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PROOFS OF CERTAIN CONJECTURES OF VUKŠIĆ CONCERNING THE INEQUALITIES FOR MEANS

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Abstract. By using the asymptotic expansion method, Vukšić conjectured inequalities between Seiffert means and convex combinations of other means. In this paper, we prove certain conjectures given by Vukšić.

1. Introduction

For $x, y > 0$ with $x \neq y$, the first and second Seiffert means $P(x, y)$ and $T(x, y)$ are defined in [13] and [14], respectively by

$$P(x,y) = \frac{x-y}{2\arcsin\frac{x-y}{x+y}} \quad \text{and} \quad T(x,y) = \frac{x-y}{2\arctan\frac{x-y}{x+y}}.$$

In what follows we will assume that the numbers x and y are positive and unequal. Let

$$H = \frac{2xy}{x+y}, \quad G = \sqrt{xy}, \quad L = \frac{x-y}{\ln x - \ln y}, \quad A = \frac{x+y}{2}, \quad Q = \sqrt{\frac{x^2+y^2}{2}}, \quad N = \frac{x^2+y^2}{x+y}$$

be the harmonic, geometric, logarithmic, arithmetic, root-square, and contraharmonic means of x and y, respectively. It is known (see [15]) that

$$H < G < L < P < A < T < Q < N.$$

There is a large number of papers studying inequalities between Seiffert means and convex combinations of other means [4–6, 11, 12, 15, 16]. For example, Chu et al. [4] established that the double inequality

$$\mu A + (1-\mu)H < P < \nu A + (1-\nu)H$$

holds if and only if $\mu \leq 2/\pi$ and $\nu \geq 5/6$. Liu and Meng [12] proved that the double inequality

$$(1-\mu)G + \mu N < P < (1-\nu)G + \nu N$$

holds if and only if $\mu \leq 2/9$ and $\nu \geq 1/\pi$. Chu et al. [5] proved that the double inequality

$$\mu Q + (1-\mu)A < T < \nu Q + (1-\nu)A \tag{1.1}$$

holds if and only if $\mu \leq (4-\pi)/\big(\pi(\sqrt{2}-1)\big)$ and $\nu \geq 2/3$. The inequality (1.1) was also proved by Witkowski [16].

Recently, Vukšić [15], by using the asymptotic expansion method, gave a systematic study of inequalities of the form

$$(1-\mu)M_1 + \mu M_3 < M_2 < (1-\nu)M_1 + \nu M_3,$$

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where M_j are chosen from the class of elementary means given above. For example, Vukšić [15, Theorem 3.5, (3.15)] proved the double inequality

$$(1-\mu)H+\mu N<T<(1-\nu)H+\nu N$$

holds if and only if $\mu \le 2/\pi$ and $\nu \ge 1/3$. See [8–10] for more details about comparison of means using asymptotic methods. Also Vukšić [15] has conjectured certain inequalities related to the first and second Seiffert means $P(x,y)$ and $T(x,y)$.

Conjecture 1.1 ([15, Conjecture 3.4]). *The following inequalities hold:*

$$\frac{\pi-2}{\pi}G+\frac{2}{\pi}A<P<\frac{1}{3}G+\frac{2}{3}A, \tag{1.2}$$

$$\frac{2}{3}G+\frac{1}{3}Q<P<\frac{\pi-\sqrt{2}}{\pi}G+\frac{\sqrt{2}}{\pi}Q, \tag{1.3}$$

$$\frac{3}{4}P+\frac{1}{4}Q<A<\frac{(\sqrt{2}-1)\pi}{\sqrt{2}\pi-2}P+\frac{\pi-2}{\sqrt{2}\pi-2}Q, \tag{1.4}$$

$$\frac{4}{5}L+\frac{1}{5}Q<P<\frac{\pi-\sqrt{2}}{\pi}L+\frac{\sqrt{2}}{\pi}Q, \tag{1.5}$$

$$\frac{7}{8}L+\frac{1}{8}N<P<\frac{\pi-1}{\pi}L+\frac{1}{\pi}N. \tag{1.6}$$

Conjecture 1.2 ([15, Conjecture 3.6]). *The following inequalities hold:*

$$\frac{1}{4}H+\frac{3}{4}T<A<\frac{4-\pi}{4}H+\frac{\pi}{4}T, \tag{1.7}$$

$$\frac{1}{9}H+\frac{8}{9}Q<T<\frac{\pi-2\sqrt{2}}{\pi}H+\frac{2\sqrt{2}}{\pi}Q, \tag{1.8}$$

$$\frac{\pi-2}{\pi}H+\frac{2}{\pi}N<T<\frac{1}{3}H+\frac{2}{3}N, \tag{1.9}$$

$$\frac{1}{6}G+\frac{5}{6}Q<T<\frac{\pi-2\sqrt{2}}{\pi}G+\frac{2\sqrt{2}}{\pi}Q, \tag{1.10}$$

$$\frac{1}{2}L+\frac{1}{2}T<A<\frac{4-\pi}{4}L+\frac{\pi}{4}T, \tag{1.11}$$

$$\frac{1}{5}L+\frac{4}{5}Q<T<\frac{\pi-2\sqrt{2}}{\pi}L+\frac{2\sqrt{2}}{\pi}Q, \tag{1.12}$$

$$\frac{2\pi-4}{\pi}A+\frac{4-\pi}{\pi}N<T<\frac{2}{3}A+\frac{1}{3}N, \tag{1.13}$$

$$\frac{(2-\sqrt{2})\pi}{2\pi-4}T+\frac{\sqrt{2}\pi-4}{2\pi-4}N<Q<\frac{3}{4}T+\frac{1}{4}N. \tag{1.14}$$

The typos of (1.12) *and* (1.13) *have been corrected.*

The aim of this paper is to offer a proof of these inequalities.

Remark 1.1. *Let $(x-y)/(x+y)=z$, and suppose $x>y$. Then $z\in(0,1)$, and the following identities hold true:*

$$\frac{P(x,y)}{A(x,y)}=\frac{z}{\arcsin z},\quad \frac{T(x,y)}{A(x,y)}=\frac{z}{\arctan z},\quad \frac{H(x,y)}{A(x,y)}=1-z^2,\quad \frac{G(x,y)}{A(x,y)}=\sqrt{1-z^2},$$
$$\frac{L(x,y)}{A(x,y)}=\frac{2z}{\ln\frac{1+z}{1-z}},\quad \frac{Q(x,y)}{A(x,y)}=\sqrt{1+z^2},\quad \frac{N(x,y)}{A(x,y)}=1+z^2.$$

The following elementary power series expansions are useful in our investigation.

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},\qquad |x|<\infty, \tag{1.15}$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!},\qquad |x|<\infty, \tag{1.16}$$

$$\tan x=\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1},\qquad |x|<\frac{\pi}{2}, \tag{1.17}$$

$$\cot x=\frac{1}{x}-\sum_{n=1}^{\infty}\frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1},\qquad 0<|x|<\pi, \tag{1.18}$$

where B_n $(n=0,1,2,\ldots)$ are Bernoulli numbers, defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}.$$

The following lemma is also needed in the sequel.

Lemma 1.1 ([2,3])**.** *Let $-\infty<a<b<\infty$, and let $f,\ g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b). Let $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)}\quad \text{and}\quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2. Proof of Conjecture 1.1

The inequalities (1.2) have been proved in [16]. We here provide an alternative proof.

Theorem 2.1. *The following double inequality hold true:*

$$\frac{\pi-2}{\pi}G+\frac{2}{\pi}A<P<\frac{1}{3}G+\frac{2}{3}A. \tag{2.1}$$

Proof. By Remark 1.1, (2.1) may be rewritten as

$$\frac{2}{\pi}<\frac{P-G}{A-G}=\frac{\frac{P}{A}-\frac{G}{A}}{1-\frac{G}{A}}=\frac{\frac{z}{\arcsin z}-\sqrt{1-z^2}}{1-\sqrt{1-z^2}}<\frac{2}{3},\qquad 0<z<1. \tag{2.2}$$

By an elementary change of variable $z=\sin x$ $(0<x<\pi/2)$, (2.2) becomes

$$\frac{2}{\pi}<\frac{\frac{\sin x}{x}-\cos x}{1-\cos x}<\frac{2}{3},\qquad 0<x<\frac{\pi}{2}. \tag{2.3}$$

For $0 \le x \le \pi/2$, let

$$f_1(x) = \begin{cases} \dfrac{\sin x}{x} - \cos x, & x \neq 0 \\ 0, & x = 0, \end{cases} \qquad f_2(x) = 1 - \cos x,$$

and let

$$f(x) = \frac{f_1(x)}{f_2(x)} = \frac{\frac{\sin x}{x} - \cos x}{1 - \cos x}, \qquad 0 < x < \frac{\pi}{2}. \tag{2.4}$$

