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FURTHER OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR THE GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF FUNCTIONS WITH BOUNDED VARIATION

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Abstract. In this paper we establish some Ostrowski and generalized trapezoid type inequalities for the Generalized Riemann-Liouville fractional integrals of functions of bounded variation. Applications for mid-point and trapezoid inequalities are provided as well. Some examples concerning the Hadamard and Harmonic fractional integrals are also given.

1. Introduction

Let $f:[a,b]\rightarrow\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)\,dt$$

for $a<x\leq b$ and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)\,dt$$

for $a\leq x<b$, where Γ is the *Gamma function.* For $\alpha=0$, they are defined as

$$J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)\text{ for }x\in(a,b).$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [1]-[6], [17]-[28] and the references therein.

The following Ostrowski type inequalities for functions of bounded variation generalize the corresponding results for the Riemann integral obtained in [9], [11], [10] and have been established recently by the author in [15] :

Theorem 1. *Let $f:[a,b]\rightarrow\mathbb{C}$ be a complex valued function of bounded variation on the real interval $[a,b]$. For any $x\in(a,b)$ we have*

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$$\begin{aligned}
&\left|J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \quad (1.1)\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}(x-t)^{\alpha-1}\bigvee_{t}^{x}(f)\,dt+\int_{x}^{b}(t-x)^{\alpha-1}\bigvee_{x}^{t}(f)\,dt\right]\\
&\leq\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}\bigvee_{a}^{x}(f)+(b-x)^{\alpha}\bigvee_{x}^{b}(f)\right]\\
&\leq\frac{1}{\Gamma(\alpha+1)}\\
&\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}\bigvee_{a}^{b}(f); \\ \\ \left((x-a)^{\alpha p}+(b-x)^{\alpha p}\right)^{1/p}\left(\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right]\left((x-a)^{\alpha}+(b-x)^{\alpha}\right),\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\left|J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)-\frac{f(x)}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\right| \quad (1.2)\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\int_{x}^{b}(b-t)^{\alpha-1}\bigvee_{x}^{t}(f)\,dt+\int_{a}^{x}(t-a)^{\alpha-1}\bigvee_{t}^{x}(f)\,dt\right]\\
&\leq\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}\bigvee_{a}^{x}(f)+(b-x)^{\alpha}\bigvee_{x}^{b}(f)\right]\\
&\leq\frac{1}{\Gamma(\alpha+1)} \quad (1.3)\\
&\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{\alpha}\bigvee_{a}^{b}(f); \\ \\ \left((x-a)^{\alpha p}+(b-x)^{\alpha p}\right)^{1/p}\left(\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right]\left((x-a)^{\alpha}+(b-x)^{\alpha}\right).\end{cases} \quad (1.4)
\end{aligned}$$

The following mid-point inequalities that can be derived from Theorem 1 are of interest as well:

$$
\begin{aligned}
(1.5)\quad & \left| J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \\
& \times \left[\int_{a}^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right)^{\alpha-1} \bigvee_{t}^{\frac{a+b}{2}} (f)\, dt + \int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right)^{\alpha-1} \bigvee_{\frac{a+b}{2}}^{t} (f)\, dt \right] \\
& \leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} (b-a)^{\alpha} \bigvee_{a}^{b} (f),
\end{aligned}
$$

and

$$
\begin{aligned}
(1.6)\quad & \left| J_{\frac{a+b}{2}+}^{\alpha} f(b) + J_{\frac{a+b}{2}-}^{\alpha} f(a) - \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{1}{\Gamma(\alpha)} \left[\int_{\frac{a+b}{2}}^{b} (b-t)^{\alpha-1} \bigvee_{\frac{a+b}{2}}^{t} (f)\, dt + \int_{a}^{\frac{a+b}{2}} (t-a)^{\alpha-1} \bigvee_{t}^{\frac{a+b}{2}} (f)\, dt \right] \\
& \leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)} (b-a)^{\alpha} \bigvee_{a}^{b} (f).
\end{aligned}
$$

In order to extend this result for other fractional integrals, we need the following definitions.

Let (a,b) with $-\infty \leq a < b \leq \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\mathrm{Re}(\alpha) > 0$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Following [19, p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ by

$$
(1.7)\qquad I_{a+,g}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{g'(t) f(t)\, dt}{[g(x) - g(t)]^{1-\alpha}},\ a < x \leq b
$$

and

$$
(1.8)\qquad I_{b-,g}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{b} \frac{g'(t) f(t)\, dt}{[g(t) - g(x)]^{1-\alpha}},\ a \leq x < b.
$$

For $g(t) = t$ we have the classical *Riemann-Liouville fractional integrals* defined above while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [19, p. 111]

$$
(1.9)\qquad H_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t},\ 0 \leq a < x \leq b
$$

and

$$
(1.10)\qquad H_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{b} \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t},\ 0 \leq a < x < b.
$$

One can consider the function $g(t)=-t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_{a}^{x}\frac{f(t)\,dt}{(x-t)^{1-\alpha}t^{\alpha+1}},\ 0\leq a<x\leq b \tag{1.11}$$

and

$$R_{b-}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_{x}^{b}\frac{f(t)\,dt}{(t-x)^{1-\alpha}t^{\alpha+1}},\ 0\leq a<x<b. \tag{1.12}$$

Also, for $g(t)=t^{p}$, $p>0$, we have the *p-Riemann-Liouville fractional integrals*

$$J_{a+,p}^{\alpha}f(x):=\frac{p}{\Gamma(\alpha)}\int_{a}^{x}\frac{t^{p-1}f(t)\,dt}{(x^{p}-t^{p})^{1-\alpha}},\ 0\leq a<x\leq b \tag{1.13}$$

and

$$J_{b-,p}^{\alpha}f(x):=\frac{p}{\Gamma(\alpha)}\int_{x}^{b}\frac{t^{p-1}f(t)\,dt}{(t^{p}-x^{p})^{1-\alpha}},\ 0\leq a\leq x<b. \tag{1.14}$$

Motivated by the above results, in this paper we establish some new Ostrowski and generalized trapezoid type inequalities for the Generalized Riemann-Liouville fractional integrals of functions of bounded variation. Applications for mid-point and trapezoid inequalities are provided as well. Some examples concerning the Hadamard and Harmonic fractional integrals are also given.

