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INEQUALITIES BETWEEN IDENTRIC MEAN AND CONVEX COMBINATIONS OF OTHER MEANS

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ABSTRACT. By using the asymptotic expansion method, Elezović conjectured inequalities between identric mean and convex combinations of other means. In this paper, we prove certain conjectures given by Elezović.

1. Introduction

Throughout this paper we assume that the numbers x and y are positive and unequal. Let

$$H = \frac{2xy}{x+y}, \quad G = \sqrt{xy}, \quad L = \frac{x-y}{\ln x - \ln y}, \quad I = \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{1/(y-x)},$$
$$A = \frac{x+y}{2}, \quad Q = \sqrt{\frac{x^2+y^2}{2}}, \quad N = \frac{x^2+y^2}{x+y}$$

be the harmonic, geometric, logarithmic, identric, arithmetic, root-square, and contraharmonic means of x and y, respectively. It is known (see [14, 17]) that

$$H < G < L < I < A < Q < N.$$

Sándor [12] proved that

$$\frac{2}{3}A + \frac{1}{3}G < I. \tag{1.1}$$

Alzer and Qiu [1] developed (1.1) to produce a double inequality. More precisely, these authors proved that the inequalities

$$\alpha A + (1-\alpha)G < I < \beta A + (1-\beta)G \tag{1.2}$$

hold if and only if

$$\alpha \le 2/3 \quad \text{and} \quad \beta \ge 2/e = 0.73575\ldots.$$

Zhu [19, Theorem 2] also considered (1.2). Subsequently, Zhu [20, Theorem 2] (see also [21, Theorem 5.4, Eq. (5.5)]) established a more general result and proved, for $0 < p \le 6/5$,

$$\alpha A^p + (1-\alpha)G^p < I^p < \beta A^p + (1-\beta)G^p \tag{1.3}$$

holds if and only if $\alpha \le 2/3$ and $\beta \ge (2/e)^p$. The choice $p = 1$ in (1.3) yields (1.2).

Trif [16] proved, for $p \ge 2$,

$$\alpha A^p + (1-\alpha)G^p < I^p < \beta A^p + (1-\beta)G^p \tag{1.4}$$

holds if and only if $\alpha \le (2/e)^p$ and $\beta \ge 2/3$.

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The choice $(p, \beta) = (2, 2/3)$ in the right-hand side of (1.4) yields

$$I^2 < \frac{2}{3}A^2 + \frac{1}{3}G^2, \tag{1.5}$$

which has been presented by Sándor and Trif [13].

Let $p > 0$. Kouba [10] proved that the inequality

$$I^p < \frac{2}{3}A^p + \frac{1}{3}G^p \tag{1.6}$$

holds if and only if $p \geq \ln(3/2)/\ln(e/2) = 1.3214\ldots$, and the reverse inequality holds if and only if $p \leq 6/5$.

Zhu [19, Theorem 3] proved that the inequalities

$$\alpha A + (1-\alpha)L < I < \beta A + (1-\beta)L \tag{1.7}$$

hold if and only if $\alpha \leq 1/2$ and $\beta \geq 2/e$. Subsequently, Zhu [21, Theorem 1.3] established a more general result and proved, for $0 < p \leq 8/5$,

$$\alpha A^p + (1-\alpha)L^p < I^p < \beta A^p + (1-\beta)L^p \tag{1.8}$$

holds if and only if $\alpha \leq 1/2$ and $\beta \geq (2/e)^p$. The choice $p = 1$ in (1.8) yields (1.7).

There is a large number of papers studying inequalities between linear combinations of means (see, for example, [3–5, 9, 11, 15, 17, 18]).

Recently, Elezović [6] proposed a new approach to this subject, using the concept of asymptotical expansion of means. More precisely, the author established inequalities of the form

$$aM_1 + bM_2 + cM_3 \geq 0 \tag{1.9}$$

where a, b, c are constants and M_k $(k = 1, 2, 3)$ are chosen from the class of elementary means given above. The inequality (1.9) is called optimal by the author if for any other choices of constants a^*, b^*, c^* such that (1.9) is valid, it follows that

$$a^*M_1 + b^*M_2 + c^*M_3 \geq aM_1 + bM_2 + cM_3 \geq 0.$$

In order to formulate such inequalities the author considers the asymptotic expansion of the means as follows:

$$M(x+s, x+t) = x + c_1(s,t) + \frac{c_2(s,t)}{x} + \frac{c_3(s,t)}{x^2} + \cdots.$$

By comparing the coefficients it is possible to derive several candidates for optimal inequalities. These possible optimal inequalities are systematically studied. The author obtained some critical values and also formulates several conjectures in connection with optimal inequalities.