Then,

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\frac{\cos x}{x} - \frac{\sin x}{x^2} + \sin x}{\sin x} = \frac{x \cot x - 1 + x^2}{x^2} =: f_3(x).$$

Using (1.18), we find

$$f_3(x) = \frac{2}{3} - \sum_{n=2}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} x^{2n-2}.$$

Differentiation yields

$$f_3'(x) = -\sum_{n=2}^{\infty} \frac{(2n-2)2^{2n}|B_{2n}|}{(2n)!} x^{2n-3} < 0.$$

Therefore, the functions $f_3(x)$ and $f_1'(x)/f_2'(x)$ are strictly decreasing on $(0, \pi/2)$. By Lemma 1.1, the function

$$f(x) = \frac{f_1(x)}{f_2(x)} = \frac{f_1(x) - f_1(0)}{f_2(x) - f_2(0)}$$

is strictly decreasing on $(0, \pi/2)$, and we have

$$\frac{2}{\pi} = f\left(\frac{\pi}{2}\right) < f(x) = \frac{\frac{\sin x}{x} - \cos x}{1 - \cos x} < \lim_{t \to 0} f(t) = \frac{2}{3}$$

for $0 < x < \pi/2$. The proof is complete. □

Remark 2.1. *Let $f(x)$ be given in* (2.4). *By the monotonicity property of $f(x)$, we here provide a proof of* (1.1).

By Remark 1.1, (1.1) *may be rewritten as*

$$\mu < \frac{\frac{z}{\arctan z} - 1}{\sqrt{1+z^2} - 1} < \nu, \qquad 0 < z < 1.$$

By an elementary change of variable $z = \tan x$ $(0 < x < \pi/4)$, we find

$$\mu < \frac{\frac{\tan x}{x} - 1}{\sec x - 1} = \frac{\frac{\sin x}{x} - \cos x}{1 - \cos x} = f(x) < \nu, \qquad 0 < x < \frac{\pi}{4}.$$

Since $f(x)$ is strictly decreasing on $(0, \pi/4)$, we obtain, for $0 < x < \pi/4$,

$$\frac{4 - \pi}{(\sqrt{2} - 1)\pi} = f\left(\frac{\pi}{4}\right) < f(x) = \frac{\frac{\tan x}{x} - 1}{\sec x - 1} < \lim_{t \to 0^+} f(t) = \frac{2}{3}.$$

Hence, (1.1) *holds if and only if $\mu \le (4 - \pi)/\big(\pi(\sqrt{2} - 1)\big)$ and $\nu \ge 2/3$.*

Theorem 2.2. *The following double inequalities hold true:*

$$\frac{2}{3}G+\frac{1}{3}Q<P<\frac{\pi-\sqrt{2}}{\pi}G+\frac{\sqrt{2}}{\pi}Q \tag{2.5}$$

and

$$\frac{3}{4}P+\frac{1}{4}Q<A<\frac{(\sqrt{2}-1)\pi}{\sqrt{2}\pi-2}P+\frac{\pi-2}{\sqrt{2}\pi-2}Q. \tag{2.6}$$

Proof. By Remark 1.1, (2.5) and (2.6) may be rewritten for $0<z<1$ as

$$\frac{1}{3}<\frac{\frac{z}{\arcsin z}-\sqrt{1-z^2}}{\sqrt{1+z^2}-\sqrt{1-z^2}}<\frac{\sqrt{2}}{\pi}\quad\text{and}\quad\frac{1}{4}<\frac{1-\frac{z}{\arcsin z}}{\sqrt{1+z^2}-\frac{z}{\arcsin z}}<\frac{\pi-2}{\sqrt{2}\pi-2},$$

respectively. By an elementary change of variable $z=\sin x$ $(0<x<\pi/2)$, these two inequalities become

$$\frac{1}{3}<F(x)<\frac{\sqrt{2}}{\pi}\quad\text{and}\quad\frac{1}{4}<H(x)<\frac{\pi-2}{\sqrt{2}\pi-2}\quad\text{for}\quad 0<x<\frac{\pi}{2},$$

where

$$F(x)=\frac{\frac{\sin x}{x}-\cos x}{\sqrt{1+\sin^2 x}-\cos x}\quad\text{and}\quad H(x)=\frac{1-\frac{\sin x}{x}}{\sqrt{1+\sin^2 x}-\frac{\sin x}{x}}.$$

Elementary calculations reveal that

$$\lim_{x\to0^+}F(x)=\frac{1}{3},\quad F\left(\frac{\pi}{2}\right)=\frac{\sqrt{2}}{\pi},\quad \lim_{x\to0^+}H(x)=\frac{1}{4},\quad H\left(\frac{\pi}{2}\right)=\frac{\pi-2}{\sqrt{2}\pi-2}.$$

In order prove (2.5) and (2.6), it suffices to show that $F(x)$ and $H(x)$ are both strictly increasing for $0<x<\pi/2$.

Differentiation yields

$$F'(x)=\frac{G_1(x)-(x-\sin x\cos x)\sqrt{1+\sin^2 x}}{2x^2\cos x\sqrt{1+\sin^2 x}\big(\sqrt{1+\tan^2 x}-\sqrt{1+\sin^2 x}\big)},$$

where

$$\begin{aligned}
G_1(x)&=x\cos x-2\sin x+\sin x\cos^2 x+2x^2\sin x\\
&=x\cos x+\frac{1}{4}\sin(3x)+2x^2\sin x-\frac{7}{4}\sin x\\
&>x\left(1-\frac{1}{2}x^2\right)+\frac{1}{4}\left(3x-\frac{1}{3!}(3x)^3+\frac{1}{5!}(3x)^5-\frac{1}{7!}(3x)^7\right)\\
&\quad+2x^2\left(x-\frac{1}{3!}x^3\right)-\frac{7}{4}\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5\right)\\
&=x^3\left(\frac{2}{3}+\frac{19}{120}x^2-\frac{243}{2240}x^4\right)>0\quad\text{for}\quad 0<x<\frac{\pi}{2}.
\end{aligned}$$

For $0<x<\pi/2$, let

$$\big(G_1(x)\big)^2-\big(x-\sin x\cos x\big)^2(1+\sin^2 x)=2\sin xG(x),$$

where

$$G(x)=2x^3\cos x+\sin x\cos^4 x+(2x^2-3)\sin x\cos^2 x+(2x^4-5x^2+2)\sin x.$$

In order to prove $F'(x) > 0$ for $0 < x < \pi/2$, it suffices to show that

$$G(x) > 0, \qquad 0 < x < \pi/2.$$

We find

$$\begin{aligned} G(x) &= 2x^3\cos x + \sin x\left(\frac{\cos(4x)+4\cos(2x)+3}{8}\right) + (2x^2-3)\sin x\left(\frac{\cos(2x)+1}{2}\right) \\ &\quad + (2x^4-5x^2+2)\sin x \\ &= 2x^3\cos x + \frac{1}{8}\sin x\cos(4x) + \big(x^2-1\big)\sin x\cos(2x) + \left(2x^4-4x^2+\frac{7}{8}\right)\sin x \\ &= 2x^3\cos x + \frac{1}{16}\sin(5x) + \left(\frac{1}{2}x^2-\frac{9}{16}\right)\sin(3x) + \left(2x^4-\frac{9}{2}x^2+\frac{11}{8}\right)\sin x \\ &= \sum_{n=4}^{\infty}(-1)^n u_n(x) = \frac{14}{135}x^9 - \frac{223}{4725}x^{11} + \frac{187}{18900}x^{13} - \frac{173471}{130977000}x^{15} + \cdots, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} u_n(x) = \Big(&15\cdot 25^n - (32n^2+16n+81)9^n \\ &+ 1536n^4 - 2304n^3 + 480n^2 + 1008n + 66\Big)\frac{x^{2n+1}}{48\cdot(2n+1)!}. \end{aligned}$$

Elementary calculations reveal that, for $0 < x < \pi/2$ and $n \geq 6$,

$$\frac{u_{n+1}(x)}{u_n(x)} = \frac{3x^2}{2n+3}\frac{\alpha_n}{\beta_n} < \frac{3\left(\pi/2\right)^2}{2n+3}\frac{\alpha_n}{\beta_n} < \frac{\alpha_n}{\beta_n},$$

where

$$\alpha_n = 125\cdot 25^n - (96n^2+240n+387)9^n + 512n^4 + 1280n^3 + 928n^2 + 400n + 262$$

and

$$\begin{aligned} \beta_n = 2(n+1)\big(&15\cdot 25^n - (32n^2+16n+81)9^n \\ &+ 1536n^4 - 2304n^3 + 480n^2 + 1008n + 66\big). \end{aligned}$$

We find, for $n \geq 6$,

$$\begin{aligned} \beta_n - \alpha_n &= (30n-95)25^n - (64n^3-46n-225)9^n + 20253302 + 17631188(n-6) \\ &\quad + 6106496(n-6)^2 + 1051840(n-6)^3 + 90112(n-6)^4 + 3072(n-6)^5 \\ &> (30n-95)25^n - (64n^3-46n-225)9^n \\ &= (30n-95)9^n\left\{\left(\frac{25}{9}\right)^n - \frac{64n^3-46n-225}{30n-95}\right\} > 0. \end{aligned}$$

The last inequality can be proved by induction on n, and we omit it.