2. Some Identities of Interest

We have the following results:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b).*

(i) *For any $x\in(a,b)$ we have*

$$\begin{aligned} &I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)\\ &=\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}f(a)+(g(b)-g(x))^{\alpha}f(b)\right]\\ &+\frac{1}{\Gamma(\alpha+1)}\left[\int_{a}^{x}(g(x)-g(t))^{\alpha}df(t)-\int_{x}^{b}(g(t)-g(x))^{\alpha}df(t)\right]. \end{aligned} \tag{2.1}$$

(ii) *For any $x\in(a,b)$ we have*

$$\begin{aligned} &I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)\\ &=\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right]f(x)\\ &+\frac{1}{\Gamma(\alpha+1)}\left[\int_{x}^{b}(g(b)-g(t))^{\alpha}df(t)-\int_{a}^{x}(g(t)-g(a))^{\alpha}df(t)\right]. \end{aligned} \tag{2.2}$$

(iii) *We have the trapezoid equality*

$$\begin{aligned}(2.3)\qquad &\frac{I_{b-,g}^{\alpha}f(a)+I_{a+,g}^{\alpha}f(b)}{2}\\ &=\frac{1}{\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\frac{f(b)+f(a)}{2}\\ &+\frac{1}{\Gamma(\alpha+1)}\int_a^b\frac{(g(b)-g(t))^{\alpha}-(g(t)-g(a))^{\alpha}}{2}df(t).\end{aligned}$$

Proof. (i) Since $f:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$ and g is continuous on $[a,b]$, then the Riemann-Stieltjes integrals

$$\int_a^x(g(x)-g(t))^{\alpha}df(t)\text{ and }\int_x^b(g(t)-g(x))^{\alpha}df(t)$$

exist and integrating by parts, we have

$$\begin{aligned}(2.4)\quad &\frac{1}{\Gamma(\alpha+1)}\int_a^x(g(x)-g(t))^{\alpha}df(t)\\ &=\frac{1}{\Gamma(\alpha)}\int_a^x(g(x)-g(t))^{\alpha-1}g'(t)f(t)dt-\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}f(a)\\ &=I_{a+,g}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}f(a)\end{aligned}$$

for $a<x\le b$ and

$$\begin{aligned}(2.5)\quad &\frac{1}{\Gamma(\alpha+1)}\int_x^b(g(t)-g(x))^{\alpha}df(t)\\ &=\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha}f(b)-\frac{1}{\Gamma(\alpha)}\int_x^b(g(t)-g(x))^{\alpha-1}g'(t)f(t)dt\\ &=\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha}f(b)-I_{b-,g}^{\alpha}f(x)\end{aligned}$$

for $a\le x<b$.

From (2.4), we then have

$$\begin{aligned}I_{a+,g}^{\alpha}f(x)&=\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}f(a)\\ &+\frac{1}{\Gamma(\alpha+1)}\int_a^x(g(x)-g(t))^{\alpha}df(t)\end{aligned}$$

for $a<x\le b$ and from (2.5) we have

$$\begin{aligned}I_{b-,g}^{\alpha}f(x)&=\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha}f(b)\\ &-\frac{1}{\Gamma(\alpha+1)}\int_x^b(g(t)-g(x))^{\alpha}df(t),\end{aligned}$$

for $a\le x<b$, which by addition give (2.1).

(ii) We have

$$I_{x+,g}^{\alpha}f(b)=\frac{1}{\Gamma(\alpha)}\int_x^b(g(b)-g(t))^{\alpha-1}g'(t)f(t)dt$$

for $a \le x < b$ and

$$I_{x-,g}^{\alpha} f(a) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(t) - g(a))^{\alpha-1} g'(t) f(t) \, dt$$

for $a < x \le b$.

Since $f : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$ and g is continuous on $[a,b]$, then the Riemann-Stieltjes integrals

$$\int_a^x (g(t) - g(a))^{\alpha} \, df(t) \text{ and } \int_x^b (g(b) - g(t))^{\alpha} \, df(t)$$

exist and integrating by parts, we have

$$\begin{aligned}(2.6)\quad & \frac{1}{\Gamma(\alpha+1)} \int_a^x (g(t) - g(a))^{\alpha} \, df(t) \\ &= \frac{1}{\Gamma(\alpha+1)} (g(x) - g(a))^{\alpha} f(x) - \frac{1}{\Gamma(\alpha)} \int_a^x (g(t) - g(a))^{\alpha-1} g'(t) f(t) \, dt \\ &= \frac{1}{\Gamma(\alpha+1)} (g(x) - g(a))^{\alpha} f(x) - I_{x-,g}^{\alpha} f(a)\end{aligned}$$

for $a < x \le b$ and

$$\begin{aligned}(2.7)\quad & \frac{1}{\Gamma(\alpha+1)} \int_x^b (g(b) - g(t))^{\alpha} \, df(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_x^b (g(b) - g(t))^{\alpha-1} g'(t) f(t) \, dt - \frac{1}{\Gamma(\alpha+1)} (g(b) - g(x))^{\alpha} f(x) \\ &= I_{x+,g}^{\alpha} f(b) - \frac{1}{\Gamma(\alpha+1)} (g(b) - g(x))^{\alpha} f(x)\end{aligned}$$

for $a \le x < b$.