Conjecture 1.1 (see [6, Conjecture 2.3])**.** *The following inequalities are valid:*

$$N + 6I \leq 7A, \quad Q + 3I \leq 4A, \quad A + L \leq 2I, \quad 2A + G \leq 3I \quad \textit{and} \quad 5A + H \geq 6I.$$

Conjecture 1.2 (see [6, Conjecture 4.2])**.** *The following inequalities are valid:*

$$(e-2)N + e \cdot I \geq 2(e-1)A \quad \textit{(typo corrected)},$$

$$(e-2)Q + (\sqrt{2}-1)eI \geq (e\sqrt{2}-2)A, \qquad 2A + (e-2)L \geq e \cdot I$$

$$2A + (e-2)G \geq e \cdot I \quad \textit{and} \quad 2A + (e-2)H \leq e \cdot I.$$

Conjecture 1.3 (see [6, Conjecture 4.3, (4.11)-(4.15)]). *The following inequalities are valid:*

$$N + (e-1)L \geq e \cdot I, \tag{1.10}$$

$$N + (e-1)G \geq e \cdot I, \tag{1.11}$$

$$N + (e-1)H \leq e \cdot I, \tag{1.12}$$

$$\sqrt{2} \cdot Q + (e-\sqrt{2})L \geq e \cdot I, \tag{1.13}$$

$$\sqrt{2} \cdot Q + (e-\sqrt{2})G \geq e \cdot I. \tag{1.14}$$

The typos of (1.10)-(1.12) have been corrected. See [7, 8, 17] for more details about comparison of means using asymptotic methods.

The aim of this paper is to offer a proof of Conjectures 1.1, 1.2 and 1.3. We also develop (1.10)-(1.14) to produce double inequalities.

Remark 1.1. *Let $\sqrt{x/y} = e^t$, and suppose $x > y$. Then $t > 0$, and the following identities hold true:*

$$\frac{H(x,y)}{G(x,y)} = \frac{1}{\cosh t}, \quad \frac{L(x,y)}{G(x,y)} = \frac{\sinh t}{t}, \quad \frac{I(x,y)}{G(x,y)} = e^{t\coth t - 1},$$
$$\frac{A(x,y)}{G(x,y)} = \cosh t, \quad \frac{Q(x,y)}{G(x,y)} = \sqrt{\cosh(2t)}, \quad \frac{N(x,y)}{G(x,y)} = \frac{\cosh(2t)}{\cosh t}.$$

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2. Proofs of Conjectures 1.1 and 1.2

Theorem 2.1. *The following double inequalities hold true:*

$$\frac{1}{7}N + \frac{6}{7}I < A < \frac{e-2}{2(e-1)}N + \frac{e}{2(e-1)}I \tag{2.1}$$

and

$$\frac{1}{4}Q + \frac{3}{4}I < A < \frac{e-2}{e\sqrt{2}-2}Q + \frac{(\sqrt{2}-1)e}{e\sqrt{2}-2}I. \tag{2.2}$$

Proof. By Remark 1.1, (2.1) and (2.2) can be written, respectively, as

$$\frac{1}{7} < f_1(t) < \frac{e-2}{2(e-1)} \quad \text{for} \quad t > 0 \tag{2.3}$$

and

$$\frac{1}{4} < f_2(t) < \frac{e-2}{e\sqrt{2}-2} \quad \text{for} \quad t > 0, \tag{2.4}$$

where

$$f_1(t) = \frac{\cosh t - e^{t\coth t - 1}}{\frac{\cosh(2t)}{\cosh t} - e^{t\coth t - 1}} \quad \text{and} \quad f_2(t) = \frac{\cosh t - e^{t\coth t - 1}}{\sqrt{\cosh(2t)} - e^{t\coth t - 1}}.$$

Elementary calculations reveal that

$$\lim_{t\to 0} f_1(t) = \frac{1}{7} = 0.14285714\ldots, \quad \lim_{t\to\infty} f_1(t) = \frac{e-2}{2(e-1)} = 0.20901164\ldots,$$
$$\lim_{t\to 0} f_2(t) = \frac{1}{4} = 0.25, \quad \lim_{t\to\infty} f_2(t) = \frac{e-2}{e\sqrt{2}-2} = 0.38947497\ldots.$$

In order prove (2.3) and (2.4), it suffices to show that $f_1(t)$ and $f_2(t)$ are both strictly increasing for $t > 0$.

