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ON SOME OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS OF ABSOLUTELY CONTINUOUS FUNCTIONS

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Abstract. In this paper we establish some new Ostrowski and trapezoid type inequalities for the generalized Riemann-Liouville fractional integrals of absolutely continuous functions. Some examples for weighted integrals and for Hadamard fractional integrals are also provided.

1. Introduction

Let $f:[a,b]\rightarrow\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)\,dt$$

for $a<x\leq b$ and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)\,dt$$

for $a\leq x<b$, where Γ is the *Gamma function.* For $\alpha=0$, they are defined as $J_{a+}^{0}f(x)=J_{b-}^{0}f(x)=f(x)$ for $x\in(a,b)$.

For several inequalities for Riemann-Liouville fractional integrals see [1]-[5], [13]-[30] and the references therein.

Let (a,b) with $-\infty\leq a<b\leq\infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and α a complex number with $\operatorname{Re}(\alpha)>0$. Also, let g be a *strictly increasing function* on (a,b), having a continuous derivative g' on (a,b). Following [21, p. 100], we introduce the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ by

$$I_{a+,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{g'(t)\,f(t)\,dt}{[g(x)-g(t)]^{1-\alpha}},\ a<x\leq b \tag{1.1}$$

and

$$I_{b-,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\frac{g'(t)\,f(t)\,dt}{[g(t)-g(x)]^{1-\alpha}},\ a\leq x<b. \tag{1.2}$$

For $g(t)=t$ we have the classical *Riemann-Liouville fractional integrals* introduced above while for the logarithmic function $g(t)=\ln t$ we have the *Hadamard*

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fractional integrals [21, p. 111]

$$H_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t)\,dt}{t},\ 0 \le a < x \le b \tag{1.3}$$

and

$$H_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \frac{f(t)\,dt}{t},\ 0 \le a < x < b. \tag{1.4}$$

One can consider the function $g(t) = -t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha} t^{\alpha+1}},\ 0 \le a < x \le b \tag{1.5}$$

and

$$R_{b-}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha} t^{\alpha+1}},\ 0 \le a < x < b. \tag{1.6}$$

Recall also the concept of *generalized mean* generated by a function. If g is a function which maps an interval I of the real line to the real numbers and is both continuous and injective then we can define the *g-mean of two numbers* $a, b \in I$ by

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a,b) = A(a,b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0,\infty)$ and $g(t) = \ln t$, then $M_g(a,b) = G(a,b) := \sqrt{ab}$, the *geometric mean*. If $I = (0,\infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a,b) = H(a,b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0,\infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a,b) = M_p(a,b) := \left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p.

If $w : (a,b) \to (0,\infty)$ is continuous, then the function $g(t) := \int_a^t w(s)\,ds$ is strictly increasing on (a,b) having a continuous derivative g' on (a,b) and then we have

$$I_{a+,W}^{\alpha} f(x) := I_{a+,g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{w(t) f(t)\,dt}{\left[\int_t^x w(s)\,ds\right]^{1-\alpha}},\ a < x \le b \tag{1.7}$$

and

$$I_{b-,W}^{\alpha} f(x) := I_{b-,g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{w(t) f(t)\,dt}{\left[\int_x^t w(s)\,ds\right]^{1-\alpha}},\ a \le x < b. \tag{1.8}$$

For $\alpha = 1$ we get the weighted integrals

$$I_{a+,W} f(x) := \int_a^x w(t) f(t)\,dt \text{ and } I_{b-,W}^{\alpha} f(x) := \int_x^b w(t) f(t)\,dt$$

where $x \in (a,b)$.

In [17], we obtained amongst other the following trapezoid type inequality

$$
\begin{aligned}
(1.9)\quad & \left| I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)-\frac{(g(x)-g(a))^{\alpha}f(a)+(g(b)-g(x))^{\alpha}f(b)}{\Gamma(\alpha+1)}\right| \\
& \leq \frac{1}{\Gamma(\alpha+2)}\left[\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}(g(x)-g(a))^{\alpha+1}+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}(g(b)-g(x))^{\alpha+1}\right] \\
& \leq \frac{1}{\Gamma(\alpha+2)}\left\|\frac{f'}{g'}\right\|_{[a,b],\infty}\left[(g(x)-g(a))^{\alpha+1}+(g(b)-g(x))^{\alpha+1}\right]
\end{aligned}
$$

for any $x\in(a,b)$ and provided $\frac{f'}{g'}\in L_{\infty}[a,b]$, while in [16] we obtained the Ostrowski type inequality

$$
\begin{aligned}
(1.10)\quad & \left| I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)-\frac{(g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}}{\Gamma(\alpha+1)}f(x)\right| \\
& \leq \frac{1}{\Gamma(\alpha+2)}\left[\left\|\frac{f'}{g'}\right\|_{[x,a],\infty}(g(x)-g(a))^{\alpha+1}+\left\|\frac{f'}{g'}\right\|_{[b,x],\infty}(g(b)-g(x))^{\alpha+1}\right] \\
& \leq \frac{1}{\Gamma(\alpha+2)}\left\|\frac{f'}{g'}\right\|_{[a,b],\infty}\left[(g(x)-g(a))^{\alpha+1}+(g(b)-g(x))^{\alpha+1}\right]
\end{aligned}
$$

if $\frac{f'}{g'}\in L_{\infty}[a,b]$ and for any $x\in(a,b)$.

