-g(a)\right)^{\alpha}\int_0^1 s^{\alpha-1}f\circ g^{-1}\left((1-s)g(a)+sg(b)\right)ds.$$

If we add these equalities, we then get (2.3). □

Corollary 1. *With the assumptions of Theorem 1, we have*

$$\begin{aligned}\frac{2^{\alpha-1}\Gamma(\alpha)}{\left(g(b)-g(a)\right)^{\alpha}}&\left[I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))\right]\\&=\frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\circ g^{-1}\left(sg(a)+(1-s)g\left(f(M_g(a,b)\right)\right)\right.\\&\qquad\left.+f\circ g^{-1}\left((1-s)g\left(f(M_g(a,b)\right)+sg(b)\right)\right]ds\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\frac{2^{\alpha-1}\Gamma(\alpha)}{\left(g(b)-g(a)\right)^{\alpha}}&\left[I_{M_g(a,b)+,g}^{\alpha}f(b)+I_{M_g(a,b)-,g}^{\alpha}f(a)\right]\\&=\frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\circ g^{-1}\left(sg\left(M_g(a,b)\right)+(1-s)g(b)\right)\right.\\&\qquad\left.+f\circ g^{-1}\left((1-s)g(a)+sg\left(M_g(a,b)\right)\right)\right]ds.\end{aligned} \tag{2.8}$$

Using the above equalities we can obtain the following results for absolutely continuous functions:

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then for any $x\in(a,b)$ we have the equalities*

$$\begin{aligned}\frac{f(a)+f(b)}{2}&-\frac{1}{2}\Gamma(\alpha+1)\left[\frac{I_{a+,g}^{\alpha}f(x)}{\left(g(x)-g(a)\right)^{\alpha}}+\frac{I_{b-,g}^{\alpha}f(x)}{\left(g(b)-g(x)\right)^{\alpha}}\right]\\&=\frac{1}{2}\left(g(b)-g(x)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left((1-s)g(x)+sg(b)\right)}{g'\circ g^{-1}\left((1-s)g(x)+sg(b)\right)}ds\\&\qquad-\frac{1}{2}\left(g(x)-g(a)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(sg(a)+(1-s)g(x)\right)}{g'\circ g^{-1}\left(sg(a)+(1-s)g(x)\right)}ds\end{aligned} \tag{2.9}$$

and

$$\frac{1}{2}\Gamma(\alpha+1)\left[\frac{I^{\alpha}_{x+,g}f(b)}{(g(b)-g(x))^{\alpha}}+\frac{I^{\alpha}_{x-,g}f(a)}{(g(x)-g(a))^{\alpha}}\right]-f(x) \tag{2.10}$$
$$=\frac{1}{2}(g(b)-g(x))\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}(sg(x)+(1-s)g(b))}{g'\circ g^{-1}(sg(x)+(1-s)g(b))}ds$$
$$-\frac{1}{2}(g(x)-g(a))\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}((1-s)g(a)+sg(x))}{g'\circ g^{-1}((1-s)g(a)+sg(x))}ds.$$

We also have

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[\frac{I^{\alpha}_{a+,g}f(b)+I^{\alpha}_{b-,g}f(a)}{2}\right] \tag{2.11}$$
$$=\frac{1}{2}(g(b)-g(a))\int_0^1\left[s^{\alpha}-(1-s)^{\alpha}\right]\frac{f'\circ g^{-1}((1-s)g(a)+sg(b))}{g'\circ g^{-1}((1-s)g(a)+sg(b))}ds.$$

Proof. Since f is absolutely continuous, then the functions defined as $[0,1]\ni f\circ g^{-1}(sg(a)+(1-s)g(x))\to\mathbb{C}$ and $[0,1]\ni f\circ g^{-1}((1-s)g(x)+sg(b))\to\mathbb{C}$ are also absolutely continuos and the following derivative over s exists almost everywhere on $[0,1]$

$$\begin{aligned}&\left(f\circ g^{-1}(sg(a)+(1-s)g(x))\right)'\\&=f'\circ g^{-1}(sg(a)+(1-s)g(x))\left(g^{-1}(sg(a)+(1-s)g(x))\right)'\\&=\frac{f'\circ g^{-1}(sg(a)+(1-s)g(x))}{g'\circ g^{-1}(sg(a)+(1-s)g(x))}(sg(a)+(1-s)g(x))'\\&=(g(a)-g(x))\frac{f'\circ g^{-1}(sg(a)+(1-s)g(x))}{g'\circ g^{-1}(sg(a)+(1-s)g(x))}\end{aligned}$$

and, similarly

$$\begin{aligned}&\left(f\circ g^{-1}((1-s)g(x)+sg(b))\right)'\\&=(g(b)-g(x))\frac{f'\circ g^{-1}((1-s)g(x)+sg(b))}{g'\circ g^{-1}((1-s)g(x)+sg(b))}\end{aligned}$$

where $x\in(a,b)$.

Utilising the integration by parts formula, we have

$$\begin{aligned}&\int_0^1 s^{\alpha-1}f\circ g^{-1}(sg(a)+(1-s)g(x))\,ds\\&=\frac{1}{\alpha}s^{\alpha}f\circ g^{-1}(sg(a)+(1-s)g(x))\Big|_0^1\\&-\frac{1}{\alpha}(g(a)-g(x))\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}(sg(a)+(1-s)g(x))}{g'\circ g^{-1}(sg(a)+(1-s)g(x))}ds\\&=\frac{1}{\alpha}f(a)-\frac{1}{\alpha}(g(a)-g(x))\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}(sg(a)+(1-s)g(x))}{g'\circ g^{-1}(sg(a)+(1-s)g(x))}ds\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 s^{\alpha-1} f\circ g^{-1}\left((1-s)\,g(x)+sg(b)\right)ds \\
&= \frac{1}{\alpha}s^{\alpha} f\circ g^{-1}\left((1-s)\,g(x)+sg(b)\right)\Big|_0^1 \\
&- \frac{1}{\alpha}\left(g(b)-g(x)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left((1-s)\,g(x)+sg(b)\right)}{g'\circ g^{-1}\left((1-s)\,g(x)+sg(b)\right)}ds \\
&= \frac{1}{\alpha}f(b) - \frac{1}{\alpha}\left(g(b)-g(x)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left((1-s)\,g(x)+sg(b)\right)}{g'\circ g^{-1}\left((1-s)\,g(x)+sg(b)\right)}ds
\end{aligned}$$

where $x\in(a,b)$.