Hence, for all $0 < x < \pi/2$ and $n \geq 6$,

$$\frac{u_{n+1}(x)}{u_n(x)} < 1.$$

Therefore, for fixed $x \in (0, \pi/2)$, the sequence $n \longmapsto u_n(x)$ is strictly decreasing for $n \geq 6$. We then obtain from (2.7) that

$$G(x) > x^9\left(\frac{14}{135} - \frac{223}{4725}x^2 + \frac{187}{18900}x^4 - \frac{173471}{130977000}x^6\right) > 0 \quad \text{and} \quad F'(x) > 0$$

for $0 < x < \pi/2$. Hence, $F(x)$ is strictly increasing for $0 < x < \pi/2$.

Differentiation yields

$$H'(x) = \frac{\sin x(1+\sin^2 x) - x\cos x(1+x\sin x) - (\sin x - x\cos x)\sqrt{1+\sin^2 x}}{\sqrt{1+\sin^2 x}\Big(x^2(1+\sin^2 x) + \sin^2 x - 2x\sin x\sqrt{1+\sin^2 x}\Big)}.$$

We now prove that

$$H'(x) > 0, \qquad 0 < x < \frac{\pi}{2}, \tag{2.8}$$

Noting that[1]

$$\sin x(1+\sin^2 x) - x\cos x(1+x\sin x) > 0, \qquad 0 < x < \frac{\pi}{2} \tag{2.9}$$

holds, it suffices to show for $0 < x < \pi/2$ that

$$\sin x(1+\sin^2 x) - x\cos x(1+x\sin x) > (\sin x - x\cos x)\sqrt{1+\sin^2 x} \tag{2.10}$$

and

$$x^2(1+\sin^2 x) + \sin^2 x > 2x\sin x\sqrt{1+\sin^2 x}. \tag{2.11}$$

Here we only prove (2.10). The proof of (2.11) is analogous.

Elementary calculations reveal that

$$\begin{aligned}
&\frac{\Big(\sin x(1+\sin^2 x) - x\cos x(1+x\sin x)\Big)^2 - (\sin x - x\cos x)^2(1+\sin^2 x)}{\sin x} \\
&= 2x^3\cos^2 x + \sin x\cos^4 x + 2x^2\sin x\cos^3 x + (x^4 - x^2 - 3)\sin x\cos^2 x \\
&\quad - 2x^2\sin(2x) + 2\sin x \\
&= \frac{1}{16}\sin(5x) + \frac{1}{4}x^2\sin(4x) + \frac{1}{16}(4x^4 - 4x^2 - 9)\sin(3x) - \frac{3}{2}x^2\sin(2x) \\
&\quad + \frac{1}{8}(2x^4 - 2x^2 + 11)\sin x + x^3\cos(2x) + x^3 \\
&= \frac{8}{135}x^9 - \frac{53}{4725}x^{11} + \frac{23}{75600}x^{13} + \frac{13087}{174636000}x^{15} - \frac{65647}{13621608000}x^{17} \\
&\quad - \frac{577}{1047816000}x^{19} + \sum_{n=10}^{\infty}(-1)^n w_n(x),
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
w_n(x) = \Big(&135\cdot 25^n - 54n(2n+1)16^n + (64n^4 - 64n^3 + 128n^2 + 88n - 729)9^n \\
&- 432n(2n+1)(n-2)4^n + 1728n^4 - 1728n^3 + 648n + 594\Big)\frac{x^{2n+1}}{432\cdot(2n+1)!}.
\end{aligned}$$

Elementary calculations reveal that, for $0 < x < \pi/2$ and $n \ge 10$,

$$\frac{w_{n+1}(x)}{w_n(x)} = \frac{9x^2 c_n}{d_n} < \frac{9(\pi/2)^2 c_n}{d_n} < \frac{23 c_n}{d_n},$$

[1]The inequality (2.9) is proved in the appendix.

where

$$\begin{aligned} c_n = &375 \cdot 25^n - 96(2n+3)(n+1)16^n + (64n^4 + 192n^3 + 320n^2 + 408n - 513)9^n \\ &- 192(n-1))(2n+3)(n+1)4^n + 192n^4 + 576n^3 + 576n^2 + 264n + 138 \end{aligned}$$

and

$$\begin{aligned} d_n = 2(n+1)(2n+3)\Big(&135 \cdot 25^n - 54n(2n+1)16^n \\ &+ (64n^4 - 64n^3 + 128n^2 + 88n - 729)9^n \\ &- 432n(2n+1)(n-2)4^n + 1728n^4 - 1728n^3 + 648n + 594\Big). \end{aligned}$$

Elementary calculations reveal that

$$\begin{aligned} d_n - 23c_n = &(540n^2 + 1350n - 7815)25^n - 12(2n+3))(n+1)(18n^2 + 9n - 184)16^n \\ &+ (256n^6 + 384n^5 - 1216n^4 - 3168n^3 - 8628n^2 - 16146n + 7425)9^n \\ &- 96(2n+3)(n+1)(18n^3 - 27n^2 - 64n + 46)4^n \\ &+ 6912n^6 + 10368n^5 - 11328n^4 - 21024n^3 - 4392n^2 + 3756n + 390. \end{aligned}$$

For $n = 10$, we have $d_{10} - 23c_{10} = 966320930464183296 > 0$. We find that for $n \geq 11$,

$$\begin{aligned} &d_n - 23c_n \\ &> (540n^2 + 1350n - 7815)16^n \\ &\quad \times \left\{ \left(\frac{25}{16}\right)^n - \frac{432n^4 + 1296n^3 - 3228n^2 - 10716n - 6624}{540n^2 + 1350n - 7815} \right\} \\ &\quad + (256n^6 + 384n^5 - 1216n^4 - 3168n^3 - 8628n^2 - 16146n + 7425)4^n \\ &\quad \times \left\{ \left(\frac{9}{4}\right)^n - \frac{3456n^5 + 3456n^4 - 20064n^3 - 29664n^2 + 3648n + 13248}{256n^6 + 384n^5 - 1216n^4 - 3168n^3 - 8628n^2 - 16146n + 7425} \right\} > 0. \end{aligned}$$

We note that, for $n \geq 11$, the inequalities

$$\left(\frac{25}{16}\right)^n - \frac{432n^4 + 1296n^3 - 3228n^2 - 10716n - 6624}{540n^2 + 1350n - 7815} > 0$$

and

$$\left(\frac{9}{4}\right)^n - \frac{3456n^5 + 3456n^4 - 20064n^3 - 29664n^2 + 3648n + 13248}{256n^6 + 384n^5 - 1216n^4 - 3168n^3 - 8628n^2 - 16146n + 7425} > 0.$$

can be proved by induction on n, and we omit them.

Hence, for all $0 < x < \pi/2$ and $n \geq 10$,

$$\frac{w_{n+1}(x)}{w_n(x)} < 1.$$

Therefore, for fixed $x \in (0, \pi/2)$, the sequence $n \longmapsto w_n(x)$ is strictly decreasing for $n \geq 10$. We then obtain from (2.12) that, for $0 < x < \pi/2$,

$$\begin{aligned}
&\frac{\Big(\sin x(1+\sin^2 x) - x\cos x(1 + x\sin x)\Big)^2 - (\sin x - x\cos x)^2(1+\sin^2 x)}{\sin x} \\
&> x^9\left(\frac{8}{135} - \frac{53}{4725}x^2\right) + x^{13}\left(\frac{23}{75600} - \frac{65647}{13621608000}x^4\right) \\
&\quad + x^{15}\left(\frac{13087}{174636000} - \frac{577}{1047816000}x^4\right) > 0.
\end{aligned}$$

Hence, (2.10) holds for $0 < x < \pi/2$. We then obtain $H'(x) > 0$ for $0 < x < \pi/2$. So, $H(x)$ is strictly increasing for $0 < x < \pi/2$. The proof is complete. □

Theorem 2.3. *The inequalities*

$$(1-\mu_1)L + \mu_1 Q < P < (1-\nu_1)\,L + \nu_1 Q \tag{2.13}$$

and

$$(1-\mu_2)L + \mu_2 N < P < (1-\nu_2)\,L + \nu_2 N \tag{2.14}$$

hold if and only if

$$\mu_1 \leq \frac{1}{5}, \quad \nu_1 \geq \frac{\sqrt{2}}{\pi}, \quad \mu_2 \leq \frac{1}{8}, \quad \nu_2 \geq \frac{1}{\pi}. \tag{2.15}$$