If $w:(a,b)\to(0,\infty)$ is continuous, then by taking the function $g(t):=\int_a^t w(s)\,ds$ and $\alpha=1$ in (1.9) and (1.10) , then we get the following weighted integral inequalities

$$
\begin{aligned}
(1.11)\quad & \left|\int_a^b w(t)f(t)\,dt-f(a)\int_a^x w(s)\,ds-f(b)\int_x^b w(s)\,ds\right| \\
& \leq \frac{1}{2}\left[\left\|\frac{f'}{w}\right\|_{[a,x],\infty}\left(\int_a^x w(s)\,ds\right)^2+\left\|\frac{f'}{w}\right\|_{[x,b],\infty}\left(\int_x^b w(s)\,ds\right)^2\right] \\
& \leq \frac{1}{2}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left[\left(\int_a^x w(s)\,ds\right)^2+\left(\int_x^b w(s)\,ds\right)^2\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(1.12)\quad & \left|\int_a^b w(t)f(t)\,dt-f(x)\int_a^b w(s)\,ds\right| \\
& \leq \frac{1}{2}\left[\left\|\frac{f'}{w}\right\|_{[a,x],\infty}\left(\int_a^x w(s)\,ds\right)^2+\left\|\frac{f'}{w}\right\|_{[x,b],\infty}\left(\int_x^b w(s)\,ds\right)^2\right] \\
& \leq \frac{1}{2}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left[\left(\int_a^x w(s)\,ds\right)^2+\left(\int_x^b w(s)\,ds\right)^2\right]
\end{aligned}
$$

for any $x\in(a,b)$.

If we take in these two inequalities $w \equiv 1$, then we recapture the well known generalized trapezoid and Ostrowski inequalities

$$\left|\int_a^b f(t)\,dt - f(a)(x-a) - f(b)(b-x)\right| \le \frac{1}{2}\left[(x-a)^2\|f'\|_{[a,x],\infty} + (b-x)^2\|f'\|_{[x,b],\infty}\right] \le \left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty} \tag{1.13}$$

and

$$\left|\int_a^b f(t)\,dt - f(x)(b-a)\right| \le \frac{1}{2}\left[\|f'\|_{[a,x],\infty}(x-a)^2 + \|f'\|_{[x,b],\infty}(b-x)^2\right] \le \left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty} \tag{1.14}$$

for any $x \in (a,b)$.

In this paper we establish some new Ostrowski and Trapezoid type inequalities for the following operators

$$\frac{\Gamma(\alpha+1)}{2}\left[\frac{I_{a+,g}^{\alpha} f(x)}{[g(x)-g(a)]^{\alpha}} + \frac{I_{b-,g}^{\alpha} f(x)}{[g(b)-g(x)]^{\alpha}}\right]$$

and

$$\frac{\Gamma(\alpha+1)}{2}\left[\frac{I_{x-,g}^{\alpha} f(a)}{[g(x)-g(a)]^{\alpha}} + \frac{I_{x+,g}^{\alpha} f(b)}{[g(b)-g(x)]^{\alpha}}\right]$$

where $x \in (a,b)$, associated to the generalized Riemann-Liouville fractional integrals of absolutely continuous functions. Some examples for weighted integrals and for Hadamard fractional integrals are also provided.

2. Some Preliminary facts

We have the following equalities of interest:

Lemma 1. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then for any $x \in (a,b)$ we have*

$$\frac{\Gamma(\alpha+1)}{2}\left[\frac{I_{a+,g}^{\alpha} f(x)}{[g(x)-g(a)]^{\alpha}} + \frac{I_{b-,g}^{\alpha} f(x)}{[g(b)-g(x)]^{\alpha}}\right] = \frac{f(a)+f(b)}{2} + \frac{1}{2[g(x)-g(a)]^{\alpha}}\int_a^x (g(x)-g(t))^{\alpha} f'(t)\,dt - \frac{1}{2[g(b)-g(x)]^{\alpha}}\int_x^b (g(t)-g(x))^{\alpha} f'(t)\,dt \tag{2.1}$$

and

$$\frac{\Gamma(\alpha+1)}{2}\left[\frac{I_{x-,g}^{\alpha}f(a)}{[g(x)-g(a)]^{\alpha}}+\frac{I_{x+,g}^{\alpha}f(b)}{[g(b)-g(x)]^{\alpha}}\right]=f(x) \tag{2.2}$$
$$+\frac{1}{2[g(b)-g(x)]^{\alpha}}\int_x^b (g(b)-g(t))^{\alpha} f'(t)\,dt$$
$$-\frac{1}{2[g(x)-g(a)]^{\alpha}}\int_a^x (g(t)-g(a))^{\alpha} f'(t)\,dt.$$

Proof. Since $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function on $[a,b]$, then the Lebesgue integrals

$$\int_a^x (g(x)-g(t))^{\alpha} f'(t)\,dt \text{ and } \int_x^b (g(t)-g(x))^{\alpha} f'(t)\,dt$$

exist and integrating by parts, we have

$$\frac{1}{\Gamma(\alpha+1)}\int_a^x (g(x)-g(t))^{\alpha} f'(t)\,dt \tag{2.3}$$
$$=\frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1} g'(t) f(t)\,dt-\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha} f(a)$$
$$=I_{a+,g}^{\alpha}f(x)-\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha} f(a)$$

for $a<x\le b$ and

$$\frac{1}{\Gamma(\alpha+1)}\int_x^b (g(t)-g(x))^{\alpha} f'(t)\,dt \tag{2.4}$$
$$=\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha} f(b)-\frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1} g'(t) f(t)\,dt$$
$$=\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha} f(b)-I_{b-,g}^{\alpha}f(x)$$

for $a\le x<b$.

From (2.3), we then have

$$\frac{\Gamma(\alpha+1)\,I_{a+,g}^{\alpha}f(x)}{(g(x)-g(a))^{\alpha}}=f(a)+\frac{1}{(g(x)-g(a))^{\alpha}}\int_a^x (g(x)-g(t))^{\alpha} f'(t)\,dt$$

for $a<x\le b$ and from (2.4) we have

$$\frac{\Gamma(\alpha+1)\,I_{b-,g}^{\alpha}f(x)}{(g(b)-g(x))^{\alpha}}=f(b)-\frac{1}{(g(b)-g(x))^{\alpha}}\int_x^b (g(t)-g(x))^{\alpha} f'(t)\,dt,$$

for $a\le x<b$, which by addition and division by 2 give (2.1).

By the definition of generalized Riemann-Liouville fractional integrals, we have

$$I_{x+,g}^{\alpha}f(b)=\frac{1}{\Gamma(\alpha)}\int_x^b (g(b)-g(t))^{\alpha-1} g'(t) f(t)\,dt$$

for $a\le x<b$ and

$$I_{x-,g}^{\alpha}f(a)=\frac{1}{\Gamma(\alpha)}\int_a^x (g(t)-g(a))^{\alpha-1} g'(t) f(t)\,dt$$

for $a<x\le b$.