If we add these equalities and divide by 2 we get

$$\begin{aligned}
&\frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\circ g^{-1}\left(sg(a)+(1-s)\,g(x)\right)+f\circ g^{-1}\left((1-s)\,g(x)+sg(b)\right)\right]ds \\
&= \frac{1}{\alpha}\frac{f(a)+f(b)}{2} - \frac{1}{2\alpha}\left[\left(g(a)-g(x)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(sg(a)+(1-s)\,g(x)\right)}{g'\circ g^{-1}\left(sg(a)+(1-s)\,g(x)\right)}ds\right. \\
&\left.+\left(g(b)-g(x)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left((1-s)\,g(x)+sg(b)\right)}{g'\circ g^{-1}\left((1-s)\,g(x)+sg(b)\right)}ds\right]
\end{aligned}$$

and by (2.1) we get

$$\begin{aligned}
&\frac{1}{2}\Gamma(\alpha)\left[\frac{I^{\alpha}_{a+,g}f(x)}{\left(g(x)-g(a)\right)^{\alpha}}+\frac{I^{\alpha}_{b-,g}f(x)}{\left(g(b)-g(x)\right)^{\alpha}}\right] \\
&= \frac{1}{\alpha}\frac{f(a)+f(b)}{2} - \frac{1}{2\alpha}\left[\left(g(a)-g(x)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(sg(a)+(1-s)\,g(x)\right)}{g'\circ g^{-1}\left(sg(a)+(1-s)\,g(x)\right)}ds\right. \\
&\left.+\left(g(b)-g(x)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left((1-s)\,g(x)+sg(b)\right)}{g'\circ g^{-1}\left((1-s)\,g(x)+sg(b)\right)}ds\right]
\end{aligned}$$

and by multiplying both sides with $\alpha>0$, we get

$$\begin{aligned}
&\frac{1}{2}\Gamma(\alpha+1)\left[\frac{I^{\alpha}_{a+,g}f(x)}{\left(g(x)-g(a)\right)^{\alpha}}+\frac{I^{\alpha}_{b-,g}f(x)}{\left(g(b)-g(x)\right)^{\alpha}}\right] \\
&= \frac{f(a)+f(b)}{2} - \frac{1}{2}\left[\left(g(a)-g(x)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(sg(a)+(1-s)\,g(x)\right)}{g'\circ g^{-1}\left(sg(a)+(1-s)\,g(x)\right)}ds\right. \\
&\left.+\left(g(b)-g(x)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left((1-s)\,g(x)+sg(b)\right)}{g'\circ g^{-1}\left((1-s)\,g(x)+sg(b)\right)}ds\right]
\end{aligned}$$

that is equivalent to the identity (2.9).

Also, we have

$$\begin{aligned}
&\left(f\circ g^{-1}\left(sg(x)+(1-s)\,g(b)\right)\right)' \\
&= \left(g(x)-g(b)\right)\frac{f'\circ g^{-1}\left(sg(x)+(1-s)\,g(b)\right)}{g'\circ g^{-1}\left(sg(x)+(1-s)\,g(b)\right)}
\end{aligned}$$

and

$$\begin{aligned}&\left(f\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(x\right)\right)\right)'\\&=\left(g\left(x\right)-g\left(a\right)\right)\frac{f'\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(x\right)\right)}{g'\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(x\right)\right)}\end{aligned}$$

where $x\in\left(a,b\right)$.

Utilising the integration by parts formula, we have

$$\begin{aligned}&\int_0^1 s^{\alpha-1}f\circ g^{-1}\left(sg\left(x\right)+\left(1-s\right)g\left(b\right)\right)ds\\&=\frac{1}{\alpha}f\left(x\right)-\frac{1}{\alpha}\left(g\left(x\right)-g\left(b\right)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(sg\left(x\right)+\left(1-s\right)g\left(b\right)\right)}{g'\circ g^{-1}\left(sg\left(x\right)+\left(1-s\right)g\left(b\right)\right)}ds\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 s^{\alpha-1}f\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(x\right)\right)ds\\&=\frac{1}{\alpha}f\left(x\right)-\frac{1}{\alpha}\left(g\left(x\right)-g\left(a\right)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(x\right)\right)}{g'\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(x\right)\right)}ds\end{aligned}$$

where $x\in\left(a,b\right)$.

If we add these equalities and divide by 2, we get

$$\begin{aligned}&\frac{1}{2}\int_0^1 s^{\alpha-1}\left[f\circ g^{-1}\left(sg\left(x\right)+\left(1-s\right)g\left(b\right)\right)+f\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(x\right)\right)\right]ds\\&=\frac{1}{\alpha}f\left(x\right)-\frac{1}{2\alpha}\left(g\left(x\right)-g\left(b\right)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(sg\left(x\right)+\left(1-s\right)g\left(b\right)\right)}{g'\circ g^{-1}\left(sg\left(x\right)+\left(1-s\right)g\left(b\right)\right)}ds\\&-\frac{1}{2\alpha}\left(g\left(x\right)-g\left(a\right)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(x\right)\right)}{g'\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(x\right)\right)}ds\end{aligned}$$

and by the equality (2.2) we get

$$\begin{aligned}&\frac{1}{2}\Gamma\left(\alpha+1\right)\left[\frac{I^{\alpha}_{x+,g}f(b)}{\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}}+\frac{I^{\alpha}_{x-,g}f(a)}{\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}}\right]\\&=f\left(x\right)-\frac{1}{2}\left(g\left(x\right)-g\left(b\right)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(sg\left(x\right)+\left(1-s\right)g\left(b\right)\right)}{g'\circ g^{-1}\left(sg\left(x\right)+\left(1-s\right)g\left(b\right)\right)}ds\\&-\frac{1}{2}\left(g\left(x\right)-g\left(a\right)\right)\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(x\right)\right)}{g'\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(x\right)\right)}ds\end{aligned}$$

where $x\in\left(a,b\right)$, that is equivalent to the identity (2.10).