Proof. We first prove (2.13) and (2.14) with $\mu_1 = \frac{1}{5}, \nu_1 = \frac{\sqrt{2}}{\pi}, \mu_2 = \frac{1}{8}, \nu_2 = \frac{1}{\pi}$, namely,

$$\frac{4}{5}L + \frac{1}{5}Q < P < \left(1 - \frac{\sqrt{2}}{\pi}\right) L + \frac{\sqrt{2}}{\pi}Q \tag{2.16}$$

and

$$\frac{7}{8}L + \frac{1}{8}N < P < \left(1 - \frac{1}{\pi}\right) L + \frac{1}{\pi}N. \tag{2.17}$$

We claim that

$$\left(1 - \frac{\sqrt{2}}{\pi}\right) G + \frac{\sqrt{2}}{\pi}Q < \left(1 - \frac{\sqrt{2}}{\pi}\right) L + \frac{\sqrt{2}}{\pi}Q < \left(1 - \frac{1}{\pi}\right) L + \frac{1}{\pi}N. \tag{2.18}$$

This claim shows that, among the second inequalities in (2.5), (2.16) and (2.17), the upper bound

$$\left(1 - \frac{\sqrt{2}}{\pi}\right) G + \frac{\sqrt{2}}{\pi}Q$$

is the best, in the sense that it is the smallest one among the three upper bounds in (2.5), (2.16) and (2.17).

Obvious, the left-hand side of (2.18) holds. We now prove the right-hand side of (2.18). Noting that $G < L$ holds, we have

$$\begin{aligned}
&\left(1 - \frac{1}{\pi}\right) L + \frac{1}{\pi}N - \left\{\left(1 - \frac{\sqrt{2}}{\pi}\right) L + \frac{\sqrt{2}}{\pi}Q\right\} \\
&\quad = \frac{1}{\pi}\left\{(\sqrt{2}-1)L + N - \sqrt{2}Q\right\} > \frac{1}{\pi}\left\{(\sqrt{2}-1)G + N - \sqrt{2}Q\right\}.
\end{aligned}$$

In order prove the right-hand side of (2.18), it suffices to show that

$$(\sqrt{2}-1)G+N>\sqrt{2}Q,$$

which can be written, by Remark 1.1, as

$$(\sqrt{2}-1)\sqrt{1-z^2}+(1+z^2)>\sqrt{2}\sqrt{1+z^2}, \qquad 0<z<1,$$

i.e.,

$$(\sqrt{2}-1)\sqrt{1-t}+(1+t)>\sqrt{2}\sqrt{1+t}, \qquad 0<t<1. \tag{2.19}$$

We find

$$\Big((\sqrt{2}-1)\sqrt{1-t}+(1+t)\Big)^2-\Big(\sqrt{2}\sqrt{1+t}\Big)^2$$
$$=2(\sqrt{2}-1)(1+t)\sqrt{1-t}-(2\sqrt{2}-2+t)(1-t)$$

and

$$\Big(2(\sqrt{2}-1)(1+t)\sqrt{1-t}\Big)^2-\Big((2\sqrt{2}-2+t)(1-t)\Big)^2$$
$$=t(1-t)\Big\{t^2+(7-4\sqrt{2})t+40-28\sqrt{2}\Big\}>0 \quad \text{for} \quad 0<t<1.$$

Hence, (2.19) holds. The claim (2.18) is proved.

By Remark 1.1, the first inequalities in (2.16) and (2.17) can be written for $0<z<1$ as

$$\frac{4}{5}\frac{2z}{\ln\frac{1+z}{1-z}}+\frac{1}{5}\sqrt{1+z^2}<\frac{z}{\arcsin z} \tag{2.20}$$

and

$$\frac{7}{8}\frac{2z}{\ln\frac{1+z}{1-z}}+\frac{1}{8}(1+z^2)<\frac{z}{\arcsin z}, \tag{2.21}$$

respectively.

We first prove (2.20) for $0<z<0.7$. From the well known continued fraction for $\ln\frac{1+x}{1-x}$ (see [7, p. 196, Eq. (11.2.4)]), we find that for $0<x<1$,

$$\frac{2x(15-4x^2)}{3(5-3x^2)}=\cfrac{2x}{1+\cfrac{-\frac{1}{3}x^2}{1+\cfrac{-\frac{4}{15}x^2}{1}}}<\ln\frac{1+x}{1-x}. \tag{2.22}$$

Using (2.22), we have

$$\frac{z}{\arcsin z}-\left(\frac{4}{5}\frac{2z}{\ln\frac{1+z}{1-z}}+\frac{1}{5}\sqrt{1+z^2}\right)>\frac{z}{\arcsin z}-\left\{\frac{4}{5}\frac{3(5-3z^2)}{15-4z^2}+\frac{1}{5}\left(1+\frac{1}{2}z^2\right)\right\}$$
$$=\frac{z}{\arcsin z}-\frac{150-65z^2-4z^4}{10(15-4z^2)}.$$

In order to prove (2.20) for $0<z<0.7$, it suffices to show that

$$\theta(z)>0 \quad \text{for} \quad 0<z<0.7,$$

where

$$\theta(z)=\frac{10z(15-4z^2)}{150-65z^2-4z^4}-\arcsin z.$$

Differentiation yields

$$\theta'(z)=\frac{10(2250-825z^2+440z^4-16z^6)}{(150-65z^2-4z^4)^2}-\frac{1}{\sqrt{1-z^2}}.$$

Elementary calculations reveal that, for $0<z<0.7$,

$$\begin{aligned}&\left(\frac{10(2250-825z^2+440z^4-16z^6)}{(150-65z^2-4z^4)^2}\right)^2-\frac{1}{1-z^2}\\ &\quad=\frac{1}{(1-z^2)\big(150-65z^2-4z^4\big)^4}\Big[120937500-251287500z^2+112209375z^4\\ &\qquad\quad-25930000z^6+z^8(1066400-42240z^2-256z^4)\Big]>0.\end{aligned}$$

We then obtain $\theta'(z)>0$ for $0<z<0.7$. Hence, $\theta(z)$ is strictly increasing for $0<z<0.7$, and we have

$$\theta(z)=\frac{10z(15-4z^2)}{150-65z^2-4z^4}-\arcsin z>\theta(0)=0\quad\text{for}\quad 0<z<0.7.$$

Therefore, (2.20) holds for $0<z<0.7$.

Second, we prove (2.20) for $0.7\le z<1$. Let

$$\omega(z)=\omega_1(z)+\omega_2(z),$$

where

$$\omega_1(z)=-\left(\frac{4}{5}\frac{2z}{\ln\frac{1+z}{1-z}}+\frac{1}{5}\sqrt{1+z^2}\right)\quad\text{and}\quad\omega_2(z)=\frac{z}{\arcsin z}.$$

Let $0.7\le r\le z\le s<1$. Since $\omega_1(z)$ is increasing and $\omega_2(z)$ is decreasing, we obtain

$$\omega(z)\ge\omega_1(r)+\omega_2(s)=:\sigma(r,s).$$

We divide the interval $[0.7,1]$ into 30 subintervals:

$$[0.7,1]=\bigcup_{k=0}^{29}\left[0.7+\frac{k}{100},0.7+\frac{k+1}{100}\right]\quad\text{for}\quad k=0,1,2,\ldots,29.$$

By direct computation we get

$$\sigma\left(0.7+\frac{k}{100},0.7+\frac{k+1}{100}\right)>0\quad\text{for}\quad k=0,1,2,\ldots,29.$$

Hence,

$$\omega(z)>0\quad\text{for}\quad z\in\left[0.7+\frac{k}{100},0.7+\frac{k+1}{100}\right]\quad\text{and}\quad k=0,1,2,\ldots,29.$$

This implies that $\omega(z)$ is positive on $[0.7,1)$. This proves (2.20) for $0.7\le z<1$. Hence, (2.20) holds for all $0<z<1$.