Since $f:[a,b]\to\mathbb{C}$ is an absolutely continuous function $[a,b]$, then the Lebesgue integrals

$$\int_a^x (g(t)-g(a))^{\alpha} f'(t)\,dt \text{ and } \int_x^b (g(b)-g(t))^{\alpha} f'(t)\,dt$$

exist and integrating by parts, we have

$$\begin{aligned}(2.5)\quad &\frac{1}{\Gamma(\alpha+1)}\int_a^x (g(t)-g(a))^{\alpha} f'(t)\,dt\\ &=\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha} f(x)-\frac{1}{\Gamma(\alpha)}\int_a^x (g(t)-g(a))^{\alpha-1} g'(t) f(t)\,dt\\ &=\frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha} f(x)-I_{x-,g}^{\alpha} f(a)\end{aligned}$$

for $a<x\le b$ and

$$\begin{aligned}(2.6)\quad &\frac{1}{\Gamma(\alpha+1)}\int_x^b (g(b)-g(t))^{\alpha} f'(t)\,dt\\ &=\frac{1}{\Gamma(\alpha)}\int_x^b (g(b)-g(t))^{\alpha-1} g'(t) f(t)\,dt-\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha} f(x)\\ &=I_{x+,g}^{\alpha} f(b)-\frac{1}{\Gamma(\alpha+1)}(g(b)-g(x))^{\alpha} f(x)\end{aligned}$$

for $a\le x<b$.

From (2.5) we get

$$\frac{\Gamma(\alpha+1) I_{x-,g}^{\alpha} f(a)}{(g(x)-g(a))^{\alpha}}=f(x)-\frac{1}{(g(x)-g(a))^{\alpha}}\int_a^x (g(t)-g(a))^{\alpha} f'(t)\,dt$$

for $a<x\le b$ and from (2.6)

$$\frac{\Gamma(\alpha+1) I_{x+,g}^{\alpha} f(b)}{(g(b)-g(x))^{\alpha}}=f(x)+\frac{1}{(g(b)-g(x))^{\alpha}}\int_x^b (g(b)-g(t))^{\alpha} f'(t)\,dt,$$

for $a\le x<b$, which by addition and division by 2 produce (2.2). □

Corollary 1. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}(2.7)\quad &\frac{2^{\alpha-1}\Gamma(\alpha+1)}{[g(b)-g(a)]^{\alpha}}\left[I_{a+,g}^{\alpha} f(M_g(a,b))+I_{b-,g}^{\alpha} f(M_g(a,b))\right]=\frac{f(a)+f(b)}{2}\\ &\quad+\frac{2^{\alpha-1}}{[g(b)-g(a)]^{\alpha}}\left[\int_a^{M_g(a,b)}\left(\frac{g(a)+g(b)}{2}-g(t)\right)^{\alpha} f'(t)\,dt\right.\\ &\qquad\left.-\int_{M_g(a,b)}^b\left(g(t)-\frac{g(a)+g(b)}{2}\right)^{\alpha} f'(t)\,dt\right]\end{aligned}$$

and

$$\begin{aligned}(2.8)\quad & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{[g(b)-g(a)]^{\alpha}}\left[I^{\alpha}_{M_g(a,b)-,g}f(a)+I^{\alpha}_{M_g(a,b)+,g}f(b)\right]=f(M_g(a,b)) \\ & +\frac{2^{\alpha-1}}{[g(b)-g(a)]^{\alpha}}\left[\int_{M_g(a,b)}^{b}(g(b)-g(t))^{\alpha}f'(t)\,dt\right. \\ & \left.-\int_{a}^{M_g(a,b)}(g(t)-g(a))^{\alpha}f'(t)\,dt\right].\end{aligned}$$

By taking the integral mean in Lemma 1 we also have:

Corollary 2. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}(2.9)\quad & \frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{b-a}\int_a^b\frac{I^{\alpha}_{a+,g}f(x)\,dx}{[g(x)-g(a)]^{\alpha}}+\frac{1}{b-a}\int_a^b\frac{I^{\alpha}_{b-,g}f(x)\,dx}{[g(b)-g(x)]^{\alpha}}\right] \\ & =\frac{f(a)+f(b)}{2}+\frac{1}{2(b-a)}\int_a^b\frac{\int_a^x(g(x)-g(t))^{\alpha}f'(t)\,dt}{[g(x)-g(a)]^{\alpha}}dx \\ & -\frac{1}{2(b-a)}\int_a^b\frac{\int_x^b(g(t)-g(x))^{\alpha}f'(t)\,dt}{[g(b)-g(x)]^{\alpha}}dx\end{aligned}$$

and

$$\begin{aligned}(2.10)\quad & \frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{b-a}\int_a^b\frac{I^{\alpha}_{x-,g}f(a)\,dx}{[g(x)-g(a)]^{\alpha}}+\frac{1}{b-a}\int_a^b\frac{I^{\alpha}_{x+,g}f(b)\,dx}{[g(b)-g(x)]^{\alpha}}\right] \\ & =\frac{1}{b-a}\int_a^b f(x)\,dx+\frac{1}{2(b-a)}\int_a^b\frac{\int_x^b(g(b)-g(t))^{\alpha}f'(t)\,dt}{[g(b)-g(x)]^{\alpha}}dx \\ & -\frac{1}{2(b-a)}\int_a^b\frac{\int_a^x(g(t)-g(a))^{\alpha}f'(t)\,dt}{[g(x)-g(a)]^{\alpha}}dx.\end{aligned}$$

Remark 1. *The equality (2.7) has been obtained in an equivalent form in* [17] *while (2.8) in* [16].