Differentiation yields

$$\frac{\Big(\cosh(2t) - e^{t\coth t - 1}\cosh t\Big)^2}{t\cosh t + \sinh t} f_1'(t) = e^{t\coth t-1} - \frac{2\cosh t \sinh t}{t\cosh t + \sinh t}.$$

We claim that

$$e^{t\coth t-1} - \frac{2\cosh t\sinh t}{t\cosh t + \sinh t} > 0, \qquad t > 0. \tag{2.5}$$

It suffices to show

$$g_1(t) = t\coth t - 1 - \ln\frac{2\cosh t\sinh t}{t\cosh t + \sinh t} > 0, \qquad t > 0.$$

Differentiation yields

$$g_1'(t) = \frac{h_1(t)}{(t\cosh t + \sinh t)\sinh^2 t\cosh t},$$

with

$$\begin{aligned} h_1(t) &= -t\cosh t\sinh t - t^2\cosh^2 t + \cosh^4 t - 1 \\ &= -\frac{1}{2}t\sinh(2t) + \left(\frac{1}{2} - \frac{1}{2}t^2\right)\cosh(2t) + \frac{1}{8}\cosh(4t) + \frac{1}{2}\cosh(2t) - \frac{1}{2}t^2 - \frac{5}{8} \\ &= \sum_{n=3}^{\infty} \frac{2^{2n-1} - (2n^2+n-2)}{(2n)!} 2^{2n-2} t^{2n} > 0. \end{aligned}$$

We then obtain $g_1'(t) > 0$ for $t > 0$. So, $g_1(t)$ is strictly increasing for $t > 0$, and we have

$$g_1(t) > \lim_{x\to 0} g_1(x) = 0 \quad \text{for} \quad t > 0.$$

Thus, the claim (2.5) is proved.

We then obtain $f_1'(t) > 0$ for $t > 0$. Hence, $f_1(t)$ is strictly increasing for $t > 0$.

Differentiation yields

$$\frac{\sinh^2 t\sqrt{\cosh(2t)}\Big(\sqrt{\cosh(2t)} - e^{t\coth t - 1}\Big)^2}{g_2(t)} f_2'(t) = e^{t\coth t - 1} - \frac{\sinh^3 t}{g_2(t)},$$

where[1]

$$g_2(t) = 2t\cosh^2 t - \sinh t\cosh t - t - \sqrt{\cosh(2t)}(t\cosh t - \sinh t) > 0, \qquad t > 0. \tag{2.6}$$

We now prove

$$f_2'(t) > 0, \qquad t > 0,$$

[1]The inequality (2.6) is proved in the appendix.

it suffices to show that

$$h_2(t) = t\coth t - 1 - \ln\frac{\sinh^3 t}{g_2(t)} > 0, \qquad t > 0.$$

Differentiation yields

$$h_2'(t) = \frac{\lambda(t)}{g_2(t)\sqrt{\cosh(2t)}\left(\sqrt{\cosh(2t)} + \cosh t\right)},$$

with

$$\begin{aligned}\lambda(t) &= 2\cosh^4 t - (2t^2+2)\cosh^2 t - t\sinh t\cosh t + t^2\\ &= -t^2\cosh(2t) + \frac{1}{4}\cosh(4t) - \frac{1}{2}t\sinh(2t) - \frac{1}{4}\\ &= \sum_{n=3}^{\infty}\frac{4^n(4^{n-1}-n^2)}{(2n)!}t^{2n} > 0.\end{aligned}$$

We then obtain $h_2'(t) > 0$ for $t > 0$. So, $h_2(t)$ is strictly increasing for $t > 0$, and we have, for $t > 0$,

$$h_2(t) > \lim_{x\to 0} h_2(x) = 0 \Longrightarrow f_2'(t) > 0.$$

Hence, $f_2(t)$ is strictly increasing for $t > 0$. The proof is complete. □

Remark 2.1. *The inequality* (2.5) *can be rewritten as*

$$I > \frac{2AL}{A+L}. \tag{2.7}$$

This shows that I is larger than the harmonic mean of A and L.