We now prove (2.21). We first prove (2.21) for $0<z<0.7$. Using (2.22), we have

$$\begin{aligned}\frac{z}{\arcsin z}-\left(\frac{7}{8}\frac{2z}{\ln\frac{1+z}{1-z}}+\frac{1}{8}(1+z^2)\right)&>\frac{z}{\arcsin z}-\left\{\frac{7}{8}\frac{3(5-3z^2)}{15-4z^2}+\frac{1}{8}(1+z^2)\right\}\\ &=\frac{z}{\arcsin z}-\frac{30-13z^2-z^4}{2(15-4z^2)}.\end{aligned}$$

In order to prove (2.21) for $0 < z < 0.7$, it suffices to show that

$$\Theta(z) > 0 \quad \text{for} \quad 0 < z < 0.7,$$

where

$$\Theta(z) = \frac{2z(15 - 4z^2)}{30 - 13z^2 - z^4} - \arcsin z.$$

Differentiation yields

$$\Theta'(z) = \frac{2(450 - 165z^2 + 97z^4 - 4z^6)}{(30 - 13z^2 - z^4)^2} - \frac{1}{\sqrt{1 - z^2}}.$$

Elementary calculations reveal that, for $0 < z < 0.7$,

$$\left(\frac{2(450 - 165z^2 + 97z^4 - 4z^6)}{(30 - 13z^2 - z^4)^2}\right)^2 - \frac{1}{1 - z^2}$$
$$= \frac{(247500 - 477300z^2) + z^4(212235 - 50128z^2) + z^8(2274 - 116z^2 - z^4)}{\left(30 - 13z^2 - z^4\right)^4(1 - z^2)} > 0.$$

We then obtain $\Theta'(z) > 0$ for $0 < z < 0.7$. Hence, $\Theta(z)$ is strictly increasing for $0 < z < 0.7$, and we have

$$\Theta(z) = \frac{2z(15 - 4z^2)}{30 - 13z^2 - z^4} - \arcsin z > \Theta(0) = 0 \quad \text{for} \quad 0 < z < 0.7.$$

Therefore, (2.21) holds for $0 < z < 0.7$.

Second, we prove (2.21) for $0.7 \le z < 1$. Let

$$y(z) = y_1(z) + y_2(z),$$

where

$$y_1(z) = -\left(\frac{7}{8}\frac{2z}{\ln\frac{1+z}{1-z}} + \frac{1}{8}(1 + z^2)\right) \quad \text{and} \quad y_2(z) = \frac{z}{\arcsin z}.$$

Let $0.7 \le r \le z \le s < 1$. Since $y_1(z)$ is increasing and $y_2(z)$ is decreasing, we obtain

$$y(z) \ge y_1(r) + y_2(s) =: \rho(r, s).$$

We divide the interval $[0.7, 1]$ into 30 subintervals:

$$[0.7, 1] = \bigcup_{k=0}^{29}\left[0.7 + \frac{k}{100}, 0.7 + \frac{k+1}{100}\right] \quad \text{for} \quad k = 0, 1, 2, \dots, 29.$$

By direct computation we get

$$\rho\left(0.7 + \frac{k}{100}, 0.7 + \frac{k+1}{100}\right) > 0 \quad \text{for} \quad k = 0, 1, 2, \dots, 29.$$

Hence,

$$y(z) > 0 \quad \text{for} \quad z \in \left[0.7 + \frac{k}{100}, 0.7 + \frac{k+1}{100}\right] \quad \text{and} \quad k = 0, 1, 2, \dots, 29.$$

This implies that $y(z)$ is positive on $[0.7, 1)$. This proves (2.21) for $0.7 \le z < 1$. Hence, (2.21) holds for all $0 < z < 1$.

We then obtain (2.13) and (2.14) with $\mu_1 = \frac{1}{5},\ \nu_1 = \frac{\sqrt{2}}{\pi}, \mu_2 = \frac{1}{8}, \nu_2 = \frac{1}{\pi}$.

Conversely, if (2.13) and (2.14) are valid, then we get

$$\mu_1 < \frac{P-L}{Q-L} = \frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{\sqrt{1+z^2} - \frac{2z}{\ln\frac{1+z}{1-z}}} < \nu_1 \quad \text{and} \quad \mu_2 < \frac{P-L}{N-L} = \frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{1+z^2 - \frac{2z}{\ln\frac{1+z}{1-z}}} < \nu_2.$$

The limit relations

$$\lim_{z\to 0} \frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{\sqrt{1+z^2} - \frac{2z}{\ln\frac{1+z}{1-z}}} = \frac{1}{5}, \quad \lim_{z\to 1} \frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{\sqrt{1+z^2} - \frac{2z}{\ln\frac{1+z}{1-z}}} = \frac{\sqrt{2}}{\pi},$$

$$\lim_{z\to 0} \frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{1+z^2 - \frac{2z}{\ln\frac{1+z}{1-z}}} = \frac{1}{8}, \quad \lim_{z\to 1} \frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{1+z^2 - \frac{2z}{\ln\frac{1+z}{1-z}}} = \frac{1}{\pi}$$

yield

$$\mu_1 \le \frac{1}{5}, \quad \nu_1 \ge \frac{\sqrt{2}}{\pi}, \quad \mu_2 \le \frac{1}{8}, \quad \nu_2 \ge \frac{1}{\pi}.$$

The proof is complete. □

3. Proof of Conjecture 1.2

Theorem 3.1. *The following double inequality holds true:*

$$\frac{2\pi-4}{\pi}A + \frac{4-\pi}{\pi}N < T < \frac{2}{3}A + \frac{1}{3}N. \tag{3.1}$$

Proof. By Remark 1.1, (3.1) may be rewritten as

$$\frac{4-\pi}{\pi} < \frac{\frac{z}{\arctan z} - 1}{z^2} < \frac{1}{3} \quad \text{for} \quad 0 < z < 1. \tag{3.2}$$

By an elementary change of variable $z = \tan x$ $(0 < x < \pi/4)$, (3.2) becomes

$$\frac{4-\pi}{\pi} < U(x) < \frac{1}{3} \quad \text{for} \quad 0 < x < \frac{\pi}{4}, \tag{3.3}$$

where

$$U(x) = \frac{\frac{\tan x}{x} - 1}{\tan^2 x}.$$

Differentiation yields

$$U'(x) = -\frac{U_1(x)}{x^2 \sin^2 x \tan x},$$

where

$$U_1(x) = x\tan x - 2x^2 + \sin^2 x = x\tan x - \frac{1}{2}\cos(2x) - 2x^2 + \frac{1}{2}$$
$$= \sum_{n=3}^{\infty} \frac{2^{2n-1}\Big(2(2^{2n}-1)|B_{2n}| - (-1)^n\Big)}{(2n)!} x^{2n}.$$

It is well known [1, p. 805] that

$$\frac{2(2n)!}{(2\pi)^{2n}} < |B_{2n}| < \frac{2(2n)!}{(2\pi)^{2n}(1-2^{1-2n})}, \qquad n \ge 1.$$

By the first inequality above, we find

$$2(2^{2n}-1)|B_{2n}| > 2(2^{2n}-1)\frac{2(2n)!}{(2\pi)^{2n}} > 1, \qquad n \geq 3.$$

We then obtain

$$U_1(x) > 0, \qquad 0 < x < \frac{\pi}{4}, \tag{3.4}$$

So, we have $U'(x) < 0$ for $0 < x < \pi/4$. Hence, $U(x)$ are strictly decreasing on $(0, \pi/4)$, and we have

$$\frac{4-\pi}{\pi} = U\left(\frac{\pi}{4}\right) < U(x) = \frac{\frac{\tan x}{x} - 1}{\tan^2 x} < \lim_{t\to 0} U(t) = \frac{1}{3}$$

for $0 < x < \pi/4$. The proof is complete. □

Remark 3.1. *Noting that $H + N = 2A$ holds,* (3.1) *can be written as* (1.9).

Theorem 3.2. *The following double inequalities hold true:*

$$\frac{1}{4}H + \frac{3}{4}T < A < \frac{4-\pi}{4}H + \frac{\pi}{4}T, \tag{3.5}$$

$$\frac{1}{9}H + \frac{8}{9}Q < T < \frac{\pi - 2\sqrt{2}}{\pi}H + \frac{2\sqrt{2}}{\pi}Q, \tag{3.6}$$

$$\frac{1}{6}G + \frac{5}{6}Q < T < \frac{\pi - 2\sqrt{2}}{\pi}G + \frac{2\sqrt{2}}{\pi}Q, \tag{3.7}$$

$$\frac{(2-\sqrt{2})\pi}{2\pi - 4}T + \frac{\sqrt{2}\pi - 4}{2\pi - 4}N < Q < \frac{3}{4}T + \frac{1}{4}N. \tag{3.8}$$

