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OSTROWSKI TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR s-GODUNOVA-LEVIN FUNCTIONS VIA KATUGAMPOLA FRACTIONAL INTEGRALS

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Abstract. In this paper, we give some fractional integral inequalities of Ostrowski type for s-Godunova-Levin functions via Katugampola fractional integrals. We also deduce some known Ostrowski type fractional integral inequalities for Riemann-Liouville fractional integrals.

1. INTRODUCTION

In 1938 Ostrowski [13] proved an inequality stated in the following result (see also [10, p.468]).

Theorem 1.1. *Let $f : I \to \mathbb{R}$ where I is interval in $\mathbb{R}$, be a mapping differentiable in I° the interior of I and $a, b \in I^\circ$, $a < b$. If $\left|f'(t)\right| \leq M$, for all $t \in [a, b]$, then we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M, x \in [a, b].$$

Ostrowski inequality gives bounds of integral average of a function f over an interval $[a, b]$ to its value $f(x)$ at point $x \in [a, b]$. Ostrowski and Ostrowski type inequalities have great importance in numerical analysis as they provide the error bound of many quaderature rules [3]. Therefore in recent years, so many such type of inequalities have been obtained and generalized (see [12, 5]) and references therein.

As fractional calculus is a generalization of classical calculus concerned with operations of integration and differentiation of fractional order so in this research article we will use Katugampola fractional integrals to

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generalize the Ostrowski type inequalities given in [12].
In [8] Laurent give definition of Riemann-Liouville fractional integrals.

Definition 1.2. [8] Let $f \in L_1[a,b]$. The Riemann-Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, x < b,$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}du$. Here $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $J^{0}_{a+}f(x) = J^{0}_{b-}f(x) = f(x)$. In case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Definition 1.3. J. Hadamard introduced the Hadamard fractional integral in [7], and is given by

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(log\frac{x}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau},$$

for $Re(\alpha) > 0$, $x > a \geq 0$.

Recently Katugampola generalized Riemann-Liouville and Hadamard fractional integrals into a single form called Katugampola fractional integrals.

Definition 1.4. [9] Let $[a,b]$ be a finite interval in $\mathbb{R}$. Then Katugampola fractional integrals of order $\alpha > 0$ for a real valued function f are defined by

$${}^{\rho}I^{\alpha}_{a+}f\left(x\right) = \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)} \int_a^x t^{\rho-1}\left(x^{\rho}-t^{\rho}\right)^{\alpha-1} f\left(t\right)dt$$

and

$${}^{\rho}I^{\alpha}_{b-}f\left(x\right) = \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)} \int_x^b t^{\rho-1}\left(t^{\rho}-x^{\rho}\right)^{\alpha-1} f\left(t\right)dt$$

with $a < x < b$ and $\rho > 0$.
Where $\Gamma\left(\alpha\right)$ is the Euler gamma function. For $\rho = 1$, Katugampola fractional integrals give Riemann-Liouville fractional integrals, while $\rho \to 0^{+}$ produces the Hadamard fractional integral. For its proof one can check [9].

The ρ-Gamma function [4] for any two positive numbers x, y denoted by ${}^{\rho}\Gamma(x,y)$, is defined by

$${}^{\rho}\Gamma(\alpha) = \int_0^{\infty} e^{-t^{\rho}}(t^{\rho})^{\alpha-\frac{1}{\rho}}dt,$$

then we have

$$\Gamma(\alpha) = \rho\ {}^{\rho}\Gamma(\alpha).$$

In this manner we can also have the following relation

$$ {}^{\rho}\beta(x, y) = \frac{{}^{\rho}\Gamma(x)\ {}^{\rho}\Gamma(y)}{{}^{\rho}\Gamma(x, y)}. \tag{1}$$

