Received 19/06/17

CHEBYSHEV TYPE INEQUALITIES BY MEANS OF COPULAS

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Abstract. A copula is a function which joins (or "couples") a bivariate distribution function to its marginal (one-dimensional) distribution functions. In this paper, we obtain Chebyshev type inequalities by utilising copulas.

1. Introduction

A copula is a function which joins (or "couples") a bivariate distribution function to its marginal (one-dimensional) distribution functions. Mathematically defined, a copula C is a function $C : [0,1]^2 \to [0,1]$ with the following properties:

(C1) $C(u,0) = C(0,u) = 0$, $C(u,1) = u$, and $C(1,u) = u$ for all $u \in [0,1]$,

(C2) $C(u_1,v_1)-C(u_1,v_2)-C(u_2,v_1)+C(u_2,v_2) \geq 0$ for every $u_1,u_1,v_1,v_2 \in [0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

Property (C2) is referred to as the 2-increasing property (or moderate growth [2]). The 2-increasing property implies the following properties for any copula C:

(C4) C is non decreasing in each variable,

(C5) C satisfies the Lipschitz condition: for all $u_1,u_2,v_1,v_2 \in [0,1]$,

$$|C(u_2,v_2) - C(u_1,v_1)| \leq |u_2 - u_1| + |v_2 - v_1|.$$

For further reading on copulas, we refer the reader to the book by Nelsen [4].

While copulas join probability distributions, t-norms join membership functions of fuzzy sets, and hence combining probabilistic information and combining fuzzy information are not so different [3]. Mathematically defined, a t-norm T is a function $T : [0,1]^2 \to [0,1]$ with the properties [3]:

(T1) Commutativity: $T(x,y) = T(y,x)$ for all $x,y \in [0,1]$,

(T2) Associativity: $T(x,T(y,z)) = T(T(x,y),z)$ for all $x,y,z \in [0,1]$,

(T3) Monotonicity: $T(x,y) \leq T(x,z)$ for all $x,y,z \in [0,1]$ with $y \leq z$,

(T4) Boundary condition: $T(x,1) = T(1,x) = x$, $T(x,0) = T(0,x) = x$ for all $x \in [0,1]$,

A copula is a t-norm if and only it is associative, conversely, a t-norm is a copula if and only if it is 1-Lipschitz [2]. The three main continuous t-norms, namely the minimum operator ($M(x,y) = \min\{x,y\}$), the algebraic product ($P(x,y) = xy$), and the Lukasiewicz t-norm ($W(x,y) = \max\{x+y-1,0\}$), are copulas.

The first importance of these copulas are given by the following inequality: for any copula C, we have

$$W(u,v) \leq C(u,v) \leq M(u,v), \quad \text{for all } u,v \in [0,1]. \tag{1.1}$$

The above inequality is referred to as the Fréchet-Hoeffding bounds for copulas and provides a basic inequality for copulas.

Inequality (1.1) also holds in the contexts of probability theory and fuzzy probability calculus [2] and is referred to as the Bell inequalities. Further inequalities

RGMIA Res. Rep. Coll. 20 (2017), Art. 85, 15 pp

for copulas of Bell-type are given in [2]. Other inequalities for copulas are given in [6] in relation to a family of continuous functions L from $[0,\infty]\times[0,\infty]$ onto $[0,\infty]$ which are nondecreasing in each variable with $\lim_{x\to\infty} L(x,x)=\infty$.

Egozcue et al. in [7] establish Grüss-type bounds for covariances by assuming the dependence structures such as quadrant dependence and quadrant dependence in expectation. They utilise copulas to illustrate these dependent structures.

In the same spirit to that of [7], it is our aim here to establish inequalities by utilising copulas. Firstly, we note the connection between the 2-increasing property and the Chebyshevian mappings. A mapping $F:[a,b]^2\to\mathbb{R}$ is called *Chebyshevian* on $[a,b]^2$ if the following inequality is satisfied:

$$F(x,x)+F(y,y)\ge F(x,y)+F(y,x),\quad \text{for all } x,y\in[a,b].$$

Let C be a copula, $x,y\in[0,1]$, and set $u_1=u_2=x$ and $v_1=v_2=y$ in property $(C2)$ (2-increasing property) above to obtain

$$C(x,x)-C(x,y)-C(y,x)+C(y,y)\ge 0$$

or equivalently,

$$C(x,x)+C(y,y)\ge C(x,y)+C(y,x), \tag{1.2}$$

i.e. C is Chebyshevian on $[0,1]^2$.

In Dragomir and Crstici [1], the relationship between two synchronous functions and Chebyshevian mappings are established. Two functions $f,g:[a,b]\to\mathbb{R}$ are *synchronous* on $[a,b]$ if they have the same monotonicity, that is,

$$(f(x)-f(y))(g(x)-g(y))\ge 0,\quad \text{for all } x,y\in[a,b].$$

The relationship between the two notions are given in the following result.

Proposition 1 (Dragomir and Crstici [1])**.** *If f,g are synchronous on $[a,b]$ and $F:[a,b]^2\to\mathbb{R}$ where $F(x,y)=f(x)g(y)$, then F is Chebyshevian on $[a,b]^2$.*

Consequently, the following Chebyshev type inequalities can be stated:

Proposition 2 (Dragomir and Crstici [1])**.** *Let $p:[a,b]\to\mathbb{R}$ be integrable and nonnegative on $[a,b]$.*

(1) Let $F:[a,b]^2\to\mathbb{R}$. If F is Chebyshevian on $[a,b]^2$, then

$$\int_a^t p(x)\,dx\int_a^t p(x)F(x,x)\,dx\ge\int_a^t\int_a^t p(x)p(y)F(x,y)\,dx\,dy \tag{1.3}$$

for all $t\in[a,b]$.