Proof. By Remark 1.1, (3.5), (3.6), (3.7) and (3.8) may be rewritten for $0 < z < 1$ as

$$\frac{3}{4} < \frac{z^2}{\frac{z}{\arctan z} - (1-z^2)} < \frac{\pi}{4}, \qquad \frac{8}{9} < \frac{\frac{z}{\arctan z} - (1-z^2)}{\sqrt{1+z^2} - (1-z^2)} < \frac{2\sqrt{2}}{\pi},$$

$$\frac{5}{6} < \frac{\frac{z}{\arctan z} - \sqrt{1-z^2}}{\sqrt{1+z^2} - \sqrt{1-z^2}} < \frac{2\sqrt{2}}{\pi}, \quad \frac{\sqrt{2}\pi - 4}{2\pi - 4} < \frac{\sqrt{1+z^2} - \frac{z}{\arctan z}}{1 + z^2 - \frac{z}{\arctan z}} < \frac{1}{4},$$

respectively. By an elementary change of variable $z = \tan x$ $(0 < x < \pi/4)$, these four inequalities become

$$\frac{3}{4} < J_1(x) < \frac{\pi}{4}, \quad \frac{8}{9} < J_2(x) < \frac{2\sqrt{2}}{\pi}, \quad \frac{5}{6} < J_3(x) < \frac{2\sqrt{2}}{\pi}, \quad \frac{\sqrt{2}\pi - 4}{2\pi - 4} < J_4(x) < \frac{1}{4}$$

for $0 < x < \pi/4$, where

$$J_1(x) = \frac{\tan^2 x}{\frac{\tan x}{x} - (1 - \tan^2 x)}, \quad J_2(x) = \frac{\frac{\tan x}{x} - (1 - \tan^2 x)}{\sec x - (1 - \tan^2 x)},$$

$$J_3(x) = \frac{\frac{\tan x}{x} - \sqrt{1 - \tan^2 x}}{\sec x - \sqrt{1 - \tan^2 x}} = \frac{\frac{\sin x}{x} - \sqrt{\cos(2x)}}{1 - \sqrt{\cos(2x)}}, \quad J_4(x) = \frac{\sec x - \frac{\tan x}{x}}{\sec^2 x - \frac{\tan x}{x}}.$$

Elementary calculations reveal that

$$\lim_{x\to 0^+} J_1(x) = \frac{3}{4}, \quad J_1\left(\frac{\pi}{4}\right) = \frac{\pi}{4}, \qquad \lim_{x\to 0^+} J_2(x) = \frac{8}{9}, \quad J_2\left(\frac{\pi}{4}\right) = \frac{2\sqrt{2}}{\pi},$$

$$\lim_{x\to 0^+} J_3(x) = \frac{5}{6}, \quad J_3\left(\frac{\pi}{4}\right) = \frac{2\sqrt{2}}{\pi}, \qquad \lim_{x\to 0^+} J_4(x) = \frac{1}{4}, \quad J_4\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}\pi - 4}{2\pi - 4}.$$

In order prove (3.5), (3.6), (3.7) and (3.8), it suffices to show that $J_1(x), J_2(x)$ and $J_3(x)$ are strictly increasing and $J_4(x)$ is strictly decreasing for $0 < x < \pi/4$.

Differentiation yields

$$J_1'(x) = \frac{\sin x \cos x U_1(x)}{U_2(x)}, \qquad 0 < x < \frac{\pi}{4},$$

where

$$U_1(x) = x \tan x + \sin^2 x - 2x^2 > 0 \quad \text{(see (3.4))}$$

and

$$U_2(x) = 2x \sin x \cos x - (4x^2 - 1) \sin^2 x \cos^2 x - 4x \cos^3 x \sin x + x^2.$$

We find

$$\begin{aligned} U_2(x) &= -\frac{1}{2}\left(x^2 - \frac{1}{4}\right)\big(1 - \cos(4x)\big) - \frac{1}{2} x \sin(4x) + x^2 \\ &= \sum_{n=3}^{\infty} (-1)^{n-1} v_n(x) = \frac{16}{9} x^6 - \frac{64}{45} x^8 + \sum_{n=5}^{\infty} (-1)^{n-1} v_n(x), \end{aligned} \tag{3.9}$$

where

$$v_n(x) = \frac{2^{4n-5}(n-2)}{n \cdot (2n-2)!} x^{2n}.$$

Elementary calculations reveal that, for $0 < x < \pi/4$ and $n \geq 5$,

$$\begin{aligned} \frac{v_{n+1}(x)}{v_n(x)} &= \frac{8(n-1)x^2}{(n+1)(2n-1)(n-2)} < \frac{8(n-1)(\pi/4)^2}{(n+1)(2n-1)(n-2)} \\ &< \frac{8(n-1)}{(n+1)(2n-1)(n-2)} < 1. \end{aligned}$$

Hence, for all $0 < x < \pi/4$ and $n \geq 5$,

$$\frac{v_{n+1}(x)}{v_n(x)} < 1.$$

Therefore, for fixed $x \in (0, \pi/4)$, the sequence $n \longmapsto v_n(x)$ is strictly decreasing for $n \geq 5$. We then obtain from (3.9) that

$$U_2(x) > x^6 \left(\frac{16}{9} - \frac{64}{45} x^2 \right) > 0, \quad 0 < x < \frac{\pi}{4}.$$

Thus, we have

$$J_1'(x) > 0, \quad 0 < x < \frac{\pi}{4}.$$

Hence, $J_1(x)$ is strictly increasing for $0 < x < \pi/4$.

Differentiation yields

$$\begin{aligned} &x^2(1 - \cos x)^2 (1 + 2\cos x)^2 J_2'(x) \\ &= 2 \sin x \cos^3 x + 2x^2 \sin x \cos^2 x - \sin x \cos x + x^2 \sin x - \sin x \cos^2 x - x + x \cos^3 x \\ &= \frac{1}{4} \sin(4x) + \left(\frac{x^2}{2} - \frac{1}{4} \right) \sin(3x) + \frac{1}{4} x \cos(3x) + \left(\frac{3x^2}{2} - \frac{1}{4} \right) \sin x + \frac{3}{4} x \cos x - x \\ &= \frac{1}{15} x^7 - \frac{1}{105} x^9 - \frac{53}{25200} x^{11} + \sum_{n=6}^{\infty} (-1)^n V_n(x), \end{aligned} \tag{3.10}$$

where

$$V_n(x) = \frac{6\cdot 16^n - (4n^2-n+3)9^n - 36n^2 - 9n + 3}{6(2n+1)!}x^{2n+1}.$$

We find that for $0 < x < \pi/4$ and $n \geq 6$,

$$\begin{aligned}\frac{V_{n+1}(x)}{V_n(x)} &= \frac{\frac{3}{2}x^2\Big(32\cdot 16^n - (12n^2+21n+18)9^n - (12n^2+27n+14)\Big)}{(n+1)(2n+3)\Big(6\cdot 16^n - (4n^2-n+3)9^n - (36n^2+9n-3)\Big)} \\ &< \frac{\frac{3}{2}(\frac{\pi}{4})^2\Big(32\cdot 16^n - (12n^2+21n+18)9^n - (12n^2+27n+14)\Big)}{(n+1)(2n+3)\Big(6\cdot 16^n - (4n^2-n+3)9^n - (36n^2+9n-3)\Big)} \\ &< \frac{32\cdot 16^n - (12n^2+21n+18)9^n - (12n^2+27n+14)}{(n+1)(2n+3)\Big(6\cdot 16^n - (4n^2-n+3)9^n - (36n^2+9n-3)\Big)} \\ &< \frac{32\cdot 16^n}{(n+1)(2n+3)\Big(6\cdot 16^n - (4n^2-n+3)9^n - (36n^2+9n-3)\Big)} \\ &= \frac{32}{(n+1)(2n+3)(6-x_n)},\end{aligned}$$

where

$$x_n = (4n^2-n+3)\left(\frac{9}{16}\right)^n + \frac{36n^2+9n-3}{16^n}.$$

Noting that the sequence $\{x_n\}$ is strictly decreasing for $n \geq 6$, we have

$$0 < x_n \leq x_6 = \frac{37465917}{8388608}.$$

We then obtain that, for $0 < x < \pi/4$ and $n \geq 6$,

$$\frac{V_{n+1}(x)}{V_n(x)} < \frac{32}{(n+1)(2n+3)\Big(6 - \frac{37465917}{8388608}\Big)} < 1.$$

Therefore, for fixed $x \in (0, \pi/4)$, the sequence $n \longmapsto V_n(x)$ is strictly decreasing for $n \geq 6$. We then obtain from (3.10) that, for $0 < x < \pi/4$,

$$x^2(1-\cos x)^2(1+2\cos x)^2 J_2'(x) > x^7\left(\frac{1}{15} - \frac{1}{105}x^2 - \frac{53}{25200}x^4\right) > 0 \quad\text{and}\quad J_2'(x) > 0.$$

Hence, $J_2(x)$ is strictly increasing for $0 < x < \pi/4$.

Differentiation yields

$$x^2\sqrt{\cos(2x)}\big(1-\sqrt{\cos(2x)}\big)^2 J_3'(x) = D_2(x) - D_1(x),$$

where

$$D_2(x) = (\sin x - x\cos x)\cos(2x) + x(x-\sin x)\sin(2x) > 0$$

and

$$D_1(x) = (\sin x - x\cos x)\sqrt{\cos(2x)} > 0$$

for $0 < x < \pi/4$.

We now prove $J_3'(x) > 0$ for $0 < x < \pi/4$, it suffices to show that $D_2(x) > D_1(x)$.