If $w:(a,b)\to(0,\infty)$ is continuous, then by taking the function $g(t):=\int_a^t w(s)\,ds$ in Lemma 1 above, then we get

$$\begin{aligned}(2.11)\quad & \frac{\Gamma(\alpha+1)}{2}\left[\frac{I^{\alpha}_{a+,W}f(x)}{\left[\int_a^x w(s)\,ds\right]^{\alpha}}+\frac{I^{\alpha}_{b-,W}f(x)}{\left[\int_x^b w(s)\,ds\right]^{\alpha}}\right]=\frac{f(a)+f(b)}{2} \\ & +\frac{1}{2\left[\int_a^x w(s)\,ds\right]^{\alpha}}\int_a^x\left(\int_t^x w(s)\,ds\right)^{\alpha}f'(t)\,dt \\ & -\frac{1}{2\left[\int_x^b w(s)\,ds\right]^{\alpha}}\int_x^b\left(\int_x^t w(s)\,ds\right)^{\alpha}f'(t)\,dt\end{aligned}$$

and

$$\frac{\Gamma(\alpha+1)}{2}\left[\frac{I^{\alpha}_{x-,W}f(a)}{\left[\int_a^x w(s)\,ds\right]^{\alpha}}+\frac{I^{\alpha}_{x+,W}f(b)}{\left[\int_x^b w(s)\,ds\right]^{\alpha}}\right]=f(x)$$
$$+\frac{1}{2\left[\int_x^b w(s)\,ds\right]^{\alpha}}\int_x^b\left(\int_t^b w(s)\,ds\right)^{\alpha}f'(t)\,dt$$
$$-\frac{1}{2\left[\int_a^x w(s)\,ds\right]^{\alpha}}\int_a^x\left(\int_a^t w(s)\,ds\right)^{\alpha}f'(t)\,dt. \tag{2.12}$$

For $\alpha=1$ we get the weighted integral equalities

$$\frac{1}{2}\left[\frac{\int_a^x w(t)f(t)\,dt}{\int_a^x w(t)\,dt}+\frac{\int_x^b w(t)f(t)\,dt}{\int_x^b w(t)\,dt}\right]=\frac{f(a)+f(b)}{2}$$
$$+\frac{1}{2}\left[\frac{1}{\int_a^x w(s)\,ds}\int_a^x\left(\int_t^x w(s)\,ds\right)f'(t)\,dt-\frac{1}{\int_x^b w(s)\,ds}\int_x^b\left(\int_x^t w(s)\,ds\right)f'(t)\,dt\right] \tag{2.13}$$

and

$$\frac{1}{2}\left[\frac{\int_a^x w(t)f(t)\,dt}{\int_a^x w(t)\,dt}+\frac{\int_x^b w(t)f(t)\,dt}{\int_x^b w(t)\,dt}\right]=f(x)$$
$$+\frac{1}{2}\left[\frac{1}{\int_x^b w(s)\,ds}\int_x^b\left(\int_t^b w(s)\,ds\right)f'(t)\,dt-\frac{1}{\int_a^x w(s)\,ds}\int_a^x\left(\int_a^t w(s)\,ds\right)f'(t)\,dt\right]. \tag{2.14}$$

In particular, if $w\equiv 1$ then we have

$$\frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]$$
$$=\frac{f(a)+f(b)}{2}+\frac{1}{2}\left[\frac{1}{x-a}\int_a^x (x-t)f'(t)\,dt-\frac{1}{b-x}\int_x^b (t-x)f'(t)\,dt\right] \tag{2.15}$$

and

$$\frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]$$
$$=f(x)+\frac{1}{2}\left[\frac{1}{b-x}\int_x^b (b-t)f'(t)\,dt-\frac{1}{x-a}\int_a^x (t-a)f'(t)\,dt\right]. \tag{2.16}$$

3. Inequalities in Terms of ∞-Norm

The following result holds:

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g'*

on (a,b). *Assume that* $\frac{f'}{g'}\in L_{\infty}[a,b]$, *then for any* $x\in(a,b)$ *we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2}\left[\frac{I_{a+,g}^{\alpha}f(x)}{[g(x)-g(a)]^{\alpha}}+\frac{I_{b-,g}^{\alpha}f(x)}{[g(b)-g(x)]^{\alpha}}\right]\right| \tag{3.1}$$
$$\leq\frac{1}{2(\alpha+1)}\left[\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}(g(x)-g(a))+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}(g(b)-g(x))\right]$$
$$\leq\frac{1}{2(\alpha+1)}\begin{cases}\left\|\frac{f'}{g'}\right\|_{[a,b],\infty}(g(b)-g(a));\\ \\ \left(\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}^{p}+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}^{p}\right)^{1/p}\left[(g(x)-g(a))^{q}+(g(b)-g(x))^{q}\right]^{1/q},\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}\right]\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(b)+g(a)}{2}\right|\right]\end{cases}$$

and

$$\left|\frac{\Gamma(\alpha+1)}{2}\left[\frac{I_{x-,g}^{\alpha}f(a)}{[g(x)-g(a)]^{\alpha}}+\frac{I_{x+,g}^{\alpha}f(b)}{[g(b)-g(x)]^{\alpha}}\right]-f(x)\right| \tag{3.2}$$
$$\leq\frac{1}{2(\alpha+1)}\left[\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}(g(x)-g(a))+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}(g(b)-g(x))\right]$$
$$\leq\frac{1}{2(\alpha+1)}\begin{cases}\left\|\frac{f'}{g'}\right\|_{[a,b],\infty}(g(b)-g(a));\\ \\ \left(\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}^{p}+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}^{p}\right)^{1/p}\left[(g(x)-g(a))^{q}+(g(b)-g(x))^{q}\right]^{1/q},\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}\right]\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(b)+g(a)}{2}\right|\right].\end{cases}$$