Theorem 2.2. *The following double inequality hold true:*

$$\frac{2}{3}A + \frac{1}{3}G < I < \frac{2}{e}A + \frac{e-2}{e}G, \tag{2.8}$$

$$\frac{1}{2}A + \frac{1}{2}L < I < \frac{2}{e}A + \frac{e-2}{e}L, \tag{2.9}$$

$$\frac{2}{e}A + \frac{e-2}{e}H < I < \frac{5}{6}A + \frac{1}{6}H. \tag{2.10}$$

Proof. By Remark 1.1, (2.8) can be written for $t > 0$ as

$$\frac{2}{3} < F(t) < \frac{2}{e},$$

where

$$F(t) = \frac{e^{t\coth t - 1} - 1}{\cosh t - 1}.$$

Differentiation yields

$$\frac{\sinh^2 t(\cosh t - 1)}{\sinh t + t}F'(t) = \frac{\sinh t(1+\cosh t)}{\sinh t + t} - e^{t\coth t - 1}.$$

For $t > 0$, let

$$F_1(t) = \ln\frac{\sinh t(1+\cosh t)}{\sinh t + t} - (t\coth t - 1).$$

Differentiation yields

$$F_1'(t) = \frac{F_2(t)}{\sinh^2 t(\sinh t + t)},$$

with

$$F_2(t) = -2\cosh^2 t + t\sinh t\cosh t + 2 + t^2 = \sum_{n=3}^{\infty} \frac{2^{2n-1}(n-2)}{(2n)!} t^{2n} > 0.$$

We then obtain $F_1'(t) > 0$ for $t > 0$. So, $F_1(t)$ is strictly increasing for $t > 0$, and we have, for $t > 0$,

$$F_1(t) > \lim_{x\to 0} F_1(x) = 0 \quad \text{and} \quad F'(t) > 0.$$

Hence, $F(t)$ is strictly increasing for $t > 0$, and we have

$$\frac{2}{3} = \lim_{x\to 0} F(x) < F(t) < \lim_{x\to\infty} F(x) = \frac{2}{e}$$

for $t > 0$. Hence, (2.8) holds.

In fact, (2.8) $\Longrightarrow$ (2.9). More precisely, the following inequalities are true:

$$\frac{1}{2}A + \frac{1}{2}L < \frac{2}{3}A + \frac{1}{3}G < I < \frac{2}{e}A + \frac{e-2}{e}G < \frac{2}{e}A + \frac{e-2}{e}L. \tag{2.11}$$

Obviously, the last inequality in (2.11) holds. The first inequality in (2.11) can be written as

$$L < \frac{A+2G}{3},$$

which was proved by Carlson [2].

By Remark 1.1, (2.9) can be written for $t > 0$ as

$$\frac{1}{2} < \frac{e^{t\coth t - 1} - \frac{\sinh t}{t}}{\cosh t - \frac{\sinh t}{t}} < \frac{2}{e}.$$

We find

$$\lim_{t\to 0^+} \frac{e^{t\coth t - 1} - \frac{\sinh t}{t}}{\cosh t - \frac{\sinh t}{t}} = \frac{1}{2} \quad \text{and} \quad \lim_{t\to\infty} \frac{e^{t\coth t - 1} - \frac{\sinh t}{t}}{\cosh t - \frac{\sinh t}{t}} = \frac{2}{e}.$$

Hence, (2.9) is valid and optimal.

By Remark 1.1, (2.10) can be written as

$$\frac{2}{e} < u(t) < \frac{5}{6}, \qquad t > 0, \tag{2.12}$$

where

$$u(t) = \frac{e^{t\coth t - 1} - \frac{1}{\cosh t}}{\cosh t - \frac{1}{\cosh t}}.$$

Elementary calculations reveal that

$$\lim_{t\to 0} u(t) = \frac{5}{6} = 0.83333333\ldots, \quad \lim_{t\to\infty} u(t) = \frac{2}{e} = 0.73575888\ldots.$$

In order prove (2.12), it suffices to show that $u(t)$ is strictly decreasing for $t > 0$.