Definition 1.5. [2] A non-negative function $f : I \to \mathbb{R}$ is said to be p-function, if for any two points $x, y \in I$ and $t \in [0, 1]$

$$f\left(tx + (1 - t)y\right) \leq f(x) + f(y).$$

Definition 1.6. [6] A function $f : I \to \mathbb{R}$ is said to be Godunova-Levin function, if for any two points $x, y \in I$ and $t \in (0, 1)$

$$f\left(tx + (1 - t)y\right) \leq \frac{f(x)}{t} + \frac{f(y)}{1 - t}.$$

Definition 1.7. [11] A function $f : I \to \mathbb{R}$ is said to be s-Godunova-Levin function of first kind, if $s \in (0, 1]$, for all $x, y \in I$ and $t \in (0, 1)$ then we have

$$f\left(tx + (1 - t)y\right) \leq \frac{f(x)}{t^s} + \frac{f(y)}{1 - t^s}.$$

Definition 1.8. [1] A function $f : I \to \mathbb{R}$ is said to be s-Godunova-Levin function of second kind, if $s \in [0, 1]$, for all $x, y \in I$ and $t \in (0, 1)$ then we have

$$f\left(tx + (1 - t)y\right) \leq \frac{f(x)}{t^s} + \frac{f(y)}{(1 - t)^s}.$$

We organize the paper in such a way that in the following section we prove some Ostrowski type fractional integral inequalities for s-Godunova-Levin functions of second kind via Katugampola fractional integrals. Also we will obtain some corollaries for p-functions and Godunova-Levin functions and deduce some known results of [12].

2. Ostrowski type fractional integral inequalities for mappings whose derivatives are s-Godunova-Levin of second kind via Katugampola fractional integrals

The following lemma (given and also proved in [4]) is very useful to obtain our results.

Lemma 2.1. *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $a < b$ such that $f' \in L_1[a, b]$, where $\rho > 0$. Then we have the* *following equality*

$$
\begin{aligned}
&\left(\frac{(x^\rho-a^\rho)^\alpha+(b^\rho-x^\rho)^\alpha}{b-a}\right)f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times\\
&[{}^\rho I^\alpha_{x^-}f(a^\rho)+{}^\rho I^\alpha_{x^+}f(b^\rho)]\\
&=\frac{\rho(x^\rho-a^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}f'(t^\rho x^\rho+(1-t^\rho)a^\rho)dt\\
&-\frac{\rho(b^\rho-x^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}f'(t^\rho x^\rho+(1-t^\rho)b^\rho)dt;\ x\in[a,b]. \qquad (2)
\end{aligned}
$$

Theorem 2.2. *Let $f:[a^\rho,\, b^\rho]\to\mathbb{R}$, $a,b\geq 0$, $a<b$ be a differentiable function on (a^ρ,b^ρ) and $f'\in L_1[a,b]$. If $\left|f'\right|$ is s-Godunova-Levin function of second kind and $\left|f'(x^\rho)\right|\leq M$, $x\in[a,b]$, then the following inequality holds*

$$
\begin{aligned}
&\left|\left(\frac{(x^\rho-a^\rho)^\alpha+(b^\rho-x^\rho)^\alpha}{b-a}\right)f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times\right.\\
&\left.[{}^\rho I^\alpha_{x^-}f(a^\rho)+{}^\rho I^\alpha_{x^+}f(b^\rho)]\right|\leq M\left[\frac{(x^\rho-a^\rho)^{\alpha+1}+(b^\rho-x^\rho)^{\alpha+1}}{b-a}\right]\times\\
&\left[\frac{1}{\alpha+1-s}+\frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{{}^\rho\Gamma(\alpha+2-s)}\right];\ x\in[a,b]. \qquad (3)
\end{aligned}
$$

Proof. Using lemma 2.1 and the fact that $\left|f'\right|$ is s-Godunova-Levin function of second kind, we have