Finally, we have

$$\begin{aligned}&\left(f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(b\right)\right)\right)'\\&=\left(g\left(a\right)-g\left(b\right)\right)\frac{f'\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(b\right)\right)}{g'\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(b\right)\right)}\end{aligned}$$

and

$$\begin{aligned}&\left(f\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(b\right)\right)\right)'\\&=\left(g\left(b\right)-g\left(a\right)\right)\frac{f'\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(b\right)\right)}{g'\circ g^{-1}\left(\left(1-s\right)g\left(a\right)+sg\left(b\right)\right)}.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_0^1 s^{\alpha-1} f \circ g^{-1}\left(sg(a)+(1-s)g(b)\right)ds \\
&= \frac{1}{\alpha} f(a) - \frac{1}{\alpha}\left(g(a)-g(b)\right)\int_0^1 s^{\alpha}\frac{f' \circ g^{-1}\left(sg(a)+(1-s)g(b)\right)}{g' \circ g^{-1}\left(sg(a)+(1-s)g(b)\right)}ds \\
&= \frac{1}{\alpha} f(a) - \frac{1}{\alpha}\left(g(a)-g(b)\right)\int_0^1 (1-s)^{\alpha}\frac{f' \circ g^{-1}\left((1-s)g(a)+sg(b)\right)}{g' \circ g^{-1}\left((1-s)g(a)+sg(b)\right)}ds
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 s^{\alpha-1} f \circ g^{-1}\left((1-s)g(a)+sg(b)\right)ds \\
&= \frac{1}{\alpha} f(b) - \frac{1}{\alpha}\left(g(b)-g(a)\right)\int_0^1 s^{\alpha}\frac{f' \circ g^{-1}\left((1-s)g(a)+sg(b)\right)}{g' \circ g^{-1}\left((1-s)g(a)+sg(b)\right)}ds.
\end{aligned}$$

If we add these equalities and divide by 2 then we get

$$\begin{aligned}
&\frac{1}{2}\int_0^1 s^{\alpha-1}\left[f \circ g^{-1}\left(sg(a)+(1-s)g(b)\right)+f \circ g^{-1}\left((1-s)g(a)+sg(b)\right)\right]ds \\
&= \frac{1}{\alpha}\frac{f(a)+f(b)}{2} \\
&- \frac{1}{2\alpha}\left(g(a)-g(b)\right)\int_0^1 \left[(1-s)^{\alpha}-s^{\alpha}\right]\frac{f' \circ g^{-1}\left((1-s)g(a)+sg(b)\right)}{g' \circ g^{-1}\left((1-s)g(a)+sg(b)\right)}ds
\end{aligned}$$

and by the equality (2.3) we get

$$\begin{aligned}
&\frac{\Gamma(\alpha+1)}{\left(g(b)-g(a)\right)^{\alpha}}\left[\frac{I_{a+,g}^{\alpha}f(b)+I_{b-,g}^{\alpha}f(a)}{2}\right] \\
&= \frac{f(a)+f(b)}{2} \\
&- \frac{1}{2}\left(g(a)-g(b)\right)\int_0^1 \left[(1-s)^{\alpha}-s^{\alpha}\right]\frac{f' \circ g^{-1}\left((1-s)g(a)+sg(b)\right)}{g' \circ g^{-1}\left((1-s)g(a)+sg(b)\right)}ds,
\end{aligned}$$

which is equivalent to (2.11). □

Corollary 2. *With the assumptions of Theorem 1 we have*

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(g(b)-g(a)\right)^{\alpha}}\left[I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))\right] \\
&= \frac{1}{4}\left(g(b)-g(a)\right) \\
&\times \int_0^1 s^{\alpha}\left[\frac{f' \circ g^{-1}\left(\frac{(1-s)g(a)+(1+s)g(b)}{2}\right)}{g' \circ g^{-1}\left(\frac{(1-s)g(a)+(1+s)g(b)}{2}\right)} - \frac{f' \circ g^{-1}\left(\frac{(1+s)g(a)+(1-s)g(b)}{2}\right)}{g' \circ g^{-1}\left(\frac{(1+s)g(a)+(1-s)g(b)}{2}\right)}\right]ds
\end{aligned} \tag{2.12}$$

and

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[I^{\alpha}_{M_g(a,b)+,g}f(b)+I^{\alpha}_{M_g(a,b)-,g}f(a)\right]-f(M_g(a,b)) \tag{2.13}$$
$$=\frac{1}{4}(g(b)-g(a))$$
$$\times\int_0^1 s^{\alpha}\left[\frac{f'\circ g^{-1}\left(\frac{sg(a)+(2-s)g(b)}{2}\right)}{g'\circ g^{-1}\left(\frac{sg(a)+(2-s)g(b)}{2}\right)}-\frac{f'\circ g^{-1}\left(\frac{(2-s)g(a)+sg(b)}{2}\right)}{g'\circ g^{-1}\left(\frac{(2-s)g(a)+sg(b)}{2}\right)}\right]ds.$$

Proof. If we take $x=M_g(a,b)=g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$ in (2.9), then we have

$$\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[I^{\alpha}_{a+,g}f(M_g(a,b))+I^{\alpha}_{b-,g}f(M_g(a,b))\right]$$
$$=\frac{1}{4}(g(b)-g(a))\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left((1-s)\frac{g(a)+g(b)}{2}+sg(b)\right)}{g'\circ g^{-1}\left((1-s)\frac{g(a)+g(b)}{2}+sg(b)\right)}ds$$
$$-\frac{1}{4}(g(b)-g(a))\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(sg(a)+(1-s)\frac{g(a)+g(b)}{2}\right)}{g'\circ g^{-1}\left(sg(a)+(1-s)\frac{g(a)+g(b)}{2}\right)}ds$$
$$=\frac{1}{4}(g(b)-g(a))\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(\frac{(1-s)g(a)+(1+s)g(b)}{2}\right)}{g'\circ g^{-1}\left(\frac{(1-s)g(a)+(1+s)g(b)}{2}\right)}ds$$
$$-\frac{1}{4}(g(b)-g(a))\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(\frac{(1+s)g(a)+(1-s)g(b)}{2}\right)}{g'\circ g^{-1}\left(\frac{(1+s)g(a)+(1-s)g(b)}{2}\right)}ds$$
$$=\frac{1}{4}(g(b)-g(a))$$
$$\times\int_0^1 s^{\alpha}\left[\frac{f'\circ g^{-1}\left(\frac{(1-s)g(a)+(1+s)g(b)}{2}\right)}{g'\circ g^{-1}\left(\frac{(1-s)g(a)+(1+s)g(b)}{2}\right)}-\frac{f'\circ g^{-1}\left(\frac{(1+s)g(a)+(1-s)g(b)}{2}\right)}{g'\circ g^{-1}\left(\frac{(1+s)g(a)+(1-s)g(b)}{2}\right)}\right]ds$$