Proof. By Lemma 1 we have for $x\in(a,b)$

$$\left|\frac{\Gamma(\alpha+1)}{2}\left[\frac{I_{a+,g}^{\alpha}f(x)}{[g(x)-g(a)]^{\alpha}}+\frac{I_{b-,g}^{\alpha}f(x)}{[g(b)-g(x)]^{\alpha}}\right]-\frac{f(a)+f(b)}{2}\right| \tag{3.3}$$
$$\leq\frac{1}{2[g(x)-g(a)]^{\alpha}}\left|\int_{a}^{x}(g(x)-g(t))^{\alpha}f'(t)\,dt\right|+\frac{1}{2[g(b)-g(x)]^{\alpha}}\left|\int_{x}^{b}(g(t)-g(x))^{\alpha}f'(t)\,dt\right|$$
$$\leq\frac{1}{2[g(x)-g(a)]^{\alpha}}\int_{a}^{x}(g(x)-g(t))^{\alpha}|f'(t)|\,dt+\frac{1}{2[g(b)-g(x)]^{\alpha}}\int_{x}^{b}(g(t)-g(x))^{\alpha}|f'(t)|\,dt=:A(x)$$

and, similarly

$$\text{(3.4)}\quad \left|\frac{\Gamma(\alpha+1)}{2}\left[\frac{I_{x-,g}^{\alpha}f(a)}{[g(x)-g(a)]^{\alpha}}+\frac{I_{x+,g}^{\alpha}f(b)}{[g(b)-g(x)]^{\alpha}}\right]-f(x)\right|$$
$$\leq \frac{1}{2[g(b)-g(x)]^{\alpha}}\int_x^b (g(b)-g(t))^{\alpha}|f'(t)|\,dt$$
$$+\frac{1}{2[g(x)-g(a)]^{\alpha}}\int_a^x (g(t)-g(a))^{\alpha}|f'(t)|\,dt =: B(x).$$

Observe that

$$\int_a^x (g(x)-g(t))^{\alpha}|f'(t)|\,dt = \int_a^x (g(x)-g(t))^{\alpha} g'(t)\left|\frac{f'(t)}{g'(t)}\right| dt$$
$$\leq \operatorname*{essup}_{t\in[a,x]}\left|\frac{f'(t)}{g'(t)}\right|\int_a^x (g(x)-g(t))^{\alpha} g'(t)\,dt$$
$$= \left\|\frac{f'}{g'}\right\|_{[a,x],\infty}\frac{(g(x)-g(t))^{\alpha+1}}{\alpha+1}$$

and, similarly

$$\int_a^x (g(t)-g(a))^{\alpha}|f'(t)|\,dt \leq \left\|\frac{f'}{g'}\right\|_{[x,b],\infty}\frac{(g(b)-g(x))^{\alpha+1}}{\alpha+1}.$$

Therefore

$$A(x) \leq \frac{1}{2[g(x)-g(a)]^{\alpha}}\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}\frac{(g(x)-g(a))^{\alpha+1}}{\alpha+1}$$
$$+\frac{1}{2[g(b)-g(x)]^{\alpha}}\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}\frac{(g(b)-g(x))^{\alpha+1}}{\alpha+1}$$
$$=\frac{1}{2(\alpha+1)}\left[\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}(g(x)-g(a))+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}(g(b)-g(x))\right],$$

which proves the first part of (3.1).

Since, by Hölder's discrete inequality

$$mn+uv \leq (m^p+u^p)^{1/p}(n^q+v^q)^{1/q}$$

where $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m, n, u, v \geq 0$, we have further

$$
\begin{aligned}
&\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}(g(x)-g(a))+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}(g(b)-g(x)) \\
&\leq\begin{cases}
\max\left\{\left\|\frac{f'}{g'}\right\|_{[a,x],\infty},\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}\right\}(g(b)-g(a)) \\[2ex]
\left(\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}^{p}+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}^{p}\right)^{1/p}\left[(g(x)-g(a))^{q}+(g(b)-g(x))^{q}\right]^{1/q}, \\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\max\{g(x)-g(a),g(b)-g(x)\}\left[\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}\right]
\end{cases} \\
&=\begin{cases}
\left\|\frac{f'}{g'}\right\|_{[a,b],\infty}(g(b)-g(a)) \\[2ex]
\left(\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}^{p}+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}^{p}\right)^{1/p}\left[(g(x)-g(a))^{q}+(g(b)-g(x))^{q}\right]^{1/q}, \\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\left[\left\|\frac{f'}{g'}\right\|_{[a,x],\infty}+\left\|\frac{f'}{g'}\right\|_{[x,b],\infty}\right]\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(b)+g(a)}{2}\right|\right],
\end{cases}
\end{aligned}
$$

which proves the last part of (3.1).

The inequality (3.2) follows in a similar way by using (3.4) and the details are omitted. □

Corollary 3. *With the assumptions of Theorem 1 we have*

$$
\begin{aligned}
(3.5)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{[g(b)-g(a)]^{\alpha}}\left[I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))\right]\right| \\
&\leq\frac{1}{4(\alpha+1)}(g(b)-g(a))\left[\left\|\frac{f'}{g'}\right\|_{[a,M_g(a,b)],\infty}+\left\|\frac{f'}{g'}\right\|_{[M_g(a,b),b],\infty}\right] \\
&\leq\frac{1}{2(\alpha+1)}(g(b)-g(a))\left\|\frac{f'}{g'}\right\|_{[a,b],\infty}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.6)\quad &\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{[g(b)-g(a)]^{\alpha}}\left[I_{M_g(a,b)-,g}^{\alpha}f(a)+I_{M_g(a,b)+,g}^{\alpha}f(b)\right]-f(M_g(a,b))\right| \\
&\leq\frac{1}{4(\alpha+1)}(g(b)-g(a))\left[\left\|\frac{f'}{g'}\right\|_{[a,M_g(a,b)],\infty}+\left\|\frac{f'}{g'}\right\|_{[M_g(a,b),b],\infty}\right] \\
&\leq\frac{1}{2(\alpha+1)}(g(b)-g(a))\left\|\frac{f'}{g'}\right\|_{[a,b],\infty}.
\end{aligned}
$$

These inequalities have been obtained in an equivalent form in [17] and [16].