$$
\begin{aligned}
&\left|\left(\frac{(x^\rho-a^\rho)^\alpha+(b^\rho-x^\rho)^\alpha}{b-a}\right)f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times\right.\\
&\left.[{}^\rho I^\alpha_{x^-}f(a^\rho)+\ {}^\rho I^\alpha_{x^+}f(b^\rho)]\right|\\
&\leq\frac{\rho(x^\rho-a^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|dt\\
&+\frac{\rho(b^\rho-x^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|dt\\
&\leq\frac{\rho(x^\rho-a^\rho)^{\alpha+1}}{b-a}\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}\left|f'(x^\rho)\right|+\frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\left|f'(a^\rho)\right|\right]dt\\
&+\frac{\rho(b^\rho-x^\rho)^{\alpha+1}}{b-a}\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}\left|f'(x^\rho)\right|+\frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\left|f'(b^\rho)\right|\right]dt\\
&\leq\frac{M\rho(x^\rho-a^\rho)^{\alpha+1}}{b-a}\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}+\frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\right]dt\\
&+\frac{M\rho(b^\rho-x^\rho)^{\alpha+1}}{b-a}\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}+\frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\right]dt\\
&=M\rho\left[\frac{(x^\rho-a^\rho)^{\alpha+1}+(b^\rho-x^\rho)^{\alpha+1}}{2(b-a)}\right]\times\\
&\int_0^1\left[t^{\alpha\rho-\rho s+\rho-1}+t^{\alpha\rho+\rho-1}(1-t^\rho)^{-s}\right]dt.\\
&=M\rho\left[\frac{(x^\rho-a^\rho)^{\alpha+1}+(b^\rho-x^\rho)^{\alpha+1}}{2(b-a)}\right]\times\\
&\left[\frac{1}{\rho(\alpha+1-s)}+\frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{\rho\ {}^\rho\Gamma(\alpha+2-s)}\right]\\
&=M\left[\frac{(x^\rho-a^\rho)^{\alpha+1}+(b^\rho-x^\rho)^{\alpha+1}}{b-a}\right]\times\\
&\left[\frac{1}{\alpha+1-s}+\frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{{}^\rho\Gamma(\alpha+2-s)}\right].
\end{aligned}
$$

Here we use (1). The proof is completed. □

Remark 2.3. (i) If we put $\rho=1$ in (3), then we get [12, Theorem 3.1].
(ii) If we put $\rho=1$ and $\alpha=1$ in (3), then we get [12, Corollary 3.1].

Corollary 2.4. *In Theorem 2.2, if we take $s = 0$, which means that $\left|f'\right|$ is p-function, then (3) becomes the following inequality*

$$\Bigg| \left(\frac{(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha}{b-a} \right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \times \left[{}^\rho I^\alpha_{x^-} f(a^\rho) + {}^\rho I^\alpha_{x^+} f(b^\rho) \right] \Bigg| \leq \frac{2M}{\alpha+1} \left[\frac{(x^\rho - a^\rho)^{\alpha+1} + (b^\rho - x^\rho)^{\alpha+1}}{b-a} \right]; \; x \in [a,b].$$

Corollary 2.5. *In Theorem 2.2, if we take $s = 1$, which means that $\left|f'\right|$ is Godunova-Levin function, then (3) becomes the following inequality*

$$\Bigg| \left(\frac{(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha}{b-a} \right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \times \left[{}^\rho I^\alpha_{x^-} f(a^\rho) + {}^\rho I^\alpha_{x^+} f(b^\rho) \right] \Bigg| \leq \frac{M(\alpha+1)}{\alpha} \left[\frac{(x^\rho - a^\rho)^{\alpha+1} + (b^\rho - x^\rho)^{\alpha+1}}{b-a} \right]; \; x \in [a,b].$$

Theorem 2.6. *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$, $a, b \geq 0$, $a < b$ be a differentiable function on (a^ρ, b^ρ) and $f' \in L_1[a,b]$. If $\left|f'\right|^q$, is s-Godunova-Levin function of second kind and $\left|f'(x^\rho)\right| \leq M$, $x \in [a,b]$ then the following inequality for Katugampola fractional integrals holds*

$$\Bigg| \left(\frac{(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha}{b-a} \right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \times \left[{}^\rho I^\alpha_{x^-} f(a^\rho) + {}^\rho I^\alpha_{x^+} f(b^\rho) \right] \Bigg| \leq M\rho \left[\frac{(x^\rho - a^\rho)^{\alpha+1} + (b^\rho - x^\rho)^{\alpha+1}}{(b-a)(1 + p(\alpha\rho + \rho - 1))^{\frac{1}{p}}} \right] \left[\frac{1}{1 - \rho s} \right]^{\frac{1}{q}}; \; x \in [a,b], \tag{4}$$

with $\frac{1}{p} + \frac{1}{q} = 1$ where $q > 1$.