Noting that (2.5) holds, we find

$$u'(t) = -\frac{e(t\cosh t + \sinh t)}{\sinh^4 t}\left(e^{t\coth t - 1} - \frac{2\sinh t\cosh t}{t\cosh t + \sinh t}\right) < 0.$$

This proves that $u(t)$ is strictly decreasing for $t > 0$. Hence, (2.10) holds. The proof is complete. $\square$

Remark 2.2. *In [1]. the left and right sides of* (2.8) *were proved, respectively. We here provide a unified proof.*

3. Proof of Conjecture 1.3

Theorem 3.1. *The following double inequalities hold true:*

$$\frac{1}{8}N + \frac{7}{8}L < I < \frac{1}{e}N + \frac{e-1}{e}L, \tag{3.1}$$

$$\frac{2}{9}N + \frac{7}{9}G < I < \frac{1}{e}N + \frac{e-1}{e}G, \tag{3.2}$$

$$\frac{1}{e}N + \frac{e-1}{e}H < I < \frac{5}{12}N + \frac{7}{12}H, \tag{3.3}$$

$$\frac{1}{5}Q + \frac{4}{5}L < I < \frac{\sqrt{2}}{e}Q + \frac{e-\sqrt{2}}{e}L, \tag{3.4}$$

$$\frac{1}{3}Q + \frac{2}{3}G < I < \frac{\sqrt{2}}{e}Q + \frac{e-\sqrt{2}}{e}G. \tag{3.5}$$

Proof. We point out that (2.10) and (3.3) are the same, by identity $N + H = 2A$.

By Remark 1.1, (3.1), (3.2), (3.4) and (3.5) can be written for $t > 0$ as

$$\frac{1}{8} < U_1(t) < \frac{1}{e}, \quad \frac{2}{9} < U_2(t) < \frac{1}{e}, \quad \frac{1}{5} < U_3(t) < \frac{\sqrt{2}}{e}, \quad \frac{1}{3} < U_4(t) < \frac{\sqrt{2}}{e},$$

respectively, where

$$U_1(t) = \frac{e^{t\coth t-1} - \frac{\sinh t}{t}}{\frac{\cosh(2t)}{\cosh t} - \frac{\sinh t}{t}}, \quad U_2(t) = \frac{e^{t\coth t-1} - 1}{\frac{\cosh(2t)}{\cosh t} - 1},$$
$$U_3(t) = \frac{e^{t\coth t-1} - \frac{\sinh t}{t}}{\sqrt{\cosh(2t)} - \frac{\sinh t}{t}}, \quad U_4(t) = \frac{e^{t\coth t-1} - 1}{\sqrt{\cosh(2t)} - 1}.$$

Elementary calculations reveal that

$$\lim_{t\to0} U_1(t) = \frac{1}{8} = 0.125, \quad \lim_{t\to\infty} U_1(t) = \frac{1}{e} = 0.367\ldots,$$
$$\lim_{t\to0} U_2(t) = \frac{2}{9} = 0.222\ldots, \quad \lim_{t\to\infty} U_2(t) = \frac{1}{e},$$
$$\lim_{t\to0} U_3(t) = \frac{1}{5}, \quad \lim_{t\to\infty} U_3(t) = \frac{\sqrt{2}}{e} = 0.5202\ldots,$$
$$\lim_{t\to0} U_4(t) = \frac{1}{3}, \quad \lim_{t\to\infty} U_4(t) = \frac{\sqrt{2}}{e}.$$

In order prove (3.1), (3.2), (3.4) and (3.5), it suffices to show that $U_k(t)$ $(k = 1, 2, 3, 4)$ are strictly increasing for $t > 0$. Following the same method as was used in the proof of Theorem 2.1, we can prove the monotonicity properties of $U_k(t)$ $(k = 1, 2, 3, 4)$. Here we only prove the monotonicity property of $U_2(t)$. The proofs of the monotonicity properties of $U_k(t)$ $(k = 1, 3, 4)$ are analogous.