From (2.6) we have

$$\begin{aligned}I_{x-,g}^{\alpha} f(a) &= \frac{1}{\Gamma(\alpha+1)} (g(x) - g(a))^{\alpha} f(x) \\ &\quad - \frac{1}{\Gamma(\alpha+1)} \int_a^x (g(t) - g(a))^{\alpha} \, df(t)\end{aligned} \tag{2.8}$$

for $a < x \le b$ and from (2.7)

$$\begin{aligned}I_{x+,g}^{\alpha} f(b) &= \frac{1}{\Gamma(\alpha+1)} (g(b) - g(x))^{\alpha} f(x) \\ &\quad + \frac{1}{\Gamma(\alpha+1)} \int_x^b (g(b) - g(t))^{\alpha} \, df(t),\end{aligned} \tag{2.9}$$

for $a \le x < b$, which by addition produce (2.2).

(iii) For $x = b$ in (2.8) we have

$$\begin{aligned}I_{b-,g}^{\alpha} f(a) &= \frac{1}{\Gamma(\alpha+1)} (g(b) - g(a))^{\alpha} f(b) \\ &\quad - \frac{1}{\Gamma(\alpha+1)} \int_a^b (g(t) - g(a))^{\alpha} \, df(t)\end{aligned}$$

while from (2.9) we have for $x = a$ that

$$I_{a+,g}^{\alpha} f(b) = \frac{1}{\Gamma(\alpha+1)} (g(b) - g(a))^{\alpha} f(a) + \frac{1}{\Gamma(\alpha+1)} \int_a^b (g(b) - g(t))^{\alpha} df(t).$$

If we add these two equalities and divide by 2, we get (2.3). □

Corollary 1. *With the assumptions of Lemma 1, we have*

$$\begin{aligned} (2.10)\quad & I_{a+,g}^{\alpha} f\left(\frac{a+b}{2}\right) + I_{b-,g}^{\alpha} f\left(\frac{a+b}{2}\right) \\ &= \frac{1}{\Gamma(\alpha+1)} \left[\left(g\left(\frac{a+b}{2}\right) - g(a) \right)^{\alpha} f(a) + \left(g(b) - g\left(\frac{a+b}{2}\right) \right)^{\alpha} f(b) \right] \\ &+ \frac{1}{\Gamma(\alpha+1)} \int_a^{\frac{a+b}{2}} \left(g\left(\frac{a+b}{2}\right) - g(t) \right)^{\alpha} df(t) \\ &- \frac{1}{\Gamma(\alpha+1)} \int_{\frac{a+b}{2}}^b \left(g(t) - g\left(\frac{a+b}{2}\right) \right)^{\alpha} df(t) \end{aligned}$$

and

$$\begin{aligned} (2.11)\quad & I_{\frac{a+b}{2}-,g}^{\alpha} f(a) + I_{\frac{a+b}{2}+,g}^{\alpha} f(b) \\ &= \frac{1}{\Gamma(\alpha+1)} \left[\left(g\left(\frac{a+b}{2}\right) - g(a) \right)^{\alpha} + \left(g(b) - g\left(\frac{a+b}{2}\right) \right)^{\alpha} \right] f\left(\frac{a+b}{2}\right) \\ &+ \frac{1}{\Gamma(\alpha+1)} \int_{\frac{a+b}{2}}^b (g(b) - g(t))^{\alpha} df(t) \\ &- \frac{1}{\Gamma(\alpha+1)} \int_a^{\frac{a+b}{2}} (g(t) - g(a))^{\alpha} df(t). \end{aligned}$$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a, b \in I$ by

$$M_g(a,b) := g^{-1}\left(\frac{g(a) + g(b)}{2}\right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a,b) = A(a,b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0,\infty)$ and $g(t) = \ln t$, then $M_g(a,b) = G(a,b) := \sqrt{ab}$, the *geometric mean*. If $I = (0,\infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a,b) = H(a,b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0,\infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a,b) = M_p(a,b) := \left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*. Finally, if $I = \mathbb{R}$ and $g(t) = \exp t$, then

$$M_g(a,b) = LME(a,b) := \ln\left(\frac{\exp a + \exp b}{2}\right),$$

the *LogMeanExp function*.

Corollary 2. *With the assumptions of Lemma 1, we have*

$$\begin{aligned}(2.12)\qquad & I_{a+,g}^{\alpha}f\left(M_g\left(a,b\right)\right)+I_{b-,g}^{\alpha}f\left(M_g\left(a,b\right)\right)\\ &=\frac{1}{2^{\alpha-1}\Gamma\left(\alpha+1\right)}\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}\frac{f\left(a\right)+f\left(b\right)}{2}\\ &+\frac{1}{\Gamma\left(\alpha+1\right)}\int_a^{M_g(a,b)}\left(\frac{g\left(a\right)+g\left(b\right)}{2}-g\left(t\right)\right)^{\alpha}df\left(t\right)\\ &-\frac{1}{\Gamma\left(\alpha+1\right)}\int_{M_g(a,b)}^{b}\left(g\left(t\right)-\frac{g\left(a\right)+g\left(b\right)}{2}\right)^{\alpha}df\left(t\right)\end{aligned}$$

and

$$\begin{aligned}(2.13)\qquad & I_{M_g(a,b)-,g}^{\alpha}f\left(a\right)+I_{M_g(a,b)+,g}^{\alpha}f\left(b\right)\\ &=\frac{1}{2^{\alpha-1}\Gamma\left(\alpha+1\right)}\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}f\left(M_g\left(a,b\right)\right)\\ &+\frac{1}{\Gamma\left(\alpha+1\right)}\int_{M_g(a,b)}^{b}\left(g\left(b\right)-g\left(t\right)\right)^{\alpha}df\left(t\right)\\ &-\frac{1}{\Gamma\left(\alpha+1\right)}\int_a^{M_g(a,b)}\left(g\left(t\right)-g\left(a\right)\right)^{\alpha}df\left(t\right).\end{aligned}$$

From a complementary view point we also have:

Lemma 2. *With the assumptions of Lemma 1, we have*

$$\begin{aligned}(2.14)\qquad & \frac{1}{2}\Gamma\left(\alpha+1\right)\left[\frac{I_{a+,g}^{\alpha}f\left(x\right)}{\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}}+\frac{I_{b-,g}^{\alpha}f\left(x\right)}{\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}}\right]=\frac{f\left(a\right)+f\left(b\right)}{2}\\ &+\frac{1}{2\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}}\int_a^x\left(g\left(x\right)-g\left(t\right)\right)^{\alpha}df\left(t\right)\\ &-\frac{1}{2\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}}\int_x^b\left(g\left(t\right)-g\left(x\right)\right)^{\alpha}df\left(t\right)\end{aligned}$$

and

$$\begin{aligned}(2.15)\qquad & \frac{1}{2}\Gamma\left(\alpha+1\right)\left[\frac{I_{x-,g}^{\alpha}f\left(a\right)}{\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}}+\frac{I_{x+,g}^{\alpha}f\left(b\right)}{\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}}\right]=f\left(x\right)\\ &+\frac{1}{2\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}}\int_x^b\left(g\left(b\right)-g\left(t\right)\right)^{\alpha}df\left(t\right)\\ &-\frac{1}{2\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}}\int_a^x\left(g\left(t\right)-g\left(a\right)\right)^{\alpha}df\left(t\right)\end{aligned}$$

for any $x\in(a,b)$.

Proof. By the above equalities (2.4) and (2.5) we have

$$\begin{aligned}\frac{I_{a+,g}^{\alpha}f\left(x\right)}{\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}}&=\frac{1}{\Gamma\left(\alpha+1\right)}f\left(a\right)\\ &+\frac{1}{\Gamma\left(\alpha+1\right)\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}}\int_a^x\left(g\left(x\right)-g\left(t\right)\right)^{\alpha}df\left(t\right)\end{aligned}$$

for $a < x \leq b$ and

$$\frac{I_{b-,g}^{\alpha} f(x)}{(g(b)-g(x))^{\alpha}} = \frac{1}{\Gamma(\alpha+1)} f(b) - \frac{1}{\Gamma(\alpha+1)(g(b)-g(x))^{\alpha}} \int_x^b (g(t)-g(x))^{\alpha} df(t)$$

and $a \leq x < b$.

If we add these two equalities and multiply by $\frac{1}{2}\Gamma(\alpha+1)$ we get (2.14).

By the equalities (2.6) and (2.7)

$$\frac{I_{x-,g}^{\alpha} f(a)}{(g(x)-g(a))^{\alpha}} = \frac{1}{\Gamma(\alpha+1)} f(x) - \frac{1}{\Gamma(\alpha+1)(g(x)-g(a))^{\alpha}} \int_a^x (g(t)-g(a))^{\alpha} df(t)$$

for $a < x \leq b$ and

$$\frac{I_{x+,g}^{\alpha} f(b)}{(g(b)-g(x))^{\alpha}} = \frac{1}{\Gamma(\alpha+1)} f(x) + \frac{1}{\Gamma(\alpha+1)(g(b)-g(x))^{\alpha}} \int_x^b (g(b)-g(t))^{\alpha} dft$$

for $a \leq x < b$.

If we add these two equalities and multiply by $\frac{1}{2}\Gamma(\alpha+1)$ we get (2.15). □

Corollary 3. *With the assumptions of Lemma 1, we have*

$$\frac{1}{2}\Gamma(\alpha+1)\left[\frac{I_{a+,g}^{\alpha} f\left(\frac{a+b}{2}\right)}{\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\alpha}} + \frac{I_{b-,g}^{\alpha} f\left(\frac{a+b}{2}\right)}{\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\alpha}}\right] = \frac{f(a)+f(b)}{2} + \frac{1}{2\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\alpha}} \int_a^{\frac{a+b}{2}} \left(g\left(\frac{a+b}{2}\right)-g(t)\right)^{\alpha} df(t) - \frac{1}{2\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\alpha}} \int_{\frac{a+b}{2}}^b \left(g(t)-g\left(\frac{a+b}{2}\right)\right)^{\alpha} df(t) \tag{2.16}$$

and

$$\frac{1}{2}\Gamma(\alpha+1)\left[\frac{I_{\frac{a+b}{2}-,g}^{\alpha} f(a)}{\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\alpha}} + \frac{I_{\frac{a+b}{2}+,g}^{\alpha} f(b)}{\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\alpha}}\right] = f\left(\frac{a+b}{2}\right) + \frac{1}{2\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\alpha}} \int_{\frac{a+b}{2}}^b (g(b)-g(t))^{\alpha} df(t) - \frac{1}{2\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\alpha}} \int_a^{\frac{a+b}{2}} (g(t)-g(a))^{\alpha} df(t) \tag{2.17}$$

for any $x \in (a,b)$.

Remark 1. *If we take $x = M_g(a,b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$ in Lemma 2, then we get the same equalities that have been stated in Corollary 2.*

3. Some General Inequalities

The following lemma is of interest in itself as well [2, p. 177], see also [12] for a generalization.

Lemma 3. *Let $f,\ u : [a,b] \to \mathbb{C}$. If f is continuous on $[a,b]$ and u is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and*

$$\left|\int_a^b f(t)\,du(t)\right| \le \int_a^b |f(t)|\,d\left(\bigvee_a^t (u)\right) \le \max_{t\in[a,b]} |f(t)| \bigvee_a^b (u), \tag{3.1}$$

where $\bigvee_a^t (u)$ denotes the total variation of u on $[a,t]$, $t \in [a,b]$.