and

$$\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[I^{\alpha}_{M_g(a,b)+,g}f(b)+I^{\alpha}_{M_g(a,b)-,g}f(a)\right]-f(M_g(a,b))$$
$$=\frac{1}{4}(g(b)-g(a))\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left(s\frac{g(a)+g(b)}{2}+(1-s)g(b)\right)}{g'\circ g^{-1}\left(s\frac{g(a)+g(b)}{2}+(1-s)g(b)\right)}ds$$
$$-\frac{1}{4}(g(b)-g(a))\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}\left((1-s)g(a)+s\frac{g(a)+g(b)}{2}\right)}{g'\circ g^{-1}\left((1-s)g(a)+s\frac{g(a)+g(b)}{2}\right)}ds$$

$$= \frac{1}{4}(g(b) - g(a)) \int_0^1 s^\alpha \frac{f' \circ g^{-1}\left(\frac{sg(a)+(2-s)g(b)}{2}\right)}{g' \circ g^{-1}\left(\frac{sg(a)+(2-s)g(b)}{2}\right)} ds$$

$$- \frac{1}{4}(g(b) - g(a)) \int_0^1 s^\alpha \frac{f' \circ g^{-1}\left(\frac{(2-s)g(a)+sg(b)}{2}\right)}{g' \circ g^{-1}\left(\frac{(2-s)g(a)+sg(b)}{2}\right)} ds$$

$$= \frac{1}{4}(g(b) - g(a))$$

$$\times \int_0^1 s^\alpha \left[\frac{f' \circ g^{-1}\left(\frac{sg(a)+(2-s)g(b)}{2}\right)}{g' \circ g^{-1}\left(\frac{sg(a)+(2-s)g(b)}{2}\right)} - \frac{f' \circ g^{-1}\left(\frac{(2-s)g(a)+sg(b)}{2}\right)}{g' \circ g^{-1}\left(\frac{(2-s)g(a)+sg(b)}{2}\right)} \right] ds,$$

which prove the corollary. □

3. Some Particular Equalities

If we take $g(t) = t$, $t \in [a, b]$ in Theorem 1, then we get the following equalities for the classical Riemann-Liouville fractional integrals

$$\frac{f(a) + f(b)}{2} - \frac{1}{2}\Gamma(\alpha + 1)\left[\frac{J_{a+}^\alpha f(x)}{(x-a)^\alpha} + \frac{J_{b-}^\alpha f(x)}{(b-x)^\alpha}\right] \tag{3.1}$$
$$= \frac{1}{2}(b - x)\int_0^1 s^\alpha f'((1-s)x + sb)\,ds - \frac{1}{2}(x-a)\int_0^1 s^\alpha f'(sa + (1-s)x)\,ds$$

and

$$\frac{1}{2}\Gamma(\alpha + 1)\left[\frac{J_{x+}^\alpha f(b)}{(b-x)^\alpha} + \frac{J_{x-}^\alpha f(a)}{(x-a)^\alpha}\right] - f(x) \tag{3.2}$$
$$= \frac{1}{2}(b - x)\int_0^1 s^\alpha f'(sx + (1-s)b)\,ds - \frac{1}{2}(x-a)\int_0^1 s^\alpha f'((1-s)a + sx)\,ds,$$

for any $x \in (a, b)$.

We also have

$$\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{(b-a)^\alpha}\left[\frac{J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)}{2}\right] \tag{3.3}$$
$$= \frac{1}{2}(b-a)\int_0^1 [s^\alpha - (1-s)^\alpha] f'((1-s)a + sb)\,ds.$$

This equality was obtained in [22].

By Corollary 1 we also have

$$\frac{f(a) + f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha + 1)}{(b-a)^\alpha}\left[J_{a+}^\alpha f\left(\frac{a+b}{2}\right) + J_{b-}^\alpha f\left(\frac{a+b}{2}\right)\right] \tag{3.4}$$
$$= \frac{1}{4}(b-a)\int_0^1 s^\alpha \left[f'\left(\frac{(1-s)a + (1+s)b}{2}\right) - f'\left(\frac{(1+s)a + (1-s)b}{2}\right)\right] ds$$

and

$$(3.5)\quad \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}+}f(b)+J^{\alpha}_{\frac{a+b}{2}-}f(a)\right]-f\left(\frac{a+b}{2}\right)$$
$$=\frac{1}{4}(g(b)-g(a))\int_0^1 s^{\alpha}\left[f'\left(\frac{sa+(2-s)b}{2}\right)-f'\left(\frac{(2-s)a+sb}{2}\right)\right]ds.$$

If we take $g(t)=\ln t$, $t\in[a,b]\subset(0,\infty)$ in Theorem 1, then we get the following equalities for the classical Hadamard fractional integrals

$$(3.6)\quad \frac{f(a)+f(b)}{2}-\frac{1}{2}\Gamma(\alpha+1)\left[\frac{H^{\alpha}_{a+}f(x)}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}}+\frac{H^{\alpha}_{b-}f(x)}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}\right]$$
$$=\frac{1}{2}\ln\left(\frac{b}{x}\right)\int_0^1 s^{\alpha}x^{1-s}b^{s}f'\left(x^{1-s}b^{s}\right)ds-\frac{1}{2}\ln\left(\frac{x}{a}\right)\int_0^1 s^{\alpha}x^{1-s}a^{s}f'\left(x^{1-s}a^{s}\right)ds$$

and

$$(3.7)\quad \frac{1}{2}\Gamma(\alpha+1)\left[\frac{H^{\alpha}_{x+}f(b)}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}+\frac{H^{\alpha}_{x-}f(a)}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}}\right]-f(x)$$
$$=\frac{1}{2}\ln\left(\frac{b}{x}\right)\int_0^1 s^{\alpha}x^{s}b^{1-s}f'\left(x^{s}b^{1-s}\right)ds-\frac{1}{2}\ln\left(\frac{x}{a}\right)\int_0^1 s^{\alpha}x^{s}a^{1-s}f'\left(x^{s}a^{1-s}\right)ds.$$