(2) Let $f,g:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$. If f and g are synchronous on $[a,b]$, then we have Chebyshev's inequality

$$\int_a^t p(x)\,dx\int_a^t p(x)f(x)g(x)\,dx\ge\int_a^t p(x)f(x)\,dx\int_a^t p(x)g(x)\,dx \tag{1.4}$$

for all $t\in[a.b]$.

If $f,g:[a,b]\to[0,1]$ are synchronous, then by Proposition 1, the product copula given by

$$P(f(x),g(y))=f(x)g(y),\quad x,y\in[a,b]$$

is Chebyshevian on $[a,b]^2$, as a consequence of the 2-increasing property. If we define a function $F:[a,b]^2\to[0,1]$ by $F(x,y)=P(f(x),g(y))=f(x)g(y)$, and if $p:[a,b]\to\mathbb{R}$ is integrable and nonnegative, then Proposition 2 gives us

$$\int_a^t p(x)\,dx\int_a^t p(x)P(f(x),g(x))\,dx\ge\int_a^t\int_a^t p(x)p(y)P(f(x),g(y))\,dx\,dy.$$

Motivated by this observation, we aim to obtain other types of Chebyshev inequalities by utilising (the general definition of) copulas instead of the product copula as demonstrated above. Specifically, we provide inequalities for the dispersion of a function f defined on a measure space (Ω,Σ,μ), with respect to a positive weight ω on Ω with $\int_\Omega\omega(t)\,d\mu(t)=1$, that is,

$$\left(\int_\Omega\omega f^2\,d\mu-\left(\int_\Omega\omega f\,d\mu\right)^2\right)^{\frac{1}{2}}.$$

2. Chebyshev type inequalities

The 2-increasing property of copulas gives us the following result.

Proposition 3. *Let $C:[0,1]^2\to[0,1]$ be a copula and $p:[0,1]\to\mathbb{R}$ be an integrable function. Then,*

(1) C is Chebyshevian on $[0,1]^2$.

(2) If p is nonnegative, then

$$\int_0^t p(x)\,dx\int_0^t p(x)C(x,x)\,dx\ge\int_0^t\int_0^t p(x)p(y)C(x,y)\,dx\,dy.$$

Proof follows by (1.2) and Proposition 2 part (1).

Now we state a more general form of this inequality. We start with the following lemma.

Lemma 1. *Let $f,g:[a,b]\to[0,1]$ be two synchronous functions and C be a copula. Then, $F:[a,b]^2\to[0,1]$ defined by*

$$F(x,y):=C(f(x),g(y))$$

is Chebyshevian on $[a,b]^2$.

Proof. Since f and g are synchronous, they have the same monotonicity on $[a,b]$. Let A be the collection of subsets of $[a,b]$ where f and g are both nondecreasing. Suppose that $x,y\in A$. Without loss of generality, let $x\le y$, and set

$$u_1=f(x),\ u_2=f(y),\ v_1=g(x),\ v_2=g(y).$$

Thus, $u_1\le u_2$ and $v_1\le v_2$ since f and g are nondecreasing. Therefore, the 2-increasing property of C gives

$$\begin{aligned}0&\le C(u_1,v_1)-C(u_1,v_2)-C(u_2,v_1)+C(u_2,v_2)\\&=C(f(x),g(x))-C(f(x),g(y))-C(f(y),g(x))+C(f(y),g(y))\\&=F(x,x)-F(x,y)-F(y,x)+F(y,y).\end{aligned}$$

Suppose that $x,y\in[a,b]\setminus A$. Without loss of generality, let $x\le y$,

$$u_1=f(y),\ u_2=f(x),\ v_1=g(y),\ v_2=g(x).$$

Thus, $u_1 \le u_2$ and $v_1 \le v_2$ since f and g are decreasing. Therefore, the 2-increasing property of C gives

$$\begin{aligned} 0 &\le C(u_1,v_1) - C(u_1,v_2) - C(u_2,v_1) + C(u_2,v_2) \\ &= C(f(y),g(y)) - C(f(y),g(x)) - C(f(x),g(y)) + C(f(x),g(x)) \\ &= F(y,y) - F(y,x) - F(x,y) + F(x,x). \end{aligned}$$

We show that F is Chebyshevian in both cases. □

Lemma 1 and Proposition 2 part (1) give us:

Theorem 1. *Let C be a copula, $f, g : [a,b] \to [0,1]$ be two synchronous functions, $p : [a,b] \to \mathbb{R}$ be an integrable function. If p is nonnegative then*

$$\int_a^t p(x)\,dx \int_a^t p(x)C(f(x),g(x))\,dx \ge \int_a^t \int_a^t p(x)p(y)C(f(x),g(y))\,dx\,dy.$$

Example 1. In this example, we obtain some Chebyshev types inequalities by choosing some examples of copulas. Let $f, g : [a,b] \to [0,1]$ be two synchronous functions, and $p : [a,b] \to \mathbb{R}$ be a nonnegative integrable function. Theorem 1 and (1.1) gives us following inequalities:

$$\begin{aligned} &\int_a^t \int_a^t p(x)p(y)\max\{f(x)+g(y)-1,0\}\,dx\,dy \\ &\le \int_a^t p(x)\,dx \int_a^t p(x)\max\{f(x)+g(x)-1,0\}\,dx \\ &\le \int_a^t p(x)\,dx \int_a^t p(x)C(f(x),g(x))\,dx \\ &\le \int_a^t p(x)\,dx \int_a^t p(x)\min\{f(x),g(x)\}\,dx. \end{aligned} \tag{2.1}$$

The first inequality follows from Theorem 1 (by choosing the W copula) and the rest follows from the Fréchet Hoeffding bound (1.1). Similarly, we have

$$\begin{aligned} &\int_a^t \int_a^t p(x)p(y)\max\{f(x)+g(y)-1,0\}\,dx\,dy \\ &\le \int_a^t \int_a^t p(x)p(y)C(f(x),g(y))\,dx\,dy \\ &\le \int_a^t \int_a^t p(x)p(y)\min\{f(x),g(y)\}\,dx\,dy \\ &\le \int_a^t p(x)\,dx \int_a^t p(x)\min\{f(x),g(x)\}\,dx. \end{aligned} \tag{2.2}$$

The last inequality follows from Theorem 1 (by choosing the M copula) and the rest follows from the Fréchet Hoeffding bounds (1.1).