We have:

Theorem 2. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then we have*

$$\begin{aligned}
&\Big| I_{a+,g}^{\alpha} f(x) + I_{b-,g}^{\alpha} f(x) \\
&\qquad - \frac{1}{\Gamma(\alpha+1)} \left[(g(x)-g(a))^{\alpha} f(a) + (g(b)-g(x))^{\alpha} f(b) \right] \Big| \\
&\le \frac{1}{\Gamma(\alpha)} \left[\int_a^x (g(x)-g(t))^{\alpha-1} g'(t) \bigvee_a^t (f)\,dt + \int_x^b (g(t)-g(x))^{\alpha-1} g'(t) \bigvee_t^b (f)\,dt \right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \left[(g(x)-g(a))^{\alpha} \bigvee_a^x (f) + (g(b)-g(x))^{\alpha} \bigvee_x^b (f) \right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \begin{cases}
\left[\frac{1}{2}(g(b)-g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\[2ex]
\left((g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p} \right)^{1/p} \\
\times \left(\left(\bigvee_a^x (f) \right)^q + \left(\bigvee_x^b (f) \right)^q \right)^{1/q} \\
\text{with } p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex]
\left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \\
\left((g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right)
\end{cases}
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{1}{2}\Gamma(\alpha+1) \left[\frac{I_{a+,g}^{\alpha} f(x)}{(g(x)-g(a))^{\alpha}} + \frac{I_{b-,g}^{\alpha} f(x)}{(g(b)-g(x))^{\alpha}} \right] \right| \\
&\le \frac{\alpha}{2(g(x)-g(a))^{\alpha}} \int_a^x (g(x)-g(t))^{\alpha-1} g'(t) \bigvee_a^t (f)\,dt \\
&+ \frac{\alpha}{2(g(b)-g(x))^{\alpha}} \int_x^b (g(t)-g(x))^{\alpha-1} g'(t) \bigvee_t^b (f)\,dt \\
&\le \frac{1}{2} \bigvee_a^b (f)
\end{aligned} \tag{3.3}$$

for any $x \in (a,b)$.

Proof. By using Lemma 3 we have

$$\left| \int_a^x (g(x) - g(t))^\alpha \, df(t) \right| \le \int_a^x (g(x) - g(t))^\alpha \, d\left(\bigvee_a^t (f) \right)$$

for $a < x \le b$ and

$$\left| \int_x^b (g(t) - g(x))^\alpha \, df(t) \right| \le \int_x^b (g(t) - g(x))^\alpha \, d\left(\bigvee_x^t (f) \right)$$

and $a \le x < b$.

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^x (g(x) - g(t))^\alpha \, d\left(\bigvee_a^t (f) \right) \\
&= (g(x) - g(t))^\alpha \bigvee_a^t (f) \Big|_a^x + \alpha \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \bigvee_a^t (f) \, dt \\
&= \alpha \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \bigvee_a^t (f) \, dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b (g(t) - g(x))^\alpha \, d\left(\bigvee_x^t (f) \right) \\
&= (g(t) - g(x))^\alpha \bigvee_x^t (f) \Big|_x^b - \alpha \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \bigvee_x^t (f) \, dt \\
&= (g(b) - g(x))^\alpha \bigvee_x^b (f) - \alpha \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \bigvee_x^t (f) \, dt \\
&= \alpha \bigvee_x^b (f) \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \, dt \\
&- \alpha \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \bigvee_x^t (f) \, dt \\
&= \alpha \int_x^b \left[\bigvee_x^b (f) - \bigvee_x^t (f) \right] (g(t) - g(x))^{\alpha-1} g'(t) \, dt \\
&= \alpha \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \bigvee_t^b (f) \, dt
\end{aligned}$$

for any $x \in (a,b)$.

By taking the modulus in the equality (2.1) we have

$$\begin{aligned}
&\left|I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)\right.\\
&\left.-\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}f(a)+(g(b)-g(x))^{\alpha}f(b)\right]\right|\\
&\leq\frac{1}{\Gamma(\alpha+1)}\left[\left|\int_{a}^{x}(g(x)-g(t))^{\alpha}df(t)\right|+\left|\int_{x}^{b}(g(t)-g(x))^{\alpha}df(t)\right|\right]\\
&\leq\frac{1}{\Gamma(\alpha+1)}\int_{a}^{x}(g(x)-g(t))^{\alpha}d\left(\bigvee_{a}^{t}(f)\right)\\
&+\frac{1}{\Gamma(\alpha+1)}\int_{x}^{b}(g(t)-g(x))^{\alpha}d\left(\bigvee_{x}^{t}(f)\right)\\
&=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(g(x)-g(t))^{\alpha-1}g'(t)\bigvee_{a}^{t}(f)\,dt\\
&+\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(g(t)-g(x))^{\alpha-1}g'(t)\bigvee_{t}^{b}(f)\,dt
\end{aligned}$$

for any $x\in(a,b)$, which proves the first part of (3.2).

Moreover, since $\bigvee_{a}^{t}(f)\leq\bigvee_{a}^{x}(f)$ for $a\leq t\leq x$ and $\bigvee_{t}^{b}(f)\leq\bigvee_{x}^{b}(f)$ for $x\leq t\leq b$, then

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\left[\int_{a}^{x}(g(x)-g(t))^{\alpha-1}g'(t)\bigvee_{a}^{t}(f)\,dt+\int_{x}^{b}(g(t)-g(x))^{\alpha-1}g'(t)\bigvee_{t}^{b}(f)\,dt\right]\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\bigvee_{a}^{x}(f)\int_{a}^{x}(g(x)-g(t))^{\alpha-1}g'(t)\,dt+\bigvee_{x}^{b}(f)\int_{x}^{b}(g(t)-g(x))^{\alpha-1}g'(t)\,dt\right]\\
&=\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}\bigvee_{a}^{x}(f)+(g(b)-g(x))^{\alpha}\bigvee_{x}^{b}(f)\right]
\end{aligned}$$

for any $x\in(a,b)$, which proves the second part of (3.2).