Elementary calculations reveal that

$$\begin{aligned}\frac{D_2^2(x)-D_1^2(x)}{2\sin x} &= -2x^3\cos^2 x+\sin x+2\sin x\cos^4 x+4x^2\sin x\cos^3 x\\ &\quad+(2x^4+x^2-3)\sin x\cos^2 x-x^2\sin(2x)\\ &= -x^3-x^3\cos(2x)+\left(\frac{1}{2}x^4+\frac{1}{4}x^2+\frac{1}{2}\right)\sin x\\ &\quad+\left(\frac{1}{2}x^4+\frac{1}{4}x^2-\frac{3}{8}\right)\sin(3x)+\frac{1}{2}x^2\sin(4x)+\frac{1}{8}\sin(5x)\\ &= \frac{13}{540}x^9+\frac{1}{9450}x^{11}-\frac{37}{20160}x^{13}+\frac{108961}{349272000}x^{15}\\ &\quad-\frac{1864237}{108972864000}x^{17}-\frac{493}{583783200}x^{19}+\sum_{n=10}^{\infty}(-1)^n X_n(x),\end{aligned}\tag{3.11}$$

where

$$\begin{aligned}X_n(x)=\Big(&135\cdot 25^n-54n(2n+1)16^n+(64n^4-64n^3-88n^2-20n-243)9^n\\ &+108n(2n-1)(2n+1)4^n+108(2n-1)(8n^3-4n^2-5n-1)\Big)\frac{x^{2n+1}}{216\cdot(2n+1)!}.\end{aligned}$$

We find that for $0<x<\pi/4$ and $n\geq 10$,

$$\frac{X_{n+1}(x)}{X_n(x)}=\left(\frac{9x^2}{2}\right)\frac{Y_n}{Z_n}<\frac{9}{2}\left(\frac{\pi}{4}\right)^2\frac{Y_n}{Z_n}<\frac{3Y_n}{Z_n},$$

where

$$\begin{aligned}Y_n=&\ 375\cdot 25^n-96(2n+3)(n+1)16^n+(64n^4+192n^3+104n^2-132n-351)9^n\\ &+48(2n+3)(2n+1)(n+1)4^n+12(2n+1)(8n^3+20n^2+11n-2)\end{aligned}$$

and

$$\begin{aligned}Z_n=(n+1)(2n+3)\Big(&135\cdot 25^n-54n(2n+1)16^n\\ &+(64n^4-64n^3-88n^2-20n-243)9^n+108n(2n-1)(2n+1)4^n\\ &+108(2n-1)(8n^3-4n^2-5n-1)\Big).\end{aligned}$$

Elementary calculations reveal that, for $0<x<\pi/4$ and $n\geq 10$,

$$\begin{aligned}Z_n-3Y_n &= (270n^2+675n-720)25^n-18(2n+3))(n+1)(6n^2+3n-16)16^n\\ &\quad+(128n^6-1162n^2-879n-496n^4+324-1248n^3+192n^5)9^n\\ &\quad+36(2n+3)(2n+1)(n+1)(6n^2-3n-4)4^n\\ &\quad+3456n^6+5184n^5-5328n^4-9504n^3-1620n^2+1260n+396\\ &> (270n^2+675n-720)25^n-18(2n+3)(n+1)(6n^2+3n-16)16^n\\ &= (270n^2+675n-720)16^n\left\{\left(\frac{25}{16}\right)^n-\frac{18(2n+3)(n+1)(6n^2+3n-16)}{270n^2+675n-720}\right\}\\ &> 0.\end{aligned}$$

The last inequality can be proved by induction on n, and we omit it.

We then obtain that, for $0 < x < \pi/4$ and $n \geq 10$,

$$\frac{X_{n+1}(x)}{X_n(x)} < 1.$$

Therefore, for fixed $x \in (0, \pi/4)$, the sequence $n \longmapsto X_n(x)$ is strictly decreasing for $n \geq 10$. We obtain from (3.11) that, for $0 < x < \pi/4$,

$$\begin{aligned}\frac{D_2^2(x) - D_1^2(x)}{2\sin x} &= x^9 \left(\frac{13}{540} + \frac{1}{9450}x^2 - \frac{37}{20160}x^4\right) \\ &\quad + x^{15}\left(\frac{108961}{349272000} - \frac{1864237}{108972864000}x^2 - \frac{493}{583783200}x^4\right) > 0.\end{aligned}$$

We then obtain that for $0 < x < \pi/4$,

$$D_2(x) > D_1(x) \quad \text{and} \quad J_3'(x) > 0.$$

Hence, $J_3(x)$ is strictly increasing for $0 < x < \pi/4$.

Differentiation yields

$$J_4'(x) = -\frac{I_1(x)}{I_2(x)},$$

where

$$I_1(x) = x^2 \sin x - \sin x \cos x + \sin x \cos^2 x + 2x \cos^2 x - x \cos^3 x - x$$

and

$$I_2(x) = x^2 - x\sin(2x) + \frac{1}{4}\sin^2(2x) = \frac{4}{9}x^6 - \frac{8}{45}x^8 + \frac{164}{4725}x^{10} - \frac{184}{42525}x^{12} + \cdots .$$

We now prove $J_4'(x) < 0$ for $0 < x < \pi/4$, it suffices to show that $I_1(x) > 0$ and $I_2(x) > 0$ for $0 < x < \pi/4$.

Elementary calculations reveal that

$$\begin{aligned}I_1(x) &= \left(x^2 + \frac{1}{4}\right)\sin x - \frac{1}{2}\sin(2x) + \frac{1}{4}\sin(3x) - \frac{3}{4}x\cos x + x\cos(2x) - \frac{1}{4}x\cos(3x) \\ &= \frac{7}{90}x^7 - \frac{41}{1890}x^9 + \sum_{n=5}^{\infty}(-1)^{n-1}W_n(x),\end{aligned} \tag{3.12}$$

where

$$W_n(x) = \frac{(n-1)9^n - 4n \cdot 4^n + 8n^2 + 7n + 1}{2 \cdot (2n+1)!}x^{2n+1}.$$

We find that, for $0 < x < \pi/4$ and $n \geq 5$,

$$\begin{aligned}\frac{W_{n+1}(x)}{W_n(x)} &= \frac{\frac{1}{2}x^2\Big(9n\cdot 9^n - (16n+16)4^n + 8n^2 + 23n + 16\Big)}{(n+1)(2n+3)\Big((n-1)9^n - 4n\cdot 4^n + 8n^2 + 7n + 1\Big)} \\ &< \frac{\frac{1}{2}(\frac{\pi}{4})^2\Big(9n\cdot 9^n - (16n+16)4^n + 8n^2 + 23n + 16\Big)}{(n+1)(2n+3)\Big((n-1)9^n - 4n\cdot 4^n + 8n^2 + 7n + 1\Big)} \\ &< \frac{9n\cdot 9^n - (16n+16)4^n + 8n^2 + 23n + 16}{(n+1)(2n+3)\Big((n-1)9^n - 4n\cdot 4^n + 8n^2 + 7n + 1\Big)} \\ &< \frac{9n\cdot 9^n + 8n^2 + 23n + 16}{(n+1)(2n+3)\Big((n-1)9^n - 4n\cdot 4^n\Big)} \\ &= \frac{9n + \frac{8n^2+23n+16}{9^n}}{(n+1)(2n+3)\Big((n-1) - 4n\left(\frac{4}{9}\right)^n\Big)}.\end{aligned}$$

Noting that the sequences $\left\{\frac{8n^2+23n+16}{9^n}\right\}$ and $\left\{4n\left(\frac{4}{9}\right)^n\right\}$ are both strictly decreasing for $n \geq 5$, we have, for $n \geq 5$,

$$0 < \frac{8n^2+23n+16}{9^n} \leq \left[\frac{8n^2+23n+16}{9^n}\right]_{n=5} = \frac{331}{59049}$$

and

$$0 < 4n\left(\frac{4}{9}\right)^n \leq \left[4n\left(\frac{4}{9}\right)^n\right]_{n=5} = \frac{20480}{59049}.$$

We then obtain that for $0 < x < \pi/4$ and $n \geq 5$,

$$\frac{W_{n+1}(x)}{W_n(x)} < \frac{9n + \frac{331}{59049}}{(n+1)(2n+3)\Big((n-1) - \frac{20480}{59049}\Big)} < 1.$$

Therefore, for fixed $x \in (0, \pi/4)$, the sequence $n \longmapsto W_n(x)$ is strictly decreasing for $n \geq 5$. We then obtain from (3.12) that, for $0 < x < \pi/4$,

$$I_1(x) > x^7\left(\frac{7}{90} - \frac{41}{1890}x^2\right) > 0.$$

Following the same method as was used in the proof of $I_1(x) > 0$, we can prove $I_2(x) > 0$ for $0 < x < \pi/4$, and we omit it. We then obtain $J_4'(x) < 0$ for $0 < x < \pi/4$. Hence, $J_4(x)$ is strictly decreasing for $0 < x < \pi/4$. The proof is complete. □