In what follows, we generalise Theorem 1 and Example 1.

Theorem 2. *Let (Ω, Σ, μ) be a measure space, $f : \Omega \to [0,1]$ be a measurable function, and C be a copula. Then, $F : \Omega^2 \to [0,1]$ defined by*

$$F(x,y) := C(f(x),f(y))$$

is Chebyshevian on Ω^2. *We also have for a non-negative integrable function* $p : \Omega \to \mathbb{R}$,

$$\int_\Omega p(x)\, d\mu(x) \int_\Omega p(x) C(f(x), f(x))\, d\mu(x)$$
$$\geq \int_\Omega \int_\Omega p(x)p(y) C(f(x), f(y))\, d\mu(x)\, d\mu(y).$$

Proof. The Chebyshevian property of F follows from the 2-increasing property of copulas. Therefore, we have

$$F(x,x) + F(y,y) \geq F(x,y) + F(y,x), \quad \text{for all } x, y \in \Omega,$$

or equivalently

$$C(f(x), f(x)) + C(f(y), f(y)) \geq C(f(x), f(y)) + C(f(y), f(x)).$$

Multiply both sides by $p(x)$ and $p(y)$ and take double integrals over Ω^2, we have

$$\int_\Omega p(x) d\mu(x) \int_\Omega p(x) C(f(x), f(x)) d\mu(x)$$
$$\geq \int_\Omega \int_\Omega p(x)p(y) C(f(x), f(y))\, d\mu(x) d\mu(y).$$

This completes the proof. □

Example 2. In this example, we obtain some Chebyshev types inequalities by choosing some examples of copulas. Let (Ω, Σ, μ) be a measure space, $f : \Omega \to [0,1]$ be a measurable function, and $p : \Omega \to \mathbb{R}$ be a nonnegative integrable function. We have the following inequalities:

$$\begin{aligned}
&\int_\Omega \int_\Omega p(x)p(y) \max\{f(x) + f(y) - 1, 0\}\, d\mu(x)\, d\mu(y) \\
&\leq \int_\Omega p(x)\, d\mu(x) \int_\Omega p(x) \max\{2f(x) - 1, 0\}\, d\mu(x) \\
&\leq \int_\Omega p(x)\, d\mu(x) \int_\Omega p(x) C(f(x), g(x))\, d\mu(x) \\
&\leq \int_\Omega p(x)\, d\mu(x) \int_\Omega p(x) \min\{f(x), g(x)\}\, d\mu(x),
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&\int_\Omega \int_\Omega p(x)p(y) \max\{f(x) + g(y) - 1, 0\}\, d\mu(x)\, d\mu(y) \\
&\leq \int_\Omega \int_\Omega p(x)p(y) C(f(x), g(y))\, d\mu(x)\, d\mu(y) \\
&\leq \int_\Omega \int_\Omega p(x)p(y) \min\{f(x), f(y)\}\, d\mu(x)\, d\mu(y) \\
&\leq \int_\Omega p(x)\, d\mu(x) \int_\Omega p(x) f(x)\, d\mu(x).
\end{aligned} \tag{2.4}$$

We also have the following result:

Theorem 3. *Let (Ω, Σ, μ) be a measure space, $f : \Omega \to [0,1]$ be a measurable function. Let ω be a positive weight on Ω with $\int_\Omega \omega(t)\, d\mu(t) = 1$. Let $C : [0,1]^2 \to [0,1]$ be a copula. We have the following inequalities:*

$$\begin{aligned} &\int_\Omega \omega C(f,f)\, d\mu + C\left(f\left(\int_\Omega \omega f\, d\mu\right), f\left(\int_\Omega \omega f\, d\mu\right)\right) \\ &\geq \int_\Omega \omega C\left(f, f\left(\int_\Omega \omega f\, d\mu\right)\right) d\mu + \int_\Omega \omega C\left(f\left(\int_\Omega \omega f\, d\mu\right), f\right) d\mu. \end{aligned} \tag{2.5}$$

Proof. The 2-increasing property of copulas gives us

$$C(x,x) + C(y,y) \geq C(x,y) + C(y,x)$$

for all $x, y \in [0,1]$. Take $x = f(t)$ and $y = \int_\Omega w(t) f(t)\, d\mu(t)$, we have

$$\begin{aligned} &C(f(t), f(t)) + C\left(f\left(\int_\Omega w(t)f(t)\, d\mu(t)\right), f\left(\int_\Omega w(t)f(t)\, d\mu(t)\right)\right) \\ &\geq C\left(f(t), f\left(\int_\Omega w(t)f(t)\, d\mu(t)\right)\right) + C\left(f\left(\int_\Omega w(t)f(t)\, d\mu(t)\right), f(t)\right). \end{aligned}$$

Multiply with $\omega(t) \geq 0$ and integrate over Ω gives the desired result. □

In the next section, we provide further inequalities of this type.