The last part of (3.2 is obvious by making use of the elementary Hölder type inequalities for positive real numbers c, d, m, $n\geq0$

$$mc+nd\leq\begin{cases}\max\{m,n\}(c+d);\\[2ex](m^{p}+n^{p})^{1/p}(c^{q}+d^{q})^{1/q} & \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1.\end{cases}$$

By the equality (2.14) we also have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{2}\Gamma(\alpha+1)\left[\frac{I_{a+,g}^{\alpha}f(x)}{(g(x)-g(a))^{\alpha}}+\frac{I_{b-,g}^{\alpha}f(x)}{(g(b)-g(x))^{\alpha}}\right]\right|\\
&\leq\frac{1}{2(g(x)-g(a))^{\alpha}}\left|\int_a^x(g(x)-g(t))^{\alpha}df(t)\right|\\
&+\frac{1}{2(g(b)-g(x))^{\alpha}}\left|\int_x^b(g(t)-g(x))^{\alpha}df(t)\right|\\
&\leq\frac{1}{2(g(x)-g(a))^{\alpha}}\int_a^x(g(x)-g(t))^{\alpha}d\left(\bigvee_a^t(f)\right)\\
&+\frac{1}{2(g(b)-g(x))^{\alpha}}\int_x^b(g(t)-g(x))^{\alpha}d\left(\bigvee_x^t(f)\right)\\
&=\frac{\alpha}{2(g(x)-g(a))^{\alpha}}\int_a^x(g(x)-g(t))^{\alpha-1}g'(t)\bigvee_a^t(f)\,dt\\
&+\frac{\alpha}{2(g(b)-g(x))^{\alpha}}\int_x^b(g(t)-g(x))^{\alpha-1}g'(t)\bigvee_t^b(f)\,dt\\
&\leq\frac{1}{2}\bigvee_a^x(f)\,dt+\frac{1}{2}\bigvee_x^b(f)=\frac{1}{2}\bigvee_a^b(f),
\end{aligned}$$

which proves the inequality (3.3). □

Remark 2. *The inequality (3.2) was obtained by a different technique in the earlier paper* [16].

Corollary 4. *With the assumptions of Theorem 2, we have*

$$\begin{aligned}
(3.4)\quad&\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))\right]\right|\\
&\leq\frac{2^{\alpha-1}\alpha}{(g(b)-g(a))^{\alpha}}\left[\int_a^{M_g(a,b)}\left(\frac{g(a)+g(b)}{2}-g(t)\right)^{\alpha-1}g'(t)\bigvee_a^t(f)\,dt\right.\\
&\left.+\int_{M_g(a,b)}^b\left(g(t)-\frac{g(a)+g(b)}{2}\right)^{\alpha-1}g'(t)\bigvee_t^b(f)\,dt\right]\\
&\leq\frac{1}{2}\bigvee_a^b(f).
\end{aligned}$$

The proof follows by either the inequality (3.2) or (3.3) by taking $x=x=M_g(a,b)=g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$.

Theorem 3. *With the assumptions of Theorem 2, we have*

$$\begin{aligned}
(3.5)\quad & \Big| I^{\alpha}_{x-,g} f(a) + I^{\alpha}_{x+,g} f(b) \\
& \qquad - \frac{1}{\Gamma(\alpha+1)} \left[(g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right] f(x) \Big| \\
& \leq \frac{1}{\Gamma(\alpha)} \left[\int_a^x (g(t)-g(a))^{\alpha-1} g'(t) \bigvee_t^x (f)\, dt + \int_x^b (g(b)-g(t))^{\alpha-1} g'(t) \bigvee_x^t (f)\, dt \right] \\
& \leq \frac{1}{\Gamma(\alpha+1)} \left[(g(x)-g(a))^{\alpha} \bigvee_a^x (f) + (g(b)-g(x))^{\alpha} \bigvee_x^b (f) \right] \\
& \qquad \frac{1}{\Gamma(\alpha+1)} \begin{cases} \left[\frac{1}{2}(g(b)-g(a)) + \left| g(x) - \frac{g(a)+g(b)}{2} \right| \right]^{\alpha} \bigvee_a^b (f); \\ \\ \left((g(x)-g(a))^{\alpha p} + (g(b)-g(x))^{\alpha p} \right)^{1/p} \\ \times \left(\left(\bigvee_a^x (f) \right)^q + \left(\bigvee_x^b (f) \right)^q \right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2} \bigvee_a^b (f) + \frac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right] \\ \left((g(x)-g(a))^{\alpha} + (g(b)-g(x))^{\alpha} \right) \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
(3.6)\quad & \left| \frac{1}{2} \Gamma(\alpha+1) \left[\frac{I^{\alpha}_{x-,g} f(a)}{(g(x)-g(a))^{\alpha}} + \frac{I^{\alpha}_{x+,g} f(b)}{(g(b)-g(x))^{\alpha}} \right] - f(x) \right| \\
& \leq \frac{\alpha}{2(g(b)-g(x))^{\alpha}} \int_x^b (g(b)-g(t))^{\alpha-1} g'(t) \bigvee_x^t (f)\, dt \\
& + \frac{\alpha}{2(g(x)-g(a))^{\alpha}} \int_a^x (g(t)-g(a))^{\alpha-1} g'(t) \bigvee_t^x (f)\, dt \\
& \leq \frac{1}{2} \bigvee_a^b (f),
\end{aligned}$$

for any $x \in (a,b)$.