If $w:(a,b)\to(0,\infty)$ is continuous, then by taking the function $g(t):=\int_a^t w(s)\,ds$ and $\alpha=1$ in Theorem 1 above, then we get the following weighted integral inequalities

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2}\left[\frac{\int_a^x w(t)f(t)\,dt}{\int_a^x w(t)\,dt}+\frac{\int_x^b w(t)f(t)\,dt}{\int_x^b w(t)\,dt}\right]\right| \tag{3.7}$$
$$\leq \frac{1}{4}\left[\left\|\frac{f'}{w}\right\|_{[a,x],\infty}\int_a^x w(t)\,dt+\left\|\frac{f'}{w}\right\|_{[x,b],\infty}\int_x^b w(t)\,dt\right]$$
$$\leq \frac{1}{4}\begin{cases}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\int_a^b w(t)\,dt; \\ \\ \left(\left\|\frac{f'}{w}\right\|_{[a,x],\infty}^p+\left\|\frac{f'}{w}\right\|_{[x,b],\infty}^p\right)^{1/p}\left[\left(\int_a^x w(t)\,dt\right)^q+\left(\int_x^b w(t)\,dt\right)^q\right]^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\left\|\frac{f'}{w}\right\|_{[a,x],\infty}+\left\|\frac{f'}{w}\right\|_{[x,b],\infty}\right]\left[\frac{1}{2}\int_a^b w(t)\,dt+\left|\int_a^x w(t)\,dt-\frac{1}{2}\int_a^b w(t)\,dt\right|\right]\end{cases}$$

and

$$\left|\frac{1}{2}\left[\frac{\int_a^x w(t)f(t)\,dt}{\int_a^x w(t)\,dt}+\frac{\int_x^b w(t)f(t)\,dt}{\int_x^b w(t)\,dt}\right]-f(x)\right| \tag{3.8}$$
$$\leq \frac{1}{4}\left[\left\|\frac{f'}{w}\right\|_{[a,x],\infty}\int_a^x w(t)\,dt+\left\|\frac{f'}{w}\right\|_{[x,b],\infty}\int_x^b w(t)\,dt\right]$$
$$\leq \frac{1}{4}\begin{cases}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\int_a^b w(t)\,dt; \\ \\ \left(\left\|\frac{f'}{w}\right\|_{[a,x],\infty}^p+\left\|\frac{f'}{w}\right\|_{[x,b],\infty}^p\right)^{1/p}\left[\left(\int_a^x w(t)\,dt\right)^q+\left(\int_x^b w(t)\,dt\right)^q\right]^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\left\|\frac{f'}{w}\right\|_{[a,x],\infty}+\left\|\frac{f'}{w}\right\|_{[x,b],\infty}\right]\left[\frac{1}{2}\int_a^b w(t)\,dt+\left|\int_a^x w(t)\,dt-\frac{1}{2}\int_a^b w(t)\,dt\right|\right]\end{cases}$$

for any $x\in(a,b)$.

Moreover, if we take in these two inequalities $w\equiv 1$, then we get the inequalities

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]\right| \tag{3.9}$$
$$\leq \frac{1}{4}\left[\|f'\|_{[a,x],\infty}(x-a)+\|f'\|_{[x,b],\infty}(b-x)\right]$$
$$\leq \frac{1}{4}\begin{cases}\|f'\|_{[a,b],\infty}(b-a); \\ \\ \left(\|f'\|_{[a,x],\infty}^p+\|f'\|_{[x,b],\infty}^p\right)^{1/p}\left[(x-a)^q+(b-x)^q\right]^{1/q}, \\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,b],\infty}\right]\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\end{cases}$$

and

$$(3.10)\quad \left|\frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]-f(x)\right|$$
$$\leq \frac{1}{4}\left[\|f'\|_{[a,x],\infty}(x-a)+\|f'\|_{[x,b],\infty}(b-x)\right]$$
$$\leq \frac{1}{4}\begin{cases}\|f'\|_{[a,b],\infty}(b-a);\\ \\ \left(\|f'\|_{[a,x],\infty}^p+\|f'\|_{[x,b],\infty}^p\right)^{1/p}\left[(x-a)^q+(b-x)^q\right]^{1/q},\\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,b],\infty}\right]\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\end{cases}$$

for any $x\in(a,b)$.

4. Inequalities in Terms of p-Norms

We also have:

Theorem 2. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Assume that $\frac{f'}{g'}\in L_p[a,b]$, where $p,\ q>1$ where $\frac{1}{p}+\frac{1}{q}=1$, then for any $x\in(a,b)$ we have*

$$(4.1)\quad \left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2}\left[\frac{I_{a+,g}^{\alpha}f(x)}{[g(x)-g(a)]^{\alpha}}+\frac{I_{b-,g}^{\alpha}f(x)}{[g(b)-g(x)]^{\alpha}}\right]\right|$$
$$\leq \frac{1}{2(\alpha q+1)^{1/q}}\left[\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,x],p}(g(x)-g(a))^{\frac{1}{q}}+\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[x,b],p}(g(b)-g(x))^{\frac{1}{q}}\right]$$
$$\leq \frac{1}{2(\alpha q+1)^{1/q}}\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,b],p}[g(b)-g(a)]^{1/q}$$

and

$$(4.2)\quad \left|\frac{\Gamma(\alpha+1)}{2}\left[\frac{I_{x-,g}^{\alpha}f(a)}{[g(x)-g(a)]^{\alpha}}+\frac{I_{x+,g}^{\alpha}f(b)}{[g(b)-g(x)]^{\alpha}}\right]-f(x)\right|$$
$$\leq \frac{1}{2(\alpha q+1)^{1/q}}\left[\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,x],p}(g(x)-g(a))^{\frac{1}{q}}+\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[x,b],p}(g(b)-g(x))^{\frac{1}{q}}\right]$$
$$\leq \frac{1}{2(\alpha q+1)^{1/q}}\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,b],p}[g(b)-g(a)]^{1/q}.$$