Theorem 3.3. *The inequalities*

$$(1-\mu_3)L + \mu_3 T < A < (1-\nu_3)L + \nu_3 T \tag{3.13}$$

and

$$(1-\mu_4)L + \mu_4 Q < T < (1-\nu_4)L + \nu_4 Q \tag{3.14}$$

hold if and only if

$$\mu_3 \leq \frac{1}{2}, \quad \nu_3 \geq \frac{\pi}{4}, \quad \mu_4 \leq \frac{4}{5}, \quad \nu_4 \geq \frac{2\sqrt{2}}{\pi}. \tag{3.15}$$

Proof. We first prove (3.13) and (3.14) with $\mu_3 = \frac{1}{2}, \nu_3 = \frac{\pi}{4}, \mu_4 = \frac{4}{5}, \nu_4 = \frac{2\sqrt{2}}{\pi}$, namely,

$$\frac{1}{2}L + \frac{1}{2}T < A < \left(1 - \frac{\pi}{4}\right)L + \frac{\pi}{4}T \tag{3.16}$$

and

$$\frac{1}{5}L + \frac{4}{5}Q < T < \left(1 - \frac{2\sqrt{2}}{\pi}\right)L + \frac{2\sqrt{2}}{\pi}Q. \tag{3.17}$$

In fact, (3.5) $\Longrightarrow$ (3.16) and (3.6) $\Longrightarrow$ (3.17). More precisely, the following inequalities are true:

$$\frac{1}{2}L + \frac{1}{2}T < \frac{1}{4}H + \frac{3}{4}T < A < \left(1 - \frac{\pi}{4}\right)H + \frac{\pi}{4}T < \left(1 - \frac{\pi}{4}\right)L + \frac{\pi}{4}T \tag{3.18}$$

and

$$\frac{1}{5}L + \frac{4}{5}Q < \frac{1}{9}H + \frac{8}{9}Q < T < \left(1 - \frac{2\sqrt{2}}{\pi}\right)H + \frac{2\sqrt{2}}{\pi}Q < \left(1 - \frac{2\sqrt{2}}{\pi}\right)L + \frac{2\sqrt{2}}{\pi}Q. \tag{3.19}$$

Obviously, the last inequalities in (3.18) and (3.19) hold. The first inequalities in (3.18) and (3.19) can be written, respectively, as

$$\frac{H+T}{2} > L \quad \text{and} \quad \frac{5H+4Q}{9} > L.$$

We now prove that

$$\frac{H+T}{2} > \frac{5H+4Q}{9} > L. \tag{3.20}$$

The first inequality in (3.20) can be written as

$$\frac{H+8Q}{9} < T,$$

which is the left-hand side of (3.6). The second inequality in (3.20) is mentioned in [8, Table 2]. It can be written, by Remark 1.1, as

$$5(1-z^2) + 4\sqrt{1+z^2} > \frac{18z}{\ln\frac{1+z}{1-z}}. \tag{3.21}$$

For $0 < z < 1$, let

$$\xi(z) = \ln\frac{1+z}{1-z} - \frac{18z}{5(1-z^2) + 4\sqrt{1+z^2}}.$$

Differentiation yields

$$\xi'(z) = \frac{2\Big((5 - 7z^2 + 52z^4)\sqrt{1+z^2} - 5 + 45z^2 - 40z^4\Big)}{(1-z^2)(4 - 4z^2 + 5\sqrt{1+z^2})^2\sqrt{1+z^2}}.$$

By an elementary change of variable $z = \sqrt{y^2-1}$ $(1 < y < \sqrt{2})$, we find

$$\begin{aligned}
&(5 - 7z^2 + 52z^4)\sqrt{1+z^2} - 5 + 45z^2 - 40z^4 \\
&\quad = 52y^5 - 40y^4 - 111y^3 + 125y^2 + 64y - 90 \\
&\quad = 81(y-1) + 72(y-1)^2 + 249(y-1)^3 + 220(y-1)^4 + 52(y-1)^5 > 0.
\end{aligned}$$

We then obtain $\xi'(z) > 0$ for $0 < z < 1$. Hence, $\xi(z)$ is strictly increasing for $0 < z < 1$, and we have

$$\ln\frac{1+z}{1-z} - \frac{18z}{5(1-z^2)+4\sqrt{1+z^2}} = \xi(z) > \xi(0) = 0.$$

This means that (3.21) holds. Hence, the second inequality in (3.20) holds.

We then obtain (3.13) and (3.14) with $\mu_3 = \frac{1}{2}, \nu_3 = \frac{\pi}{4}, \mu_4 = \frac{4}{5}, \nu_4 = \frac{2\sqrt{2}}{\pi}$.

Conversely, if (3.13) and (3.14) are valid, then we get

$$\mu_3 < \frac{A-L}{T-L} = \frac{1-\frac{2z}{\ln\frac{1+z}{1-z}}}{\frac{z}{\arctan z} - \frac{2z}{\ln\frac{1+z}{1-z}}} < \nu_3 \quad \text{and} \quad \mu_4 < \frac{T-L}{Q-L} = \frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{\sqrt{1+z^2} - \frac{2z}{\ln\frac{1+z}{1-z}}} < \nu_4.$$

The limit relations

$$\lim_{z\to 0}\frac{A-L}{T-L} = \frac{1-\frac{2z}{\ln\frac{1+z}{1-z}}}{\frac{z}{\arctan z} - \frac{2z}{\ln\frac{1+z}{1-z}}} = \frac{1}{2}, \quad \lim_{z\to 1}\frac{A-L}{T-L} = \frac{1-\frac{2z}{\ln\frac{1+z}{1-z}}}{\frac{z}{\arctan z} - \frac{2z}{\ln\frac{1+z}{1-z}}} = \frac{\pi}{4},$$

$$\lim_{z\to 0}\frac{T-L}{Q-L} = \frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{\sqrt{1+z^2} - \frac{2z}{\ln\frac{1+z}{1-z}}} = \frac{4}{5}, \quad \lim_{z\to 1}\frac{T-L}{Q-L} = \frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{\sqrt{1+z^2} - \frac{2z}{\ln\frac{1+z}{1-z}}} = \frac{2\sqrt{2}}{\pi}$$

yield

$$\mu_3 \le \frac{1}{2}, \quad \nu_3 \ge \frac{\pi}{4}, \quad \mu_4 \le \frac{4}{5}, \quad \nu_4 \ge \frac{2\sqrt{2}}{\pi}.$$

The proof is complete. □

Appendix A: A proof of (2.9)

Elementary calculations reveal that

$$\begin{aligned}&\sin x(1+\sin^2 x) - x\cos x(1+x\sin x)\\ &\qquad = \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{1}{42}x^7 - \frac{1}{1134}x^9 + \sum_{n=5}^{\infty}(-1)^{n-1}h_n(x),\end{aligned}$$

where

$$h_n(x) = \frac{3^{2n+1} - (2n^2+n)2^{2n+1} + 8n - 3}{4\cdot(2n+1)!}x^{2n+1}.$$

Elementary calculations reveal that, for $0 < x < \pi/2$ and $n \ge 5$,

$$\frac{h_{n+1}(x)}{h_n(x)} = \frac{x^2 a_n}{b_n} < \frac{(\pi/2)^2 a_n}{b_n} < \frac{3a_n}{b_n},$$

where

$$a_n = 27\cdot 9^n - (16n^2 + 40n + 24)4^n + 8n + 5$$

and

$$b_n = 2(n+1)(2n+3)\Big(3\cdot 9^n - (4n^2+2n)4^n + 8n - 3\Big).$$

We find, for $n \geq 5$,

$$\begin{aligned} b_n - 3a_n &= (12n^2 + 30n - 63)9^n - 4(2n+3)(2n-3)(n+2)(n+1)4^n \\ &\quad + 32n^3 + 68n^2 - 6n - 33 \\ &> (12n^2 + 30n - 63)9^n - 4(2n+3)(2n-3)(n+2)(n+1)4^n \\ &= (12n^2 + 30n - 63)4^n \left\{ \left(\frac{9}{4}\right)^n - \frac{4(2n+3)(2n-3)(n+2)(n+1)}{12n^2 + 30n - 63} \right\} > 0. \end{aligned}$$

The last inequality can be proved by induction on n, and we omit it.

Hence, for all $0 < x < \pi/2$ and $n \geq 5$,

$$\frac{h_{n+1}(x)}{h_n(x)} < 1.$$

Therefore, for fixed $x \in (0, \pi/2)$, the sequence $n \longmapsto h_n(x)$ is strictly decreasing for $n \geq 5$. We then obtain, for $0 < x < \pi/2$,

$$\sin x(1 + \sin^2 x) - x \cos x(1 + x \sin x) > x^3 \left(\frac{1}{3} + \frac{2}{15}x^2 - \frac{1}{42}x^4 - \frac{1}{1134}x^6 \right) > 0.$$

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