We also have

$$(3.8)\quad \frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}\left[\frac{H^{\alpha}_{a+}f(b)+H^{\alpha}_{b-}f(a)}{2}\right]$$
$$=\frac{1}{2}\ln\left(\frac{b}{a}\right)\int_0^1\left[s^{\alpha}-(1-s)^{\alpha}\right]a^{1-s}b^{s}f'\left(a^{1-s}b^{s}\right)ds.$$

By Corollary 1 we also have

$$(3.9)\quad \frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}\left[H^{\alpha}_{a+}f(\sqrt{ab})+H^{\alpha}_{b-}f(\sqrt{ab})\right]$$
$$=\frac{1}{4}\ln\left(\frac{b}{a}\right)\int_0^1 s^{\alpha}\left[\sqrt{a^{1-s}b^{1+s}}f'\left(\sqrt{a^{1-s}b^{1+s}}\right)-\sqrt{a^{1+s}b^{1-s}}f'\left(\sqrt{a^{1+s}b^{1-s}}\right)\right]ds$$

and

$$(3.10)\quad \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}\left[H^{\alpha}_{\sqrt{ab}+}f(b)+H^{\alpha}_{\sqrt{ab}-}f(a)\right]-f\left(\sqrt{ab}\right)$$
$$=\frac{1}{4}\ln\left(\frac{b}{a}\right)\int_0^1 s^{\alpha}\left[\sqrt{a^{s}b^{2-s}}f'\left(\sqrt{a^{s}b^{2-s}}\right)-\sqrt{a^{2-s}b^{s}}f'\left(\sqrt{a^{2-s}b^{s}}\right)\right]ds.$$

4. Some Inequalities

We have:

Theorem 2. *Let $f:[a,b]\to\mathbb{C}$ be a differentiable function on (a,b) and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If*

$\left|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right|$ *is convex on* $(g(a), g(b))$, *then for any* $x \in (a,b)$ *we have the inequalities*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{2}\Gamma(\alpha+1)\left[\frac{I^{\alpha}_{a+,g}f(x)}{(g(x)-g(a))^{\alpha}} + \frac{I^{\alpha}_{b-,g}f(x)}{(g(b)-g(x))^{\alpha}}\right]\right| \tag{4.1}$$
$$\leq \frac{1}{2(\alpha+2)}\left\{(g(b)-g(x))\left[\frac{1}{\alpha+1}\left|\frac{f'(x)}{g'(x)}\right| + \left|\frac{f'(b)}{g'(b)}\right|\right]\right.$$
$$\left. + (g(x)-g(a))\left[\left|\frac{f'(a)}{g'(a)}\right| + \frac{1}{\alpha+1}\left|\frac{f'(x)}{g'(x)}\right|\right]\right\}$$
$$\leq \frac{1}{\alpha+2}\left[\frac{g(b)-g(a)}{2} + \left|g(x) - \frac{g(a)+g(b)}{2}\right|\right]$$
$$\times\left[\frac{1}{\alpha+1}\left|\frac{f'(x)}{g'(x)}\right| + \frac{1}{2}\left(\left|\frac{f'(a)}{g'(a)}\right| + \left|\frac{f'(b)}{g'(b)}\right|\right)\right]$$

and

$$\left|\frac{1}{2}\Gamma(\alpha+1)\left[\frac{I^{\alpha}_{x+,g}f(b)}{(g(b)-g(x))^{\alpha}} + \frac{I^{\alpha}_{x-,g}f(a)}{(g(x)-g(a))^{\alpha}}\right] - f(x)\right| \tag{4.2}$$
$$\leq \frac{1}{2(\alpha+2)}\left\{(g(b)-g(x))\left[\left|\frac{f'(x)}{g'(x)}\right| + \frac{1}{\alpha+1}\left|\frac{f'(b)}{g'(b)}\right|\right]\right.$$
$$\left. + (g(x)-g(a))\left[\left|\frac{f'(x)}{g'(x)}\right| + \frac{1}{\alpha+1}\left|\frac{f'(a)}{g'(a)}\right|\right]\right\}$$
$$\leq \frac{1}{\alpha+2}\left[\frac{g(b)-g(a)}{2} + \left|g(x) - \frac{g(a)+g(b)}{2}\right|\right]$$
$$\times\left[\left|\frac{f'(x)}{g'(x)}\right| + \frac{1}{2(\alpha+1)}\left(\left|\frac{f'(a)}{g'(a)}\right| + \left|\frac{f'(b)}{g'(b)}\right|\right)\right]$$

We also have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[\frac{I^{\alpha}_{a+,g}f(b) + I^{\alpha}_{b-,g}f(a)}{2}\right]\right| \tag{4.3}$$
$$\leq \frac{1}{\alpha+1}\frac{2^{\alpha}-1}{2^{\alpha+1}}(g(b)-g(a))\left(\left|\frac{f'(a)}{g'(a)}\right| + \left|\frac{f'(b)}{g'(b)}\right|\right).$$

Proof. By taking the modulus in the equality (2.9) we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{2}\Gamma(\alpha+1)\left[\frac{I^{\alpha}_{a+,g}f(x)}{(g(x)-g(a))^{\alpha}} + \frac{I^{\alpha}_{b-,g}f(x)}{(g(b)-g(x))^{\alpha}}\right]\right| \tag{4.4}$$
$$\leq \frac{1}{2}(g(b)-g(x))\left|\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}((1-s)g(x)+sg(b))}{g'\circ g^{-1}((1-s)g(x)+sg(b))}ds\right|$$
$$+\frac{1}{2}(g(x)-g(a))\left|\int_0^1 s^{\alpha}\frac{f'\circ g^{-1}(sg(a)+(1-s)g(x))}{g'\circ g^{-1}(sg(a)+(1-s)g(x))}ds\right|$$
$$\leq \frac{1}{2}(g(b)-g(x))\int_0^1 s^{\alpha}\left|\frac{f'\circ g^{-1}((1-s)g(x)+sg(b))}{g'\circ g^{-1}((1-s)g(x)+sg(b))}\right|ds$$
$$+\frac{1}{2}(g(x)-g(a))\int_0^1 s^{\alpha}\left|\frac{f'\circ g^{-1}(sg(a)+(1-s)g(x))}{g'\circ g^{-1}(sg(a)+(1-s)g(x))}\right|ds$$
$$=: A(x)$$