Proof. By Hölder's weighted integral inequality

$$\left|\int_c^d u(t)\,v(t)\,w(t)\,dt\right|\leq\left(\int_c^d|u(t)|^p\,w(t)\,dt\right)^{1/p}\left(\int_c^d|v(t)|^q\,w(t)\,dt\right)^{1/q}$$

where p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $w\geq 0$ a.e. on $[c,d]$, we also have

$$\begin{aligned}
&\int_a^x (g(x)-g(t))^{\alpha}\left|\frac{f'(t)}{g'(t)}\right| g'(t)\,dt\\
&\leq \left(\int_a^x \left|\frac{f'(t)}{g'(t)}\right|^p g'(t)\,dt\right)^{1/p}\left(\int_a^x (g(x)-g(t))^{\alpha q} g'(t)\,dt\right)^{1/q}\\
&=\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,x],p}\frac{(g(x)-g(a))^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{1/q}}
\end{aligned}$$

and similarly,

$$\int_x^b (g(t)-g(x))^{\alpha}\left|\frac{f'(t)}{g'(t)}\right| g'(t)\,dt\leq\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[x,b],p}\frac{(g(b)-g(x))^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{1/q}}.$$

Therefore

$$\begin{aligned}
A(x)&=\frac{1}{2[g(x)-g(a)]^{\alpha}}\int_a^x (g(x)-g(t))^{\alpha}|f'(t)|\,dt\\
&\quad+\frac{1}{2[g(b)-g(x)]^{\alpha}}\int_x^b (g(t)-g(x))^{\alpha}|f'(t)|\,dt\\
&\leq\frac{1}{2[g(x)-g(a)]^{\alpha}}\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,x],p}\frac{(g(x)-g(a))^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{1/q}}\\
&\quad+\frac{1}{2[g(b)-g(x)]^{\alpha}}\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[x,b],p}\frac{(g(b)-g(x))^{\alpha+\frac{1}{q}}}{(\alpha q+1)^{1/q}}\\
&=\frac{1}{2(\alpha q+1)^{1/q}}\left[\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,x],p}(g(x)-g(a))^{\frac{1}{q}}+\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[x,b],p}(g(b)-g(x))^{\frac{1}{q}}\right],
\end{aligned}\tag{4.3}$$

which together with (3.3) proves the first part of (4.1).

By Hölder's elementary inequality, we also have for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$\begin{aligned}
&\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,x],p}(g(x)-g(a))^{\frac{1}{q}}+\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[x,b],p}(g(b)-g(x))^{\frac{1}{q}}\\
&\leq\left(\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,x],p}^p+\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[x,b],p}^p\right)^{1/p}\left[\left((g(x)-g(a))^{\frac{1}{q}}\right)^q+\left((g(b)-g(x))^{\frac{1}{q}}\right)^q\right]^{1/q}\\
&=\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,b],p}[g(b)-g(a)]^{1/q},
\end{aligned}$$

which proves the last part of (4.1).

The inequality (4.2) follows in a similar way and we omit the details. □

Corollary 4. *With the assumptions of Theorem 2 we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{[g(b)-g(a)]^{\alpha}}\left[I_{a+,g}^{\alpha}f(M_g(a,b))+I_{b-,g}^{\alpha}f(M_g(a,b))\right]\right| \tag{4.4}$$
$$\leq \frac{1}{2^{1/q+1}(\alpha q+1)^{1/q}}(g(b)-g(a))^{\frac{1}{q}}\left[\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,M_g(a,b)],p}+\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[M_g(a,b),b],p}\right]$$
$$\leq \frac{1}{2(\alpha q+1)^{1/q}}\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,b],p}[g(b)-g(a)]^{1/q}$$

and

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{[g(b)-g(a)]^{\alpha}}\left[I_{M_g(a,b)-,g}^{\alpha}f(a)+I_{M_g(a,b)+,g}^{\alpha}f(b)\right]-f(M_g(a,b))\right| \tag{4.5}$$
$$\leq \frac{1}{2^{1/q+1}(\alpha q+1)^{1/q}}(g(b)-g(a))^{\frac{1}{q}}\left[\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,M_g(a,b)],p}+\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[M_g(a,b),b],p}\right]$$
$$\leq \frac{1}{2(\alpha q+1)^{1/q}}\left\|\frac{f'}{(g')^{\frac{1}{q}}}\right\|_{[a,b],p}[g(b)-g(a)]^{1/q}.$$

If $w:(a,b)\to(0,\infty)$ is continuous, then by taking the function $g(t):=\int_a^t w(s)\,ds$ and $\alpha=1$ in Theorem 2 above, then we get the following weighted integral inequalities

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2}\left[\frac{\int_a^x w(t)f(t)\,dt}{\int_a^x w(t)\,dt}+\frac{\int_x^b w(t)f(t)\,dt}{\int_x^b w(t)\,dt}\right]\right| \tag{4.6}$$
$$\leq \frac{1}{2(q+1)^{1/q}}\left[\left\|\frac{f'}{(w)^{\frac{1}{q}}}\right\|_{[a,x],p}\left(\int_a^x w(t)\,dt\right)^{\frac{1}{q}}+\left\|\frac{f'}{(w)^{\frac{1}{q}}}\right\|_{[x,b],p}\left(\int_x^b w(t)\,dt\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{1}{2(q+1)^{1/q}}\left\|\frac{f'}{(w)^{\frac{1}{q}}}\right\|_{[a,b],p}\left[\int_a^b w(t)\,dt\right]^{1/q}$$

and

$$\left|\frac{1}{2}\left[\frac{\int_a^x w(t)f(t)\,dt}{\int_a^x w(t)\,dt}+\frac{\int_x^b w(t)f(t)\,dt}{\int_x^b w(t)\,dt}\right]-f(x)\right| \tag{4.7}$$
$$\leq \frac{1}{2(q+1)^{1/q}}\left[\left\|\frac{f'}{(w)^{\frac{1}{q}}}\right\|_{[a,x],p}\left(\int_a^x w(t)\,dt\right)^{\frac{1}{q}}+\left\|\frac{f'}{(w)^{\frac{1}{q}}}\right\|_{[x,b],p}\left(\int_x^b w(t)\,dt\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{1}{2(q+1)^{1/q}}\left\|\frac{f'}{(w)^{\frac{1}{q}}}\right\|_{[a,b],p}\left[\int_a^b w(t)\,dt\right]^{1/q}$$

for any $x\in(a,b)$, where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Moreover, if we take in these two inequalities $w \equiv 1$, then we get the inequalities