3. More inequalities

We denote the following

$$\begin{aligned} E_\omega(f) &:= \int_\Omega \omega f\, d\mu, \\ K_\omega(C; f, g) &:= \int_\Omega \int_\Omega \omega(x)\omega(y) C(f(x), g(y))\, d\mu(x)\, d\mu(y), \\ H_\omega(f) &:= \int_\Omega \omega \left| f - \int_\Omega \omega f\, d\mu \right| d\mu = \int_\Omega \omega \left| f - E_\omega(f) \right| d\mu, \end{aligned}$$

where $\omega : \Omega \to [0, \infty)$ is μ-integrable with $\int_\Omega \omega d\mu = 1$, $f, g : \Omega \to [0,1]$ are μ-measurable and $f, g \in L_\omega(\Omega)$, and $C : [0,1]^2 \to [0,1]$ is a copula.

We denote by $D_\omega(f)$ the dispersion of a function f defined on a measure space (Ω, Σ, μ), with respect to a positive weight ω on Ω with $\int_\Omega \omega(t)\, d\mu(t) = 1$, that is,

$$D_\omega(f) := \left(\int_\Omega \omega f^2\, d\mu - \left(\int_\Omega \omega f\, d\mu\right)^2\right)^{\frac{1}{2}}. \tag{3.1}$$

Theorem 4. *Let (Ω, Σ, μ) be a measure space, $f, g : \Omega \to [0,1]$ be measurable functions. Let ω be a positive weight on Ω with $\int_\Omega \omega(t)\, d\mu(t) = 1$. Let $C : [0,1]^2 \to [0,1]$ be a copula. We have the following inequalities:*

$$\begin{aligned} &\left| K_\omega(C; f, g) - C\left(E_\omega(f), E_\omega(g)\right) \right| \\ &\leq \int_\Omega \int_\Omega \omega(x)\omega(y) \left| C(f(x), g(y)) - C\left(E_\omega(f), E_\omega(g)\right) \right| d\mu(x)\, d\mu(y) \\ &\leq H_\omega(f) + H_\omega(g) \leq D_\omega(f) + D_\omega(g). \end{aligned}$$

Proof. Firstly, we have

$$
\begin{aligned}
&|K_\omega(C;f,g) - C\left(E_\omega(f), E_\omega(g)\right)| \\
&= \left| \int_\Omega \int_\Omega \omega(x)\omega(y)\big(C(f(x),g(y)) - C\left(E_\omega(f), E_\omega(g)\right)\big)\, d\mu(x)\, d\mu(y) \right| \\
&\leq \int_\Omega \int_\Omega \omega(x)\omega(y) \left|C(f(x),g(y)) - C\left(E_\omega(f), E_\omega(g)\right)\right| \, d\mu(x)\, d\mu(y).
\end{aligned}
$$

From the Lipschitz property of copulas, we have

$$
\begin{aligned}
&\left| C(f(x),g(y)) - C\left(\int_\Omega \omega f \, d\mu, \int_\Omega \omega g \, d\mu \right) \right| \\
&\leq \left| f(x) - \int_\Omega \omega f \, d\mu \right| + \left| g(y) - \int_\Omega \omega g \, d\mu \right|.
\end{aligned}
$$

Multiply with $\omega(x)\omega(y) \geq 0$ and integrate twice over Ω give:

$$
\begin{aligned}
&\int_\Omega \int_\Omega \omega(x)\omega(y) \left| C(f(x),g(y)) - C\left(\int_\Omega \omega f \, d\mu, \int_\Omega \omega g \, d\mu \right) \right| \, d\mu(x)\, d\mu(y) \\
&\leq \int_\Omega \omega \left| f - \int_\Omega \omega f \, d\mu \right| d\mu + \int_\Omega \omega \left| g - \int_\Omega \omega g \, d\mu \right| d\mu = H_\omega(f) + H_\omega(g).
\end{aligned}
$$

Finally, Schwarz's inequality gives:

$$
\begin{aligned}
&\left(\int_\Omega \omega \left| f - \int_\Omega \omega f \, d\mu \right| d\mu \right)^2 \\
\leq\ & \left(\int_\Omega \omega \left(f - \int_\Omega \omega f \, d\mu \right)^2 d\mu \right) \left(\int_\Omega \omega \, d\mu \right) \\
=\ & \int_\Omega \omega f^2 \, d\mu - 2 \int_\Omega \omega f \left(\int_\Omega \omega f \, d\mu \right) d\mu + \int_\Omega \omega \left(\int_\Omega \omega f \, d\mu \right)^2 d\mu \\
=\ & \int_\Omega \omega f^2 \, d\mu - 2 \left(\int_\Omega \omega f \, d\mu \right)^2 + \left(\int_\Omega \omega f \, d\mu \right)^2 \\
=\ & \int_\Omega \omega f^2 \, d\mu - \left(\int_\Omega \omega f \, d\mu \right)^2,
\end{aligned}
$$

that is,

$$
\int_\Omega \omega \left| f - \int_\Omega \omega f \, d\mu \right| d\mu \leq \left(\int_\Omega \omega f^2 \, d\mu - \left(\int_\Omega \omega f \, d\mu \right)^2 \right)^{\frac{1}{2}} = D_\omega(f).
$$

This completes the proof. □

Corollary 1. *Let (Ω, Σ, μ) be a measure space, $f, g : \Omega \to [0,1]$ be measurable functions. Let ω be a positive weight on Ω with $\int_\Omega \omega(t)\, d\mu(t) = 1$. Let $C : [0,1]^2 \to [0,1]$ be a copula. If f and g satisfy:*

$$
0 \leq m_f \leq f \leq M_f \leq 1, \quad \textit{and} \quad 0 \leq m_g \leq g \leq M_g \leq 1,
$$

then we have the inequalities:

$$
\begin{aligned}
|K_\omega(C;f,g) - C\left(E_\omega(f), E_\omega(g)\right)| &\leq D_\omega(f) + D_\omega(g) \\
&\leq \frac{1}{2}(M_f - m_f) + \frac{1}{2}(M_g - m_g) \leq 1.
\end{aligned}
$$

The proof follows from Theorem 4 and a Grüss type inequality:

$$D_\omega(f) \leq \frac{1}{2}(M-m) \leq \frac{1}{2},$$

for f with the property that $0 \leq m \leq f \leq M \leq 1$. We omit the details.