Since $\left|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right|$ is convex on $(g(a), g(b))$, hence

$$\left|\frac{f'\circ g^{-1}((1-s)g(x)+sg(b))}{g'\circ g^{-1}((1-s)g(x)+sg(b))}\right| \le (1-s)\left|\frac{f'(x)}{g'(x)}\right| + s\left|\frac{f'(b)}{g'(b)}\right|$$

and

$$\left|\frac{f'\circ g^{-1}(sg(a)+(1-s)g(x))}{g'\circ g^{-1}(sg(a)+(1-s)g(x))}\right| \le s\left|\frac{f'(a)}{g'(a)}\right| + (1-s)\left|\frac{f'(x)}{g'(x)}\right|$$

for any $x \in (a,b)$ and $s \in [0,1]$.

Therefore

$$\begin{aligned}
&\int_0^1 s^\alpha \left|\frac{f'\circ g^{-1}((1-s)g(x)+sg(b))}{g'\circ g^{-1}((1-s)g(x)+sg(b))}\right| ds \\
&\le \left|\frac{f'(x)}{g'(x)}\right| \int_0^1 (1-s)s^\alpha ds + \left|\frac{f'(b)}{g'(b)}\right| \int_0^1 s^{\alpha+1} ds \\
&= \frac{1}{\alpha+2}\left[\frac{1}{\alpha+1}\left|\frac{f'(x)}{g'(x)}\right| + \left|\frac{f'(b)}{g'(b)}\right|\right]
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 s^\alpha \left|\frac{f'\circ g^{-1}(sg(a)+(1-s)g(x))}{g'\circ g^{-1}(sg(a)+(1-s)g(x))}\right| ds \\
&\le \left|\frac{f'(a)}{g'(a)}\right| \int_0^1 s^{\alpha+1} ds + \left|\frac{f'(x)}{g'(x)}\right| \int_0^1 (1-s)s^\alpha ds \\
&= \frac{1}{\alpha+2}\left[\left|\frac{f'(a)}{g'(a)}\right| + \frac{1}{\alpha+1}\left|\frac{f'(x)}{g'(x)}\right|\right]
\end{aligned}$$

for any $x \in (a,b)$.

We then have

$$\begin{aligned}
A(x) &\le \frac{1}{2}(g(b)-g(x))\frac{1}{\alpha+2}\left[\frac{1}{\alpha+1}\left|\frac{f'(x)}{g'(x)}\right| + \left|\frac{f'(b)}{g'(b)}\right|\right] \\
&+ \frac{1}{2}(g(x)-g(a))\frac{1}{\alpha+2}\left[\left|\frac{f'(b)}{g'(b)}\right| + \frac{1}{\alpha+1}\left|\frac{f'(x)}{g'(x)}\right|\right] \\
&= \frac{1}{2(\alpha+2)}\left\{(g(b)-g(x))\left[\frac{1}{\alpha+1}\left|\frac{f'(x)}{g'(x)}\right| + \left|\frac{f'(b)}{g'(b)}\right|\right]\right. \\
&\left. + (g(x)-g(a))\left[\left|\frac{f'(a)}{g'(a)}\right| + \frac{1}{\alpha+1}\left|\frac{f'(x)}{g'(x)}\right|\right]\right\}
\end{aligned}$$

for any $x \in (a,b)$, which proves the first part of (4.1). The second part is obvious.

By the convexity of $\left|\frac{f'\circ g^{-1}}{g'\circ g^{-1}}\right|$ on $(g(a), g(b))$ we also have

$$\begin{aligned}
&\left|\frac{1}{2}\Gamma(\alpha+1)\left[\frac{I^{\alpha}_{x+,g}f(b)}{(g(b)-g(x))^{\alpha}}+\frac{I^{\alpha}_{x-,g}f(a)}{(g(x)-g(a))^{\alpha}}\right]-f(x)\right| \\
&\quad\le \frac{1}{2}(g(b)-g(x))\int_0^1 s^{\alpha}\left|\frac{f'\circ g^{-1}(sg(x)+(1-s)g(b))}{g'\circ g^{-1}(sg(x)+(1-s)g(b))}\right| ds \\
&\quad+\frac{1}{2}(g(x)-g(a))\int_0^1 s^{\alpha}\left|\frac{f'\circ g^{-1}((1-s)g(a)+sg(x))}{g'\circ g^{-1}((1-s)g(a)+sg(x))}\right| ds \\
&\quad\le \frac{1}{2}(g(b)-g(x))\int_0^1 s^{\alpha}\left[s\left|\frac{f'(x)}{g'(x)}\right|+(1-s)\left|\frac{f'(b)}{g'(b)}\right|\right] ds \\
&\quad+\frac{1}{2}(g(x)-g(a))\int_0^1 s^{\alpha}\left[(1-s)\left|\frac{f'(a)}{g'(a)}\right|+s\left|\frac{f'(x)}{g'(x)}\right|\right] ds \\
&\quad=\frac{1}{2}(g(b)-g(x))\frac{1}{\alpha+2}\left[\left|\frac{f'(x)}{g'(x)}\right|+\frac{1}{\alpha+1}\left|\frac{f'(b)}{g'(b)}\right|\right] \\
&\quad+\frac{1}{2}(g(x)-g(a))\frac{1}{\alpha+2}\left[\left|\frac{f'(x)}{g'(x)}\right|+\frac{1}{\alpha+1}\left|\frac{f'(a)}{g'(a)}\right|\right] \\
&\quad=\frac{1}{2(\alpha+2)}\left\{(g(b)-g(x))\left[\left|\frac{f'(x)}{g'(x)}\right|+\frac{1}{\alpha+1}\left|\frac{f'(b)}{g'(b)}\right|\right]\right. \\
&\qquad\qquad\left.+(g(x)-g(a))\left[\left|\frac{f'(x)}{g'(x)}\right|+\frac{1}{\alpha+1}\left|\frac{f'(a)}{g'(a)}\right|\right]\right\},
\end{aligned}$$

which proves the first part of (4.2). The second part is obvious.