Proof. By using Lemma 3 we have

$$\left| \int_a^x (g(t)-g(a))^{\alpha} df(t) \right| \leq \int_a^x (g(t)-g(a))^{\alpha} d\left(\bigvee_a^t (f) \right)$$

and

$$\left| \int_x^b (g(b)-g(t))^{\alpha} df(t) \right| \leq \int_x^b (g(b)-g(t))^{\alpha} d\left(\bigvee_x^t (f) \right).$$

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^x (g(t)-g(a))^\alpha d\left(\bigvee_a^t (f)\right) \\
&= (g(t)-g(a))^\alpha \bigvee_a^t (f)\Bigg|_a^x - \alpha \int_a^x (g(t)-g(a))^{\alpha-1} g'(t) \bigvee_a^t (f)\, dt \\
&= (g(x)-g(a))^\alpha \bigvee_a^x (f) - \alpha \int_a^x (g(t)-g(a))^{\alpha-1} g'(t) \bigvee_a^t (f)\, dt \\
&= \alpha \bigvee_a^x (f) \int_a^x (g(t)-g(a))^{\alpha-1} g'(t)\, dt \\
&- \alpha \int_a^x (g(t)-g(a))^{\alpha-1} g'(t) \bigvee_a^t (f)\, dt \\
&= \alpha \int_a^x \left[\bigvee_a^x (f) - \bigvee_a^t (f)\right] (g(t)-g(a))^{\alpha-1} g'(t)\, dt \\
&= \alpha \int_a^x (g(t)-g(a))^{\alpha-1} g'(t) \bigvee_t^x (f)\, dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b (g(b)-g(t))^\alpha d\left(\bigvee_x^t (f)\right) \\
&= (g(b)-g(t))^\alpha \bigvee_x^t (f)\Bigg|_x^b + \alpha \int_x^b (g(b)-g(t))^{\alpha-1} g'(t) \bigvee_x^t (f)\, dt \\
&= \alpha \int_x^b (g(b)-g(t))^{\alpha-1} g'(t) \bigvee_x^t (f)\, dt
\end{aligned}$$

for any $x \in (a,b)$.

Using the equality (2.2) we have

$$\begin{aligned}
&\left|I_{x-,g}^\alpha f(a) + I_{x+,g}^\alpha f(b)\right. \\
&\left.-\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^\alpha + (g(b)-g(x))^\alpha\right] f(x)\right| \\
&\le \frac{1}{\Gamma(\alpha+1)}\left[\left|\int_x^b (g(b)-g(t))^\alpha df(t)\right| + \left|\int_a^x (g(t)-g(a))^\alpha df(t)\right|\right] \\
&\le \frac{1}{\Gamma(\alpha+1)} \int_a^x (g(t)-g(a))^\alpha d\left(\bigvee_a^t (f)\right) \\
&+ \frac{1}{\Gamma(\alpha+1)} \int_x^b (g(b)-g(t))^\alpha d\left(\bigvee_x^t (f)\right)
\end{aligned} \tag{3.7}$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_a^x (g(t) - g(a))^{\alpha-1} g'(t) \bigvee_t^x (f)\, dt$$
$$+ \frac{1}{\Gamma(\alpha)} \int_x^b (g(b) - g(t))^{\alpha-1} g'(t) \bigvee_x^t (f)\, dt$$
$$\leq \frac{1}{\Gamma(\alpha+1)} \left[(g(x) - g(a))^{\alpha} \bigvee_a^x (f) + (g(b) - g(x))^{\alpha} \bigvee_x^b (f) \right],$$

for $x \in (a, b)$, which proves (3.5).

By the equality (2.15) we also have

$$\left| \frac{1}{2} \Gamma(\alpha+1) \left[\frac{I_{x-,g}^{\alpha} f(a)}{(g(x) - g(a))^{\alpha}} + \frac{I_{x+,g}^{\alpha} f(b)}{(g(b) - g(x))^{\alpha}} \right] - f(x) \right|$$
$$\leq \frac{1}{2(g(b) - g(x))^{\alpha}} \left| \int_x^b (g(b) - g(t))^{\alpha} df(t) \right|$$
$$+ \frac{1}{2(g(x) - g(a))^{\alpha}} \left| \int_a^x (g(t) - g(a))^{\alpha} df(t) \right|$$
$$\leq \frac{1}{2(g(b) - g(x))^{\alpha}} \int_x^b (g(b) - g(t))^{\alpha} d\left(\bigvee_x^t (f) \right)$$
$$+ \frac{1}{2(g(x) - g(a))^{\alpha}} \int_a^x (g(t) - g(a))^{\alpha} d\left(\bigvee_a^t (f) \right)$$
$$\leq \frac{\alpha}{2(g(b) - g(x))^{\alpha}} \int_x^b (g(b) - g(t))^{\alpha-1} g'(t) \bigvee_x^t (f)\, dt$$
$$+ \frac{\alpha}{2(g(x) - g(a))^{\alpha}} \int_a^x (g(t) - g(a))^{\alpha-1} g'(t) \bigvee_t^x (f)\, dt$$
$$\leq \frac{1}{2} \bigvee_x^b (f) + \frac{1}{2} \bigvee_a^x (f) = \frac{1}{2} \bigvee_a^b (f),$$

which proves (3.6). □

Remark 3. *The inequality (3.5) was obtained by a different technique in the earlier paper* [16].

Corollary 5. *With the assumptions of Theorem 2, we have*

$$\begin{aligned}
(3.8)\qquad & \left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(g(b)-g(a))^{\alpha}}\left[I^{\alpha}_{M_g(a,b)-,g}f(a)+I^{\alpha}_{M_g(a,b)+,g}f(b)\right]-f(M_g(a,b))\right| \\
& \leq \frac{2^{\alpha-1}\alpha}{(g(b)-g(a))^{\alpha}}\left[\int_{M_g(a,b)}^{b}(g(b)-g(t))^{\alpha-1}g'(t)\bigvee_{M_g(a,b)}^{t}(f)\,dt\right. \\
& \left.+\int_{a}^{M_g(a,b)}(g(t)-g(a))^{\alpha-1}g'(t)\bigvee_{t}^{M_g(a,b)}(f)\,dt\right] \\
& \leq \frac{1}{2}\bigvee_{a}^{b}(f).
\end{aligned}$$

The proof follows by either the inequality (3.5) or (3.6) by taking $x = x = M_g(a,b) = g^{-1}\left(\frac{g(a)+g(b)}{2}\right)$.