Recall the notation:

$$\begin{aligned} E_\omega(f) &:= \int_\Omega \omega f \, d\mu, \\ K_\omega(C; f, g) &:= \int_\Omega \int_\Omega \omega(x)\omega(y) C(f(x), g(y)) \, d\mu(x) \, d\mu(y), \end{aligned}$$

and introduce the following notation:

$$\begin{aligned} K_\omega(C; f) &:= \int_\Omega \int_\Omega \omega(x)\omega(y) C(f(x), f(y)) \, d\mu(x) \, d\mu(y) \\ L_\omega(C; f, g) &:= \int_\Omega \omega C\left(f, \int_\Omega \omega g \, d\mu\right) d\mu \\ L_\omega(C, f) &:= \int_\Omega \omega C\left(f, \int_\Omega \omega f \, d\mu\right) d\mu. \end{aligned}$$

Theorem 5. *Let $\omega : \Omega \to [0, \infty)$ be μ-integrable with $\int_\Omega \omega d\mu = 1$. Let $f, g : \Omega \to [0,1]$ be μ-measurable and $f, g \in L_\omega(\Omega)$. If $C : [0,1]^2 \to [0,1]$ is a copula, then*

$$\max\{E_\omega(f) + E_\omega(g) - 1, 0\} \leq K_\omega(C; f, g) \leq \min\{E_\omega(f), E_\omega(g)\}. \tag{3.2}$$

In particular, we have

$$\max\{2E_\omega(f) - 1, 0\} \leq K_\omega(C; f) \leq E_\omega(f). \tag{3.3}$$

We also have

$$\begin{aligned} \max\{E_\omega(f) + E_\omega(g) - 1, 0\} &\leq \int_\Omega \omega \max\{f + E_\omega(g) - 1, 0\} \, d\mu \\ &\leq L_\omega(C; f, g) \\ &\leq \int_\Omega \omega \min\{f, E_\omega(g)\} \, d\mu \\ &\leq \min\{E_\omega(f), E_\omega(g)\}. \end{aligned} \tag{3.4}$$

In particular,

$$\begin{aligned} \max\{2E_\omega(f) - 1, 0\} &\leq \int_\Omega \omega \max\{f + E_\omega(f) - 1, 0\} \\ &\leq L_\omega(C, f) \\ &\leq \int_\Omega \omega \min\{f, E_\omega(f)\} \, d\mu \leq E_\omega(f). \end{aligned} \tag{3.5}$$

Proof. We know that for any μ-ω-integrable functions k and l, we have

$$\int_X \omega \min\{k, l\} d\mu \leq \min\left\{\int_X \omega k \, d\mu, \int_X \omega l \, d\mu\right\}, \tag{3.6}$$

and

$$\int_X \omega \max\{k, l\} d\mu \geq \max\left\{\int_X \omega k \, d\mu, \int_X \omega l \, d\mu\right\}. \tag{3.7}$$

Using the Fréchet-Hoeffding bounds (1.1), we obtain

$$\max\{f(x)+g(y)-1,0\} \le C(f(x),g(y)) \le \min\{f(x),g(y)\}, \tag{3.8}$$

for all $x, y \in \Omega$. If we multiply (3.8) by $w(x)w(y) \ge 0$ and integrate twice over Ω, then we get

$$\begin{aligned}
&\int_\Omega \int_\Omega \omega(x)\omega(y) \max\{f(x)+g(y)-1,0\}\, d\mu(x)\, d\mu(y) \\
&\le \int_\Omega \int_\Omega \omega(x)\omega(y) C(f(x),g(y))\, d\mu(x)\, d\mu(y) \\
&\le \int_\Omega \int_\Omega \omega(x)\omega(y) \min\{f(x),g(y)\}\, d\mu(x)\, d\mu(y).
\end{aligned} \tag{3.9}$$

By (3.6) and (3.7), we get

$$\begin{aligned}
&\int_\Omega \int_\Omega \omega(x)\omega(y) \min\{f(x),g(y)\}\, d\mu(x)\, d\mu(y) \\
&\le \min\left\{\int_\Omega \omega f\, d\mu, \int_\Omega \omega g\, d\mu\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\max\left\{\int_\Omega \omega f\, d\mu + \int_\Omega \omega g\, d\mu - 1, 0\right\} \\
&\le \int_\Omega \int_\Omega \omega(x)\omega(y) \max\{f(x)+g(y)-1,0\}\, d\mu(x)\, d\mu(y).
\end{aligned}$$

This proves (3.2). We obtain (3.3) by setting $f \equiv g$ in (3.2).