Proof. Using lemma 2.1 and then Holder's inequality, we have

$$\left|\left(\frac{(x^\rho-a^\rho)^\alpha+(b^\rho-x^\rho)^\alpha}{b-a}\right)f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times\right.$$

$$\left.\left[{}^\rho I^\alpha_{x^-}f(a^\rho)+\ {}^\rho I^\alpha_{x^+}f(b^\rho)\right]\right|$$

$$\le\frac{\rho(x^\rho-a^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|dt$$

$$+\frac{\rho(b^\rho-x^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|dt$$

$$\le\frac{\rho(x^\rho-a^\rho)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p(\alpha\rho+\rho-1)}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}$$

(5)

$$+\frac{\rho(b^\rho-x^\rho)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p(\alpha\rho+\rho-1)}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f'\right|^q$ is s-Godunova-Levin function of second kind and $\left|f'(x^\rho)\right|\le M$, we get

$$\int_0^1\left|f'(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|^q dt\times$$

$$\le\int_0^1\left[\frac{1}{(t^\rho)^s}\left|f'(x^\rho)\right|^q+\frac{1}{(1-t^\rho)^s}\left|f'(a^\rho)\right|^q\right]dt$$

$$\le M^q\int_0^1\left[\frac{1}{(t^\rho)^s}+\frac{1}{(1-t^\rho)^s}\right]dt=\frac{1}{1-\rho s} \tag{6}$$

similarly

$$\int_0^1\left|f'(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|^q dt\le\frac{1}{1-\rho s}. \tag{7}$$

We also have

$$\int_0^1 t^{p(\alpha\rho+\rho-1)}dt=\frac{1}{1+p(\alpha\rho+\rho-1)}. \tag{8}$$

Using (6), (7) and (8) in (5) we can get (4). □

Remark 2.7. (i) If we put $\rho=1$ in (4), then we get [12, Theorem 3.2].
(ii) If we put $\rho=1$ and $\alpha=1$ in (4), then we get [12, Corollary 3.2].

Corollary 2.8. *In Theorem* 2.6*, if we take* $s=0$*, which means that* $\left|f'\right|$ *is p-function, then (4) becomes the following inequality*

$$\left|\left(\frac{(x^\rho-a^\rho)^\alpha+(b^\rho-x^\rho)^\alpha}{b-a}\right)f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times\right.$$
$$\left.\left[{}^\rho I^\alpha_{x^-}f(a^\rho)+{}^\rho I^\alpha_{x^+}f(b^\rho)\right]\right|$$
$$\leq\frac{M}{(1+p(\alpha\rho+\rho-1))^{\frac{1}{p}}}\left[\frac{(x^\rho-a^\rho)^{\alpha+1}+(b^\rho-x^\rho)^{\alpha+1}}{b-a}\right];\ x\in[a,b].$$

Corollary 2.9. *In Theorem* 2.6*, if we take* $s=1$*, which means that* $\left|f'\right|$ *is Godunova-Levin function, then (4) becomes the following inequality*

$$\left|\left(\frac{(x^\rho-a^\rho)^\alpha+(b^\rho-x^\rho)^\alpha}{b-a}\right)f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times\right.$$
$$\left.\left[{}^\rho I^\alpha_{x^-}f(a^\rho)+{}^\rho I^\alpha_{x^+}f(b^\rho)\right]\right|\leq\frac{M\rho}{(1+p(\alpha\rho+\rho-1))^{\frac{1}{p}}}\times$$
$$\left[\frac{(x^\rho-a^\rho)^{\alpha+1}+(b^\rho-x^\rho)^{\alpha+1}}{b-a}\right]\left[\frac{1+\alpha}{\alpha\rho}\right]^{\frac{1}{q}};\ x\in[a,b].$$