Differentiation yields

$$\frac{(\cosh t-1)(2\cosh t+1)^2\sinh^2 t}{2t\cosh^2 t+\cosh^2 t\sinh t+t\cosh t+\cosh t\sinh t+\sinh t}U_2'(t)$$
$$=\frac{(\cosh t+1)(2\cosh^2 t+1)\sinh t}{2t\cosh^2 t+\cosh^2 t\sinh t+t\cosh t+\cosh t\sinh t+\sinh t}-e^{t\coth t-1}.$$

In order prove $U_2'(t)>0$ for $t>0$, it suffices to show that

$$V_2(t)>0, \qquad t>0,$$

where

$$V_2(t)=\ln\left(\frac{(\cosh t+1)(2\cosh^2 t+1)\sinh t}{2t\cosh^2 t+\cosh^2 t\sinh t+t\cosh t+\cosh t\sinh t+\sinh t}\right)-(t\coth t-1).$$

Differentiation yields

$$V_2'(t)=\frac{(2\cosh t+1)W_2(t)}{\sinh^2 t(2\cosh^2 t+1)(2t\cosh^2 t+\cosh^2 t\sinh t+t\cosh t+\cosh t\sinh t+\sinh t)},$$

with[2]

$$W_2(t)=-2\cosh^5 t+2t\cosh^4 t\sinh t+(2t^2-2)\cosh^3 t-t\cosh^2 t\sinh t$$
$$+(t^2+4)\cosh t+2t\sinh t>0 \quad \text{for} \quad t>0. \tag{3.6}$$

We then obtain $V_2'(t)>0$ for $t>0$. So, $V_2(t)$ is strictly increasing for $t>0$. and we have, for $t>0$,

$$V_2(t)>\lim_{t\to 0}V_2(t)=0 \quad \text{and} \quad U_2'(t)>0.$$

Hence, $U_2(t)$ is strictly increasing for $t>0$. The proof is complete. □

Appendix A: A proof of (2.6)

Elementary calculations reveal that

$$\Big(2t\cosh^2 t-\sinh t\cosh t-t\Big)^2-\Big(\sqrt{\cosh(2t)}(t\cosh t-\sinh t)\Big)^2$$
$$=\sinh^2 t(2t^2\cosh^2 t-\cosh^2 t-t^2+1)=t^2\cosh(2t)+\frac{1}{2}-\frac{1}{2}\cosh(2t)$$
$$=\sum_{n=2}^{\infty}\frac{(2n+1)(n-1)2^{2n-1}}{(2n)!}t^{2n}>0.$$

Noting that

$$2t\cosh^2 t-\sinh t\cosh t-t=t\cosh(2t)-\frac{1}{2}\sinh(2t)=\frac{1}{2}\cosh(2t)\Big(2t-\tanh(2t)\Big)>0$$

and

$$t\cosh t-\sinh t=\cosh t(t-\tanh t)>0$$

holds for $t>0$, we obtain, for $t>0$,

$$g_2(t)=2t\cosh^2 t-\sinh t\cosh t-t-\sqrt{\cosh(2t)}(t\cosh t-\sinh t)>0.$$

Appendix B: A proof of (3.6)

[2]The inequality (3.6) is proved in the appendix.

$$\begin{aligned}
W_2(t) &= -\frac{1}{8}\Big(\cosh(5t)+5\cosh(3t)+10\cosh t\Big)+\frac{1}{4}t\sinh t\Big(\cosh(4t)+4\cosh(2t)+3\Big)\\
&\quad+\left(\frac{1}{2}t^2-\frac{1}{2}\right)\Big(\cosh(3t)+3\cosh t\Big)-\frac{1}{2}t\sinh t\Big(\cosh(2t)+1\Big)\\
&\quad+(t^2+4)\cosh t+2t\sinh t\\
&= -\frac{1}{8}\cosh(5t)+\left(\frac{1}{2}t^2-\frac{9}{8}\right)\cosh(3t)+\left(\frac{5}{2}t^2+\frac{5}{4}\right)\cosh t+\frac{9}{4}t\sinh t\\
&\quad+\frac{1}{4}t\sinh t\cosh(4t)+\frac{1}{2}t\sinh t\cosh(2t)\\
&= \frac{1}{8}t\sinh(5t)-\frac{1}{8}\cosh(5t)+\frac{1}{8}t\sinh(3t)+\left(\frac{1}{2}t^2-\frac{9}{8}\right)\cosh(3t)\\
&\quad+2t\sinh t+\left(\frac{5}{2}t^2+\frac{5}{4}\right)\cosh t\\
&= \sum_{n=3}^{\infty}\frac{(18n-45)\cdot 25^n+(80n^2-10n-405)\cdot 9^n+3600n^2-360n+450}{360\cdot(2n)!}t^{2n}>0.
\end{aligned}$$

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