From (1.1), we also have

$$\max\left\{f + \int_\Omega \omega g\, d\mu - 1, 0\right\} \le C\left(f, \int_\Omega \omega g\, d\mu\right) \le \min\left\{f, \int_\Omega \omega g\, d\mu\right\}. \tag{3.10}$$

If we multiply (3.10) by $w \ge 0$ and integrate over Ω, then we get

$$\begin{aligned}
\int_\Omega \omega \max\left\{f + \int_\Omega \omega g\, d\mu - 1, 0\right\} d\mu \quad &\le \quad \int_\Omega \omega C\left(f, \int_\Omega \omega g\, d\mu\right) d\mu \\
&\le \quad \int_\Omega \omega \min\{f, \int_\Omega \omega g\, d\mu\}.
\end{aligned} \tag{3.11}$$

Since

$$\int_\Omega \omega \min\left\{f, \int_\Omega \omega g\, d\mu\right\} \le \min\left\{\int_\Omega \omega f\, d\mu, \int_\Omega \omega g\, d\mu\right\} \tag{3.12}$$

and

$$\max\left\{\int_\Omega \omega f\, d\mu + \int_\Omega \omega g\, d\mu - 1, 0\right\} \le \int_\Omega \omega \max\left\{f + \int_\Omega \omega g\, d\mu - 1, 0\right\} d\mu. \tag{3.13}$$

By (3.11), (3.12), and (3.13), we get (3.4). Finally, we obtain (3.5) by setting $f \equiv g$ in (3.4). $\square$

Lemma 2. *If $C : [0,1]^2 \to [0,1]$ is a copula, then we have*

$$\begin{aligned} 0 \le \frac{1}{2}|u-v| &\le \frac{1}{2}(u+v) - C(u,v) \\ &\le \frac{1}{2}|u-v| + \frac{1}{2} - \max\left\{\left|\frac{1}{2}-u\right|, \left|\frac{1}{2}-v\right|\right\} \\ &\le \frac{1}{2}|u-v| + \frac{1}{2}, \end{aligned} \tag{3.14}$$

for any $u, v \in [0,1]$.

Proof. Using the Fréchet-Hoeffding bounds (1.1) and the fact that

$$\min\{a,b\} = \frac{1}{2}(a+b-|a+b|), \quad \max\{a,b\} = \frac{1}{2}(a+b+|a-b|),$$

thus we have

$$\frac{1}{2}(u+v-1+|u+v-1|) \le C(u,v) \le \frac{1}{2}(u+v-|u-v|),$$

for any $u, v \in [0,1]$. This inequality is equivalent to

$$\frac{1}{2}|u-v| \le \frac{1}{2}(u+v) - C(u,v) \le \frac{1}{2}(1-|u+v-1|). \tag{3.15}$$

Applying the reverse triangle inequality, we have

$$|u+v-1| = |u-v+2v-1| = |u-v-(1-2v)| \ge |1-2v| - |u-v|,$$

for any $u, v \in [0,1]$. Similarly,

$$|u+v-1| \ge |1-2u| - |u-v|,$$

for any $u, v \in [0,1]$. Therefore,

$$-|u+v-1| \le |u-v| - |1-2v|,$$

and

$$-|u+v-1| \le |u-v| - |1-2u|,$$

giving that

$$-|u+v-1| \le |u-v| - \max\{|1-2u|, |1-2v|\},$$

for all $u, v \in [0,1]$. From (3.15), we then obtain

$$\frac{1}{2}|u-v| \le \frac{1}{2}(u+v) - C(u,v) \le \frac{1}{2} + \frac{1}{2}|u-v| - \max\left\{\left|\frac{1}{2}-u\right|, \left|\frac{1}{2}-v\right|\right\} \tag{3.16}$$

for all $u, v \in [0,1]$. □

Consider the quantities

$$I_\omega(f,g) := \int_\Omega \int_\Omega \omega(x)\omega(y)|f(x)-g(y)|\, d\mu(x)\, d\mu(y),$$

and

$$I_\omega(f) := \int_\Omega \int_\Omega \omega(x)\omega(y)|f(x)-f(y)|\, d\mu(x)\, d\mu(y) = I_\omega(f,f).$$

By the properties of modulus, we have

$$I_\omega(f,g) \ge \int_\Omega \omega \left| f - \int_\Omega \omega g \, d\mu \right| d\mu =: H_\omega(f,g),$$

and

$$I_\omega(f) \geq \int_\Omega \omega \left| f - \int_\Omega \omega f \, d\mu \right| d\mu = H_\omega(f).$$

By Schwarz's inequality, we also have

$$\begin{aligned} I_\omega(f,g) &\leq \left(\int_\Omega \int_\Omega \omega(x)\omega(y)(f(x)-g(x))^2 \, d\mu(x) \, d\mu(y) \right)^{\frac{1}{2}} \\ &= \left(\int_\Omega \omega f^2 \, d\mu - 2 \int_\Omega \omega f \, d\mu \int_\Omega \omega g \, d\mu + \int_\Omega \omega g^2 \, d\mu \right)^{\frac{1}{2}}, \end{aligned}$$

and

$$I_\omega(f) \leq \sqrt{2} \left(\int_\Omega \omega f^2 \, d\mu - \left(\int_\Omega \omega f \, d\mu \right)^2 \right)^{\frac{1}{2}} = \sqrt{2} D_\omega(f).$$

We have the following result:

Theorem 6. *Let $\omega : \Omega \to [0,\infty)$ be μ-integrable with $\int_\Omega \omega \, d\mu = 1$. Let $f, g : \Omega \to [0,1]$ be μ-measurable and such that $f, g \in L_\omega(\Omega)$. If $C : [0,1]^2 \to [0,1]$ is a copula, then (with the notation in Theorem 5), we have*

$$\begin{aligned} \frac{1}{2} I_\omega(f,g) &\leq \frac{1}{2}\left(E_\omega(f) + E_\omega(g)\right) - K_\omega(C; f, g) \\ &\leq \frac{1}{2} I_\omega(f,g) + \frac{1}{2} - \max\left\{ E_\omega\left(\left| \frac{1}{2} - f \right| \right), E_\omega\left(\left| \frac{1}{2} - g \right| \right) \right\} \\ &\leq \frac{1}{2} I_\omega(f,g) + \frac{1}{2}. \end{aligned} \tag{3.17}$$