Theorem 2.10. *Let* $f:[a^\rho,b^\rho]\to\mathbb{R}$*,* $a,b\geq 0$*,* $a<b$ *be a differentiable function on* (a^ρ,b^ρ) *and* $f'\in L_1[a,b]$*. If* $\left|f'\right|^q$ *is s-Godunova-Levin function of second kind and* $\left|f'(x^\rho)\right|\leq M$*,* $x\in[a,b], q\geq 1$*, then the following inequality for Katugampola fractional integrals holds*

$$\left|\left(\frac{(x^\rho-a^\rho)^\alpha+(b^\rho-x^\rho)^\alpha}{b-a}\right)f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times\right.$$
$$\left.\left[{}^\rho I^\alpha_{x^-}f(a^\rho)+{}^\rho I^\alpha_{x^+}f(b^\rho)\right]\right|$$
$$\leq\frac{M\rho}{(\alpha\rho+\rho)^{1-\frac{1}{q}}}\left[\frac{(x^\rho-a^\rho)^{\alpha+1}+(b^\rho-x^\rho)^{\alpha+1}}{b-a}\right]\times$$
$$\left(\frac{1}{\rho(\alpha-s+1)}+\frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{\rho\ {}^\rho\Gamma(\alpha-s+2)}\right)^{\frac{1}{q}};\ x\in[a,b]. \tag{9}$$

Proof. Using lemma 2.1 and power mean inequality, we have

$$\left|\left(\frac{(x^\rho-a^\rho)^\alpha+(b^\rho-x^\rho)^\alpha}{b-a}\right)f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\times\right.$$
$$\left.\left[{}^\rho I^\alpha_{x^-}f(a^\rho)+\ {}^\rho I^\alpha_{x^+}f(b^\rho)\right]\right|$$
$$\leq\frac{\rho(x^\rho-a^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|dt$$
$$+\frac{\rho(b^\rho-x^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|dt$$
$$\leq\frac{\rho(x^\rho-a^\rho)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha\rho+\rho-1}dt\right)^{1-\frac{1}{q}}\times$$
$$\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{\rho(b^\rho-x^\rho)^{\alpha+1}}{b-a}\left(\int_0^1 t^{\alpha\rho+\rho-1}dt\right)^{1-\frac{1}{q}}\times$$
$$\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|^q dt\right)^{\frac{1}{q}}. \tag{10}$$

Since $\left|f^{'}\right|^q$ is s-Godunova-Levin function of second kind and $\left|f^{'}(x^\rho)\right|\leq M$, we get

$$\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|^q dt$$
$$\leq\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}\left|f^{'}(x^\rho)\right|^q+\frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\left|f^{'}(a^\rho)\right|^q\right]dt$$
$$\leq M^q\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}+\frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\right]dt\times$$
$$=M^q\left[\frac{1}{\rho(\alpha-s+1)}+\frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{\rho\ {}^\rho\Gamma(\alpha-s+2)}\right] \tag{11}$$

similarly

$$\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|^q dt\times$$
$$\leq M^q\left[\frac{1}{\rho(\alpha-s+1)}+\frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{\rho\ {}^\rho\Gamma(\alpha-s+2)}\right]. \tag{12}$$

Using (11) and (12) in (10) we can attain (9). □

Remark 2.11. (i) If we put $\rho = 1$ in (9), then we get [12, Theorem 3.3].
(ii) If we put $\rho = 1$ and $\alpha = 1$ in (9), then we get [12, Corollary 3.3].

Corollary 2.12. *In Theorem 2.10, if we take $s = 0$, which means that $\left|f'\right|$ is p-function, then (9) becomes the following inequality*

$$\left| \left(\frac{(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha}{b-a} \right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \times \right.$$
$$\left. [{}^\rho I^\alpha_{x^-} f(a^\rho) + {}^\rho I^\alpha_{x^+} f(b^\rho)] \right|$$
$$\leq \frac{M\rho}{(\alpha\rho+\rho)^{1-\frac{1}{q}}} \left[\frac{(x^\rho - a^\rho)^{\alpha+1} + (b^\rho - x^\rho)^{\alpha+1}}{b-a} \right] \left[\frac{2}{\rho(\alpha+1)} \right]^{\frac{1}{q}}; \; x \in [a,b].$$

Corollary 2.13. *In Theorem 2.10, if we take $s = 1$, which means that $\left|f'\right|$ is Godunova-Levin function, then (9) becomes the following inequality*

$$\left| \left(\frac{(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha}{b-a} \right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \times \right.$$
$$\left. [{}^\rho I^\alpha_{x^-} f(a^\rho) + {}^\rho I^\alpha_{x^+} f(b^\rho)] \right|$$
$$\leq \frac{M\rho}{(\alpha\rho+\rho)^{1-\frac{1}{q}}} \left[\frac{(x^\rho - a^\rho)^{\alpha+1} + (b^\rho - x^\rho)^{\alpha+1}}{b-a} \right] \left[\frac{1+\alpha}{\alpha\rho} \right]^{\frac{1}{q}}; \; x \in [a,b].$$

We use the following lemma to establish some new results. Its proof is similar to Lemma 2.1.