Finally, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[\frac{I^{\alpha}_{a+,g}f(b)+I^{\alpha}_{b-,g}f(a)}{2}\right]\right| \\
&\quad\le \frac{1}{2}(g(b)-g(a))\int_0^1 |s^{\alpha}-(1-s)^{\alpha}|\left|\frac{f'\circ g^{-1}((1-s)g(a)+sg(b))}{g'\circ g^{-1}((1-s)g(a)+sg(b))}\right| ds \\
&\quad\le \frac{1}{2}(g(b)-g(a))\int_0^1 |s^{\alpha}-(1-s)^{\alpha}|\left[(1-s)\left|\frac{f'(a)}{g'(a)}\right|+s\left|\frac{f'(b)}{g'(b)}\right|\right] ds.
\end{aligned}\tag{4.5}$$

Observe that

$$\begin{aligned}
&\int_0^1 |s^{\alpha}-(1-s)^{\alpha}|\left[(1-s)\left|\frac{f'(a)}{g'(a)}\right|+s\left|\frac{f'(b)}{g'(b)}\right|\right] ds \\
&=\int_0^{1/2} ((1-s)^{\alpha}-s^{\alpha})\left[(1-s)\left|\frac{f'(a)}{g'(a)}\right|+s\left|\frac{f'(b)}{g'(b)}\right|\right] ds \\
&+\int_{1/2}^1 (s^{\alpha}-(1-s)^{\alpha})\left[(1-s)\left|\frac{f'(a)}{g'(a)}\right|+s\left|\frac{f'(b)}{g'(b)}\right|\right] ds
\end{aligned}$$

$$
\begin{aligned}
&= \left|\frac{f'(a)}{g'(a)}\right| \int_0^{1/2} (1-s)^{\alpha+1} ds - \left|\frac{f'(a)}{g'(a)}\right| \int_0^{1/2} s^{\alpha}(1-s)\, ds \\
&+ \left|\frac{f'(b)}{g'(b)}\right| \int_0^{1/2} (1-s)^{\alpha} s ds - \left|\frac{f'(b)}{g'(b)}\right| \int_0^{1/2} s^{\alpha+1} ds \\
&+ \left|\frac{f'(a)}{g'(a)}\right| \int_{1/2}^{1} s^{\alpha}(1-s)\, ds - \left|\frac{f'(a)}{g'(a)}\right| \int_{1/2}^{1} (1-s)^{\alpha+1} ds \\
&+ \left|\frac{f'(b)}{g'(b)}\right| \int_{1/2}^{1} s^{\alpha+1} ds - \left|\frac{f'(b)}{g'(b)}\right| \int_{1/2}^{1} (1-s)^{\alpha} s ds \\
&= B.
\end{aligned}
$$

Also

$$
\int_0^{1/2} (1-s)^{\alpha} s ds = \int_{1/2}^{1} (1-u)\, u^{\alpha} du
$$

and

$$
\int_{1/2}^{1} (1-s)^{\alpha} s ds = \int_0^{1/2} (1-u)\, u^{\alpha} du
$$

Then

$$
\begin{aligned}
B &= \left|\frac{f'(a)}{g'(a)}\right| \int_0^{1/2} (1-s)^{\alpha+1} ds - \left|\frac{f'(a)}{g'(a)}\right| \int_0^{1/2} s^{\alpha}(1-s)\, ds \\
&+ \left|\frac{f'(b)}{g'(b)}\right| \int_{1/2}^{1} (1-s)\, s^{\alpha} du - \left|\frac{f'(b)}{g'(b)}\right| \int_0^{1/2} s^{\alpha+1} ds \\
&+ \left|\frac{f'(a)}{g'(a)}\right| \int_{1/2}^{1} s^{\alpha}(1-s)\, ds - \left|\frac{f'(a)}{g'(a)}\right| \int_{1/2}^{1} (1-s)^{\alpha+1} ds \\
&+ \left|\frac{f'(b)}{g'(b)}\right| \int_{1/2}^{1} s^{\alpha+1} ds - \left|\frac{f'(b)}{g'(b)}\right| \int_0^{1/2} (1-s)\, s^{\alpha} du
\end{aligned}
$$

By performing the required calculations we obtain

$$
B = \frac{1}{\alpha+1} \frac{2^{\alpha}-1}{2^{\alpha}} \left(\left|\frac{f'(a)}{g'(a)}\right| + \left|\frac{f'(b)}{g'(b)}\right| \right),
$$

which proves the desired result (4.3). □

Corollary 3. *With the assumptions of Theorem 2 we have*

$$
\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}} \left[I^{\alpha}_{a+,g} f(M_g(a,b)) + I^{\alpha}_{b-,g} f(M_g(a,b)) \right] \right| \\
&\quad \le \frac{1}{2(\alpha+2)} [g(b)-g(a)] \left[\frac{1}{\alpha+1} \left| \frac{f'(M_g(a,b))}{g'(M_g(a,b))} \right| + \frac{1}{2} \left(\left|\frac{f'(a)}{g'(a)}\right| + \left|\frac{f'(b)}{g'(b)}\right| \right) \right]
\end{aligned} \tag{4.6}
$$

and

$$
\begin{aligned}
&\left| \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}} \left[I^{\alpha}_{M_g(a,b)+,g} f(b) + I^{\alpha}_{M_g(a,b)-,g} f(a) \right] - f(M_g(a,b)) \right| \\
&\quad \le \frac{1}{2(\alpha+2)} [g(b)-g(a)] \left[\left| \frac{f'(M_g(a,b))}{g'(M_g(a,b))} \right| + \frac{1}{2(\alpha+1)} \left(\left|\frac{f'(a)}{g'(a)}\right| + \left|\frac{f'(b)}{g'(b)}\right| \right) \right].
\end{aligned} \tag{4.7}
$$

The following particular cases are of interest:

Proposition 1. *Let $f:[a,b]\to\mathbb{C}$ be a differentiable function on (a,b). If $|f'|$ is convex on (a,b), then for any $x\in(a,b)$ we have the inequalities for the classical Riemann-Liouville fractional integrals*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2}\Gamma(\alpha+1)\left[\frac{J_{a+}^{\alpha}f(x)}{(x-a)^{\alpha}}+\frac{J_{b-}^{\alpha}f(x)}{(b-a)^{\alpha}}\right]\right| \tag{4.8}$$
$$\leq\frac{1}{2(\alpha+2)}\left\{(b-x)\left[\frac{1}{\alpha+1}|f'(x)|+|f'(b)|\right]+(b-x)\left[|f'(a)|+\frac{1}{\alpha+1}|f'(x)|\right]\right\}$$
$$\leq\frac{1}{\alpha+2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left[\frac{1}{\alpha+1}|f'(x)|+\frac{1}{2}(|f'(a)|+|f'(b)|)\right]$$

and

$$\left|\frac{1}{2}\Gamma(\alpha+1)\left[\frac{J_{x+}^{\alpha}f(b)}{(b-x)^{\alpha}}+\frac{J_{x-}^{\alpha}f(a)}{(x-a)^{\alpha}}\right]-f(x)\right| \tag{4.9}$$
$$\leq\frac{1}{2(\alpha+2)}\left\{(b-x)\left[|f'(x)|+\frac{1}{\alpha+1}|f'(b)|\right]+(x-a)\left[|f'(x)|+\frac{1}{\alpha+1}|f'(a)|\right]\right\}$$
$$\leq\frac{1}{\alpha+2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left[|f'(x)|+\frac{1}{2(\alpha+1)}(|f'(a)|+|f'(b)|)\right]$$

We also have, see [22] *where this inequality was obtained earlier*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\frac{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)}{2}\right]\right|\leq\frac{1}{\alpha+1}\frac{2^{\alpha}-1}{2^{\alpha+1}}(b-a)(|f'(a)|+|f'(b)|). \tag{4.10}$$

In particular, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \tag{4.11}$$
$$\leq\frac{1}{2(\alpha+2)}(b-a)\left[\frac{1}{\alpha+1}\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{1}{2}(|f'(a)|+|f'(b)|)\right]$$

and

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}+}^{\alpha}f(b)+J_{\frac{a+b}{2}-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right)\right| \tag{4.12}$$
$$\leq\frac{1}{2(\alpha+2)}(b-a)\left[\left|f'\left(\frac{a+b}{2}\right)\right|+\frac{1}{2(\alpha+1)}(|f'(a)|+|f'(b)|)\right].$$

The following results for the classical Hadamard fractional integrals H_{a+}^{α} and H_{b-}^{α} also hold:

Proposition 2. *Let $f:[a,b]\to\mathbb{C}$ be a differentiable function on $(a,b)\subset(0,\infty)$. If the function $\varphi(t):=|(f'\circ\exp)(t)|\exp(t)$ is convex on $(\ln a,\ln b)$, then for any*

$x \in (a, b)$ *we have the inequalities*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{2}\Gamma(\alpha+1)\left[\frac{H^{\alpha}_{a+}f(x)}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}} + \frac{H^{\alpha}_{b-}f(x)}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}\right]\right| \tag{4.13}$$
$$\leq \frac{1}{2(\alpha+2)}\left\{\left(\ln\left(\frac{b}{x}\right)\right)\left[\frac{1}{\alpha+1}|xf'(x)| + |bf'(b)|\right] + \left(\ln\left(\frac{x}{a}\right)\right)\left[|af'(a)| + \frac{1}{\alpha+1}|xf'(x)|\right]\right\}$$
$$\leq \frac{1}{\alpha+2}\left[\frac{1}{2}\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{x}{\sqrt{ab}}\right)\right|\right]\left[\frac{1}{\alpha+1}|xf'(x)| + \frac{1}{2}\left(|af'(a)| + |bf'(b)|\right)\right]$$

and

$$\left|\frac{1}{2}\Gamma(\alpha+1)\left[\frac{H^{\alpha}_{x+}f(b)}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}} + \frac{H^{\alpha}_{x-}f(a)}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}}\right] - f(x)\right| \tag{4.14}$$
$$\leq \frac{1}{2(\alpha+2)}\left\{\left(\ln\left(\frac{b}{x}\right)\right)\left[|xf'(x)| + \frac{1}{\alpha+1}|bf'(b)|\right] + \left(\ln\left(\frac{x}{a}\right)\right)\left[|xf'(x)| + \frac{1}{\alpha+1}|af'(a)|\right]\right\}$$
$$\leq \frac{1}{\alpha+2}\left[\frac{1}{2}\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{x}{\sqrt{ab}}\right)\right|\right]\left[|xf'(x)| + \frac{1}{2(\alpha+1)}\left(|af'(a)| + |af'(b)|\right)\right]$$

We also have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}\left[\frac{H^{\alpha}_{a+}f(b) + H^{\alpha}_{b-}f(a)}{2}\right]\right| \tag{4.15}$$
$$\leq \frac{1}{\alpha+1}\frac{2^{\alpha}-1}{2^{\alpha+1}}\ln\left(\frac{b}{a}\right)\left(|af'(a)| + |bf'(b)|\right).$$

In particular,

$$\left|\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}\left[H^{\alpha}_{a+}f(\sqrt{ab}) + H^{\alpha}_{b-}f(\sqrt{ab})\right]\right| \tag{4.16}$$
$$\leq \frac{1}{2(\alpha+2)}\ln\left(\frac{b}{a}\right)\left[\frac{1}{\alpha+1}\sqrt{ab}\left|f'\left(\sqrt{ab}\right)\right| + \frac{1}{2}\left(|af'(a)| + |bf'(b)|\right)\right]$$

and

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left[\ln\left(\frac{b}{a}\right)\right]^{\alpha}}\left[H^{\alpha}_{\sqrt{ab}+}f(b) + H^{\alpha}_{\sqrt{ab}-}f(a)\right] - f\left(\sqrt{ab}\right)\right| \tag{4.17}$$
$$\leq \frac{1}{2(\alpha+2)}\ln\left(\frac{b}{a}\right)\left[\sqrt{ab}\left|f'\left(\sqrt{ab}\right)\right| + \frac{1}{2(\alpha+1)}\left(|af'(a)| + |bf'(b)|\right)\right].$$

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