In particular, we have

$$\begin{aligned} \frac{1}{2} I_\omega(f) &\leq E_\omega(f) - K_\omega(C; f) \\ &\leq \frac{1}{2} I_\omega(f) + \frac{1}{2} - E_\omega\left(\left| \frac{1}{2} - f \right| \right) \leq \frac{1}{2} I_\omega(f) + \frac{1}{2}. \end{aligned} \tag{3.18}$$

We also have

$$\begin{aligned} \frac{1}{2} H_\omega(f,g) &\leq \frac{1}{2}\left(E_\omega(f) + E_\omega(g)\right) - L_\omega(C; f, g) \\ &\leq \frac{1}{2} H_\omega(f,g) + \frac{1}{2} - \max\left\{ E_\omega\left(\left| \frac{1}{2} - f \right| \right), \left| \frac{1}{2} - E_\omega(g) \right| \right\} \\ &\leq \frac{1}{2} H_\omega(f,g) + \frac{1}{2}. \end{aligned} \tag{3.19}$$

In particular, we have

$$\begin{aligned} \frac{1}{2} H_\omega(f) &\leq E_\omega(f) - L_\omega(C; f) \\ &\leq \frac{1}{2} H_\omega(f) + \frac{1}{2} - E_\omega\left(\left| \frac{1}{2} - f \right| \right) \\ &\leq \frac{1}{2} H_\omega(f) + \frac{1}{2}. \end{aligned} \tag{3.20}$$

Proof. From Lemma 2 we have

$$\begin{aligned}\frac{1}{2}|f(x)-g(y)| &\leq \frac{1}{2}(f(x)+g(y))-C(f(x),g(y)) \\ &\leq \frac{1}{2}|f(x)-g(y)|+\frac{1}{2}-\max\left\{\left|\frac{1}{2}-f(x)\right|,\left|\frac{1}{2}-g(y)\right|\right\} \qquad (3.21)\end{aligned}$$

for any $x,y\in\Omega$. We multiply (3.22) by $\omega(x)\omega(y)\geq 0$ and integrate to get

$$\begin{aligned}\frac{1}{2}I_\omega(f,g) &= \frac{1}{2}\int_\Omega\int_\Omega \omega(x)\omega(y)|f(x)-g(y)|\,d\mu(x)\,d\mu(y) \\ &\leq \frac{1}{2}\left(\int_\Omega \omega f\,d\mu+\int_\Omega \omega g\,d\mu\right)-\int_\Omega\int_\Omega \omega(x)\omega(y)C(f(x),g(y))\,d\mu(x)\,d\mu(y) \\ &\leq \frac{1}{2}\int_\Omega\int_\Omega \omega(x)\omega(y)|f(x)-g(y)|\,d\mu(x)\,d\mu(y)+\frac{1}{2} \\ &\quad -\int_\Omega\int_\Omega \omega(x)\omega(y)\max\left\{\left|\frac{1}{2}-f(x)\right|,\left|\frac{1}{2}-g(y)\right|\right\}\,d\mu(x)\,d\mu(y) \\ &\leq \frac{1}{2}I_\omega(f,g)+\frac{1}{2}-\max\left\{\int_\Omega \omega\left|\frac{1}{2}-f\right|\,d\mu,\int_\Omega \omega\ \left|\frac{1}{2}-g\right|\,d\mu\right\}.\end{aligned}$$

Again, from Lemma 2 we have

$$\begin{aligned}\frac{1}{2}\left|f-\int_\Omega \omega g\,d\mu\right| &\leq \frac{1}{2}\left(f+\int_\Omega \omega g\,d\mu\right)-C\left(f,\int_\Omega \omega g\,d\mu\right) \\ &\leq \frac{1}{2}\left|f-\int_\Omega \omega g\,d\mu\right|+\frac{1}{2}-\max\left\{\left|\frac{1}{2}-f\right|,\left|\frac{1}{2}-\int_\Omega \omega g\,d\mu\right|\right\}. \qquad (3.22)\end{aligned}$$

If we multiply (3.22) by $\omega\geq 0$ and integrate, then we get

$$\begin{aligned}&\frac{1}{2}H_\omega(f,g) \\ &= \frac{1}{2}\int_\Omega \omega\left|f-\int_\Omega \omega g\,d\mu\right|\,d\mu \\ &\leq \frac{1}{2}\left(\int_\Omega \omega f\,d\mu+\int_\Omega \omega g\,d\mu\right)-L_\omega(C;f,g) \\ &\leq \frac{1}{2}\int_\Omega \omega\left|f-\int_\Omega \omega g\,d\mu\right|\,d\mu+\frac{1}{2}-\int_\Omega \omega\max\left\{\left|\frac{1}{2}-f\right|,\left|\frac{1}{2}-\int_\Omega \omega g\,d\mu\right|\right\}\,d\mu \\ &\leq \frac{1}{2}H_\omega(f,g)+\frac{1}{2}-\max\left\{\int_\Omega \omega\left|\frac{1}{2}-f\right|\,d\mu,\left|\frac{1}{2}-\int_\Omega \omega g\,d\mu\right|\right\} \\ &\leq \frac{1}{2}H_\omega(f,g)+\frac{1}{2}.\end{aligned}$$