Lemma 2.14. *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable mapping on (a^ρ, b^ρ) with $a^\rho < b^\rho$ such that $f' \in L_1[a^\rho, b^\rho]$, where $\rho > 0$. Then we have the following equality*

$$f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right]$$
$$= \frac{\rho(x^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + (1-t^\rho)a^\rho) dt$$
$$- \frac{(b^\rho - x^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + (1-t^\rho)b^\rho) dt; \; x \in [a,b]. \tag{13}$$

Theorem 2.15. *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$, $a, b \geq 0$, $a < b$ be a differentiable function on (a^ρ, b^ρ) and $f' \in L_1[a,b]$. If $\left|f'\right|$ is s-Godunova-Levin function of second kind and $\left|f'(x^\rho)\right| \leq M$, $x \in [a,b]$, then the following* *inequality holds*

$$\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho-a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$
$$\leq \frac{M(b^\rho-a^\rho)}{2}\left[\frac{1}{\alpha-s+1} + \frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{{}^\rho\Gamma(\alpha-s+2)}\right];\ x\in[a,b]. \tag{14}$$

Proof. Using lemma 2.8 and s-Godunova-Levin function of second kind of $\left|f'\right|$ we proceed as follows

$$\begin{aligned}
&\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho-a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right| \\
&\leq \frac{\rho(x^\rho-a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|dt \\
&+ \frac{\rho(b^\rho-x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|dt \\
&\leq \frac{\rho(x^\rho-a^\rho)}{2}\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}\left|f'(x^\rho)\right| + \frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\left|f'(a^\rho)\right|\right]dt \\
&+ \frac{\rho(b^\rho-x^\rho)}{2}\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}\left|f'(x^\rho)\right| + \frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\left|f'(b^\rho)\right|\right]dt \\
&\leq \frac{M\rho(x^\rho-a^\rho)}{2}\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s} + \frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\right]dt \\
&+ \frac{M\rho(b^\rho-x^\rho)}{2}\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s} + \frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\right]dt \\
&= M\rho\left[\frac{(x^\rho-a^\rho)+(b^\rho-x^\rho)}{2}\right]\int_0^1\left[t^{\alpha\rho-\rho s+\rho-1} + t^{\alpha\rho+\rho-1}(1-t^\rho)^{-s}\right]dt. \\
&= M\rho\left[\frac{(x^\rho-a^\rho)+(b^\rho-x^\rho)}{2}\right]\left[\frac{1}{\rho(\alpha-s+1)} + \frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{\rho\ {}^\rho\Gamma(\alpha-s+2)}\right] \\
&= \frac{M(b^\rho-a^\rho)}{2}\left[\frac{1}{\alpha-s+1} + \frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{{}^\rho\Gamma(\alpha-s+2)}\right].
\end{aligned}$$

Here we use (1). The proof is completed. □

Corollary 2.16. *In Theorem 3.2, if we take $s=0$, which means that $\left|f'\right|$ is p-function, then (14) becomes the following inequality*

$$\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho-a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$
$$\leq \frac{M(b^\rho-a^\rho)}{\alpha+1};\ x\in[a,b].$$

Corollary 2.17. *In Theorem 3.2, if we take $s = 1$, which means that $\left|f^{'}\right|$ is Godunova-Levin function, then (14) becomes the following inequality*

$$\left| f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right| \leq \frac{M(\alpha+1)(b^\rho - a^\rho)}{2\alpha}; \; x \in [a,b].$$