4. Some Examples

If we take $g(t) = t$, $t \in [a,b]$ in (3.2) and (3.5), then we recapture the inequalities from Theorem 1. From (3.3) we get for the classical Riemann-Liouville fractional integrals the following inequalities

$$\begin{aligned}
(4.1)\qquad & \left|\frac{f(a)+f(b)}{2}-\frac{1}{2}\Gamma(\alpha+1)\left[\frac{J^{\alpha}_{a+}f(x)}{(x-a)^{\alpha}}+\frac{J^{\alpha}_{b-}f(x)}{(b-x)^{\alpha}}\right]\right| \\
& \leq \frac{\alpha}{2}\left[\frac{1}{(x-a)^{\alpha}}\int_{a}^{x}(x-t)^{\alpha-1}\bigvee_{a}^{t}(f)\,dt+\frac{1}{(b-x)^{\alpha}}\int_{x}^{b}(t-x)^{\alpha-1}\bigvee_{t}^{b}(f)\,dt\right] \\
& \leq \frac{1}{2}\bigvee_{a}^{b}(f)
\end{aligned}$$

while from (3.6) we get

$$\begin{aligned}
(4.2)\qquad & \left|\frac{1}{2}\Gamma(\alpha+1)\left[\frac{J^{\alpha}_{x-}f(a)}{(x-a)^{\alpha}}+\frac{J^{\alpha}_{x+}f(b)}{(b-x)^{\alpha}}\right]-f(x)\right| \\
& \leq \frac{\alpha}{2}\left[\frac{1}{(x-a)^{\alpha}}\int_{a}^{x}(t-a)^{\alpha-1}\bigvee_{t}^{x}(f)\,dt+\frac{1}{(b-x)^{\alpha}}\int_{x}^{b}(b-t)^{\alpha-1}\bigvee_{x}^{t}(f)\,dt\right] \\
& \leq \frac{1}{2}\bigvee_{a}^{b}(f),
\end{aligned}$$

for any $x \in (a,b)$.

Consider the function $g(t)=\ln t$, $t\in[a,b]\subset(0,\infty)$, then by (3.3) we have for Hadamard fractional integrals

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2}\Gamma(\alpha+1)\left[\frac{H_{a+}^{\alpha}f(x)}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}}+\frac{H_{b-}^{\alpha}f(x)}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}\right]\right| \tag{4.3}$$
$$\leq\frac{\alpha}{2}\left[\frac{1}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}}\int_{a}^{x}\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{1}{t}\bigvee_{a}^{t}(f)\,dt\right.$$
$$\left.+\frac{1}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}\int_{x}^{b}\left(\ln\left(\frac{t}{x}\right)\right)^{\alpha-1}\frac{1}{t}\bigvee_{t}^{b}(f)\,dt\right]$$
$$\leq\frac{1}{2}\bigvee_{a}^{b}(f)$$

while from (3.6) we get

$$\left|\frac{1}{2}\Gamma(\alpha+1)\left[\frac{H_{x-}^{\alpha}f(a)}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}}+\frac{H_{x+}^{\alpha}f(b)}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}\right]-f(x)\right| \tag{4.4}$$
$$\leq\frac{\alpha}{2}\left[\frac{1}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}\int_{x}^{b}\left[\ln\left(\frac{b}{t}\right)\right]^{\alpha-1}\frac{1}{t}\bigvee_{x}^{t}(f)\,dt\right.$$
$$\left.+\frac{1}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}}\int_{a}^{x}\left[\ln\left(\frac{t}{a}\right)\right]^{\alpha-1}\frac{1}{t}\bigvee_{t}^{x}(f)\,dt\right]$$
$$\leq\frac{1}{2}\bigvee_{a}^{b}(f),$$

for any $x\in(a,b)$.

If we take the function $g(t)=-t^{-1}$, $t\in[a,b]\subset(0,\infty)$, then by (3.3) we have for Harmonic fractional integrals

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2}\Gamma(\alpha+1)\,x^{\alpha}\left[\frac{a^{\alpha}R_{a+}^{\alpha}f(x)}{(x-a)^{\alpha}}+\frac{b^{\alpha}R_{b-}^{\alpha}f(x)}{(b-x)^{\alpha}}\right]\right| \tag{4.5}$$
$$\leq\frac{x\alpha}{2}\left[\frac{a^{\alpha}}{(x-a)^{\alpha}}\int_{a}^{x}\frac{(x-t)^{\alpha-1}}{t^{\alpha+1}}\bigvee_{a}^{t}(f)\,dt+\frac{b^{\alpha}}{(b-x)^{\alpha}}\int_{x}^{b}\frac{(t-x)^{\alpha-1}}{t^{\alpha+1}}\bigvee_{t}^{b}(f)\,dt\right]$$
$$\leq\frac{1}{2}\bigvee_{a}^{b}(f)$$

while from (3.6) we get

$$\left|\frac{1}{2}\Gamma(\alpha+1)x^{\alpha}\left[\frac{a^{\alpha}R_{x-}^{\alpha}f(a)}{(x-a)^{\alpha}}+\frac{b^{\alpha}I_{x+}^{\alpha}f(b)}{(b-x)^{\alpha}}\right]-f(x)\right| \tag{4.6}$$
$$\leq \frac{\alpha x^{\alpha}}{2}\left[\frac{b}{(b-x)^{\alpha}}\int_x^b \frac{(b-t)^{\alpha-1}}{t^{\alpha+1}}\bigvee_x^t(f)\,dt+\frac{a}{(x-a)^{\alpha}}\int_a^x \frac{(t-a)^{\alpha-1}}{t^{\alpha+1}}\bigvee_t^x(f)\,dt\right]$$
$$\leq \frac{1}{2}\bigvee_a^b(f),$$

for any $x \in (a,b)$.

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