$$\begin{aligned}(4.8)\quad & \left| \frac{f(a)+f(b)}{2} - \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right]\right| \\ & \le \frac{1}{2(q+1)^{1/q}}\left[\|f'\|_{[a,x],p}(x-a)^{\frac{1}{q}} + \|f'\|_{[x,b],p}(b-x)^{\frac{1}{q}}\right] \\ & \le \frac{1}{2(q+1)^{1/q}}\|f'\|_{[a,b],p}(b-a)^{1/q}\end{aligned}$$

and

$$\begin{aligned}(4.9)\quad & \left| \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right] - f(x)\right| \\ & \le \frac{1}{2(q+1)^{1/q}}\left[\|f'\|_{[a,x],p}(x-a)^{\frac{1}{q}} + \|f'\|_{[x,b],p}(b-x)^{\frac{1}{q}}\right] \\ & \le \frac{1}{2(q+1)^{1/q}}\|f'\|_{[a,b],p}(b-a)^{1/q}.\end{aligned}$$

These inequalities have been obtained in an equivalent form in [17] and [16].

5. Application for Hadamard Fractional Integrals

If we take $g(t) = \ln t$, $t > 0$ in Theorem 1 and denote with $e(t) = t$, the identity function, then we get the following inequalities for the Hadamard fractional integrals for an absolutely continuous function $f : [a,b] \to \mathbb{C}$ with the property that $ef' \in L_\infty[a,b]$,

$$\begin{aligned}(5.1)\quad & \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2}\left[\frac{H^\alpha_{a+}f(x)}{\left[\ln\left(\frac{x}{a}\right)\right]^\alpha} + \frac{H^\alpha_{b-}f(x)}{\left[\ln\left(\frac{b}{x}\right)\right]^\alpha}\right]\right| \\ & \le \frac{1}{2(\alpha+1)}\left[\|ef'\|_{[a,x],\infty}\ln\left(\frac{x}{a}\right) + \|ef'\|_{[x,b],\infty}\ln\left(\frac{b}{x}\right)\right] \\ & \le \frac{1}{2(\alpha+1)}\begin{cases} \|ef'\|_{[a,b],\infty}\ln\left(\frac{b}{a}\right); \\ \\ \left(\|ef'\|^p_{[a,x],\infty} + \|ef'\|^p_{[x,b],\infty}\right)^{1/p}\left[\left[\ln\left(\frac{x}{a}\right)\right]^q + \left[\ln\left(\frac{b}{x}\right)\right]^q\right]^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\|ef'\|_{[a,x],\infty} + \|ef'\|_{[x,b],\infty}\right]\left[\frac{1}{2}\ln\left(\frac{b}{a}\right) + \left|\ln\left(\frac{x}{\sqrt{ab}}\right)\right|\right]\end{cases}\end{aligned}$$

and

$$\left|\frac{\Gamma(\alpha+1)}{2}\left[\frac{H^{\alpha}_{x-,}f(a)}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}}+\frac{H^{\alpha}_{x+}f(b)}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}\right]-f(x)\right| \tag{5.2}$$
$$\leq \frac{1}{2(\alpha+1)}\left[\|ef'\|_{[a,x],\infty}\ln\left(\frac{x}{a}\right)+\|ef'\|_{[x,b],\infty}\ln\left(\frac{b}{x}\right)\right]$$
$$\leq \frac{1}{2(\alpha+1)}\begin{cases}\|ef'\|_{[a,b],\infty}\ln\left(\frac{b}{a}\right); \\ \left(\|ef'\|^{p}_{[a,x],\infty}+\|ef'\|^{p}_{[x,b],\infty}\right)^{1/p}\left[\left[\ln\left(\frac{x}{a}\right)\right]^{q}+\left[\ln\left(\frac{b}{x}\right)\right]^{q}\right]^{1/q}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\|ef'\|_{[a,x],\infty}+\|ef'\|_{[x,b],\infty}\right]\left[\frac{1}{2}\ln\left(\frac{b}{a}\right)+\left|\ln\left(\frac{x}{\sqrt{ab}}\right)\right|\right]\end{cases}$$

for any $x\in(a,b)\subset(0,\infty)$.

If $e^{1/q}f'\in L_p[a,b]$, where $p,\ q>1$ where $\frac{1}{p}+\frac{1}{q}=1$, then for any $x\in(a,b)$ we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2}\left[\frac{H^{\alpha}_{a+}f(x)}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}}+\frac{H^{\alpha}_{b-}f(x)}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}\right]\right| \tag{5.3}$$
$$\leq \frac{1}{2(\alpha q+1)^{1/q}}\left[\left\|e^{1/q}f'\right\|_{[a,x],p}\left[\ln\left(\frac{x}{a}\right)\right]^{\frac{1}{q}}+\left\|e^{1/q}f'\right\|_{[x,b],p}\left[\ln\left(\frac{b}{x}\right)\right]^{\frac{1}{q}}\right]$$
$$\leq \frac{1}{2(\alpha q+1)^{1/q}}\left\|e^{1/q}f'\right\|_{[a,b],p}\left[\ln\left(\frac{b}{a}\right)\right]^{1/q}$$

and

$$\left|\frac{\Gamma(\alpha+1)}{2}\left[\frac{H^{\alpha}_{x-,}f(a)}{\left[\ln\left(\frac{x}{a}\right)\right]^{\alpha}}+\frac{H^{\alpha}_{x+}f(b)}{\left[\ln\left(\frac{b}{x}\right)\right]^{\alpha}}\right]-f(x)\right| \tag{5.4}$$
$$\leq \frac{1}{2(\alpha q+1)^{1/q}}\left[\left\|e^{1/q}f'\right\|_{[a,x],p}\left[\ln\left(\frac{x}{a}\right)\right]^{\frac{1}{q}}+\left\|e^{1/q}f'\right\|_{[x,b],p}\left[\ln\left(\frac{b}{x}\right)\right]^{\frac{1}{q}}\right]$$
$$\leq \frac{1}{2(\alpha q+1)^{1/q}}\left\|e^{1/q}f'\right\|_{[a,b],p}\left[\ln\left(\frac{b}{a}\right)\right]^{1/q}.$$

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