Theorem 2.18. *Let $f : [a^\rho, b^\rho] \to \mathbb{R}$, $a, b \geq 0$, $a < b$ be a differentiable function on (a^ρ, b^ρ) and $f^{'} \in L_1[a,b]$. If $\left|f^{'}\right|^q$, is s-Godunova-Levin function of second kind and $\left|f^{'}(x^\rho)\right| \leq M$, $x \in [a,b]$ then the following inequality for Katugampola fractional integrals holds*

$$\left| f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right| \leq \frac{M\rho(b^\rho - a^\rho)}{2(1+p(\alpha\rho+\rho-1))^{\frac{1}{p}}} \left[\frac{1}{1-\rho s} \right]^{\frac{1}{q}}; \; x \in [a,b], \tag{15}$$

with $\frac{1}{p} + \frac{1}{q} = 1$ where $q > 1$.

Proof. Using lemma 2.8 and then Holder's inequality, we have

$$\begin{aligned}
&\left| f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right| \\
&\leq \frac{\rho(x^\rho - a^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left| f^{'}(t^\rho x^\rho + (1-t^\rho)a^\rho) \right| dt \\
&+ \frac{\rho(b^\rho - x^\rho)}{2} \int_0^1 t^{\alpha\rho+\rho-1} \left| f^{'}(t^\rho x^\rho + (1-t^\rho)b^\rho) \right| dt \\
&\leq \frac{\rho(x^\rho - a^\rho)}{2} \left(\int_0^1 t^{p(\alpha\rho+\rho-1)} dt \right)^{\frac{1}{p}} \times \\
&\left(\int_0^1 \left| f^{'}(t^\rho x^\rho + (1-t^\rho)a^\rho) \right|^q dt \right)^{\frac{1}{q}} \\
&+ \frac{\rho(b^\rho - x^\rho)}{2} \left(\int_0^1 t^{p(\alpha\rho+\rho-1)} dt \right)^{\frac{1}{p}} \times \\
&\left(\int_0^1 \left| f^{'}(t^\rho x^\rho + (1-t^\rho)b^\rho) \right|^q dt \right)^{\frac{1}{q}}.
\end{aligned} \tag{16}$$

Since $\left|f^{'}\right|^q$ is s-Godunova-Levin function of second kind and $\left|f^{'}(x^\rho)\right| \leq M$, we get

$$\begin{aligned} &\int_0^1 \left|f^{'}(t^\rho x^\rho + (1-t^\rho)a^\rho)\right|^q dt \\ &\leq \int_0^1 \left[\frac{1}{(t^\rho)^s}\left|f^{'}(x^\rho)\right|^q + \frac{1}{(1-t^\rho)^s}\left|f^{'}(a^\rho)\right|^q\right] dt \\ &\leq M^q \int_0^1 \left[\frac{1}{(t^\rho)^s} + \frac{1}{(1-t^\rho)^s}\right] dt = \frac{M^q}{1-\rho s} \qquad (17) \end{aligned}$$

similarly

$$\int_0^1 \left|f^{'}(t^\rho x^\rho + (1-t^\rho)b^\rho)\right|^q dt \leq \frac{M^q}{1-\rho s}. \qquad (18)$$

We also have

$$\int_0^1 t^{p(\alpha\rho+\rho-1)} dt = \frac{1}{1+p(\alpha\rho+\rho-1)}. \qquad (19)$$

Using (17), (18) and (19) in (16) we can get (15). □

Corollary 2.19. *In Theorem* 3.5*, if we take* $s = 0$*, which means that* $\left|f^{'}\right|$ *is p-function, then (15) becomes the following inequality*

$$\begin{aligned} &\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha}\right]\right| \\ &\leq \frac{M\rho(b^\rho - a^\rho)}{2(p(\alpha\rho+\rho-1)+1)^{\frac{1}{p}}};\ x \in [a,b]. \end{aligned}$$

Corollary 2.20. *In Theorem* 3.5*, if we take* $s = 1$*, which means that* $\left|f^{'}\right|$ *is Godunova-Levin function, then (15) becomes the following inequality*

$$\begin{aligned} &\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha}\right]\right| \\ &\leq \frac{M\rho(b^\rho - a^\rho)}{2(p(\alpha\rho+\rho-1)+1)^{\frac{1}{p}}}\left[\frac{1}{1-\rho}\right]^{\frac{1}{q}};\ x \in [a,b]. \end{aligned}$$