We obtain the particular cases by setting $f\equiv g$. $\square$

Remark 1. *We denote the following quantities:*

$$\begin{aligned}
E_\omega &:= \int_0^1 t\omega(t)\,dt,\\
I_\omega &:= \int_0^1\int_0^1 \omega(x)\omega(y)|x-y|\,dx\,dy,\\
H_\omega &:= \int_0^1 \omega(t)\left|t-\int_0^1 t\omega(t)\,dt\right|\,dt = \int_0^1 \omega(t)|t-E_\omega|\,dt,\\
K_\omega(C) &:= \int_0^1\int_0^1 \omega(x)\omega(y)C(x,y)\,dx\,dy\\
L_\omega(C) &:= \int_0^1 \omega(t)C\left(t,\int_\Omega t\omega(t)\,dt\right)\,dt.
\end{aligned}$$

Some particular instances of interest:

(a) Let $\Omega=[0,1]$, $\omega:[0,1]\to[0,\infty)$, $\int_0^1\omega(t)\,dt=1$, $f(t)=g(t)=t$ $(t\in[0,1])$. *Then by* (3.3) *we get*

$$\begin{aligned}
&\max\left\{2\int_0^1 t\omega(t)\,dt-1,0\right\}\\
&\le \int_0^1\int_0^1 \omega(x)\omega(y)C(x,y)\,dx\,dy =: K_\omega(C)\le \int_0^1 t\omega(t)\,dt.
\end{aligned}$$

that is,

$$\max\{2E_\omega-1,0\}\le K_\omega(C)\le E_\omega.$$

By Theorem 6, we have

$$\begin{aligned}
\frac12 I_\omega &\le E_\omega-K_\omega(C)\\
&\le \frac12 I_\omega+\frac12-\int_0^1\omega(t)\left|\frac12-t\right|\,dt\le \frac12 I_\omega+\frac12
\end{aligned}$$

and

$$\begin{aligned}
\frac12 H_\omega &\le E_\omega-L_\omega(C)\\
&\le \frac12 H_\omega+\frac12-\int_0^1\omega(t)\left|\frac12-t\right|\,dt\le \frac12 H_\omega+\frac12
\end{aligned}$$

(b) Take $\Omega=[0,1]$, $\omega(t)=1$ $(t\in[0,1])$ *to get*

$$\begin{aligned}
\max\left\{\int_0^1 f(t)\,dt+\int_0^1 g(t)\,dt-1,0\right\} &\le \int_0^1\int_0^1 C(f(x),g(y))\,dx\,dy\\
&\le \min\left\{\int_0^1 f(t)\,dt,\int_0^1 g(t)\,dt\right\}.
\end{aligned}$$

When $f\equiv g$, *we get*

$$\begin{aligned}
\max\left\{2\int_0^1 f(t)\,dt-1,0\right\} &\le \int_0^1\int_0^1 C(f(x),f(y))\,dx\,dy\\
&\le \int_0^1 f(t)\,dt.
\end{aligned}$$

By Theorem 6, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\int_\Omega |f(x)-g(y)|\,d\mu(x)\,d\mu(y)\\
&\le \frac{1}{2}\left(\int_\Omega f\,d\mu+\int_\Omega g\,d\mu\right)-\int_\Omega\int_\Omega C(f(x),g(y))\,d\mu(x)\,d\mu(y)\\
&\le \frac{1}{2}\int_\Omega\int_\Omega |f(x)-g(y)|\,d\mu(x)\,d\mu(y)+\frac{1}{2}-\max\left\{\int_\Omega\left|\frac{1}{2}-f\right|\,d\mu,\int_\Omega\left|\frac{1}{2}-g\right|\,d\mu\right\}\\
&\le \frac{1}{2}\int_\Omega\int_\Omega |f(x)-g(y)|\,d\mu(x)\,d\mu(y)+\frac{1}{2}.
\end{aligned}$$

When $f\equiv g$, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\int_\Omega |f(x)-f(y)|\,d\mu(x)\,d\mu(y)\\
&\le \int_\Omega f\,d\mu-\int_\Omega\int_\Omega C(f(x),f(y))\,d\mu(x)\,d\mu(y)\\
&\le \frac{1}{2}\int_\Omega\int_\Omega |f(x)-f(y)|\,d\mu(x)\,d\mu(y)+\frac{1}{2}-\int_\Omega\left|\frac{1}{2}-f\right|\,d\mu\\
&\le \frac{1}{2}\int_\Omega\int_\Omega |f(x)-f(y)|\,d\mu(x)\,d\mu(y)+\frac{1}{2}.
\end{aligned}$$

We also have

$$\begin{aligned}
&\int_\Omega\left|f-\int_\Omega g\,d\mu\right|\,d\mu\\
&\le \frac{1}{2}\left(\int_\Omega f\,d\mu+\int_\Omega g\,d\mu\right)-\int_\Omega C\left(f,\int_\Omega g\,d\mu\right)\,d\mu\\
&\le \int_\Omega\left|f-\int_\Omega g\,d\mu\right|\,d\mu+\frac{1}{2}-\max\left\{\int_\Omega\left|\frac{1}{2}-f\right|\,d\mu,\left|\frac{1}{2}-\int_\Omega g\,d\mu\right|\right\}\\
&\le \int_\Omega\left|f-\int_\Omega g\,d\mu\right|\,d\mu+\frac{1}{2},
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega\left|f-\int_\Omega f\,d\mu\right|\,d\mu\\
&\le \int_\Omega f\,d\mu-\int_\Omega C\left(f,\int_\Omega f\,d\mu\right)\,d\mu\\
&\le \int_\Omega\left|f-\int_\Omega f\,d\mu\right|\,d\mu+\frac{1}{2}-\int_\Omega\left|\frac{1}{2}-f\right|\,d\mu\\
&\le \int_\Omega\left|f-\int_\Omega f\,d\mu\right|\,d\mu+\frac{1}{2}.
\end{aligned}$$

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