Theorem 2.21. *Let* $f : [a^\rho, b^\rho] \to \mathbb{R}$*,* $a, b \geq 0$*,* $a < b$ *be a differentiable function on* (a^ρ, b^ρ) *and* $f^{'} \in L_1[a,b]$*. If* $\left|f^{'}\right|^q$ *is s-Godunova-Levin function of second kind and* $\left|f^{'}(x^\rho)\right| \leq M$*,* $x \in [a,b], q \geq 1$*, then the* *following inequality for Katugampola fractional integrals holds*

$$\left|f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x^+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$

(20)

$$\leq\frac{M\rho(b^\rho-a^\rho)}{2(\alpha\rho+\rho)^{1-\frac{1}{q}}}\left(\frac{1}{\rho(\alpha-s+1)}+\frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{\rho\ {}^\rho\Gamma(\alpha-s+2)}\right)^{\frac{1}{q}};\ x\in[a,b].$$

Proof. Using lemma 2.8 and power mean inequality, we have

$$\begin{aligned}&\left|f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x^+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|\\
&\leq\frac{\rho(x^\rho-a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|dt\\
&+\frac{\rho(b^\rho-x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|dt\\
&\leq\frac{\rho(x^\rho-a^\rho)}{2}\left(\int_0^1 t^{\alpha\rho+\rho-1}dt\right)^{1-\frac{1}{q}}\times\\
&\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{\rho(b^\rho-x^\rho)}{2}\left(\int_0^1 t^{\alpha\rho+\rho-1}dt\right)^{1-\frac{1}{q}}\times\\
&\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}\tag{21}$$

Since $\left|f'\right|^q$ is s-Godunova-Levin function of second kind on $[a^\rho,b^\rho]$ and $\left|f'(x^\rho)\right|\leq M$, we get

$$\begin{aligned}&\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)a^\rho)\right|^q dt\\
&\leq\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}\left|f'(x^\rho)\right|^q+\frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\left|f'(a^\rho)\right|^q\right]dt\\
&\leq M^q\int_0^1\left[\frac{t^{\alpha\rho+\rho-1}}{(t^\rho)^s}+\frac{t^{\alpha\rho+\rho-1}}{(1-t^\rho)^s}\right]dt\\
&=M^q\left[\frac{1}{\rho(\alpha-s+1)}+\frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{\rho\ {}^\rho\Gamma(\alpha-s+2)}\right]\end{aligned}\tag{22}$$

similarly

$$\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+(1-t^\rho)b^\rho)\right|^q dt$$
$$\leq M^q\left[\frac{1}{\rho(\alpha-s+1)}+\frac{{}^\rho\Gamma(\alpha+1)\ {}^\rho\Gamma(1-s)}{\rho\ {}^\rho\Gamma(\alpha-s+2)}\right]. \tag{23}$$

Using (22) and (23) in (21) we can attain (20). □

Corollary 2.22. *In Theorem* 3.8*, if we take* $s=0$*, which means that* $\left|f'\right|$ *is p-function, then (20) becomes the following inequality*

$$\left|f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x^+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$
$$\leq\frac{M\rho(b^\rho-a^\rho)}{2(\rho(\alpha+1))^{1-\frac{1}{q}}}\left[\frac{2}{\rho(\alpha+1)}\right]^{\frac{1}{q}};\ x\in[a,b].$$

Corollary 2.23. *In Theorem* 3.8*, if we take* $s=1$*, which means that* $\left|f'\right|$ *is Godunova-Levin function, then (20) becomes the following inequality*

$$\left|f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x^+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|$$
$$\leq\frac{M\rho(b^\rho-a^\rho)}{2(\rho(\alpha+1))^{1-\frac{1}{q}}}\left[\frac{1+\alpha}{\rho\alpha}\right]^{\frac{1}{q}};\ x\in[a,b].$$

Conclusion. All results proved in this research paper can also be deduced for Hadamard fractional integrals just by taking limits when parameter $\rho\to 0^+$.

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