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FEJÉR-HADAMARD TYPE INEQUALITIES FOR m-CONVEX FUNCTIONS VIA CAPUTO FRACTIONAL DERIVATIVES

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ABSTRACT. In this paper we prove certain Fejér Hadamard type inequalities for m-convex functions via Caputo fractional derivatives and several known results are deduced. We deduce Fejér Hadamard-type inequalities for convex functions via Caputo fractional derivatives. As special cases we obtain Hadamard inequalities for m-convex functions via Caputo fractional derivatives.

1. Introduction

Definition 1. *A function $f : [0, b] \to \mathbb{R}$, $b > 0$, is called m-convex, $0 \leq m \leq 1$, if for any $x, y \in [0, b]$ and $t \in [0, 1]$ we have*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

For $m = 1$ we have the definition of convex function.

The following inequality for a convex function $f : I \to \mathbb{R}$ holds;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $a, b \in I$ with $a < b$. It is well known as Hadamard inequality.

In [5], Fejér established the following inequality which is the weighted generalization of Hadamard inequality (1.1).

Theorem 1. *Let $f : I \to \mathbb{R}$ be a convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : I \to \mathbb{R}$ is nonnegative, integrable and symmetric function about $\frac{a+b}{2}$.

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It is well known as Fejér-Hadamard inequality.
For more information related to (1.1) and (1.2) one can consult [1, 7, 10, 13]. In the following we give the definition of Caputo fractional derivatives [9].

Definition 2. *Let* $\alpha > 0$ *and* $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$. *The Caputo fractional derivatives of order* α *are defined as follows:*

$$(1.3)\qquad {}^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, x > a$$

and

$$(1.4)\qquad {}^{C}D^{\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, x < b.$$

If $\alpha = n \in \{1, 2, 3, ...\}$ *and usual derivative of order* n *exists, then Caputo fractional derivative* $({}^{C}D^{\alpha}_{a+}f)(x)$ *coincides with* $f^{(n)}(x)$. *In particular we have*

$$(1.5)\qquad ({}^{C}D^{0}_{a+}f)(x) = ({}^{C}D^{0}_{b-}f)(x) = f(x)$$

where $n = 1$ *and* $\alpha = 0$.

In this paper we assume that $\|g^{(n)}\|_{\infty} = sup_{t\in[a,b]}|g^{(n)}(x)|$, where $g : [a, b] \to \mathbb{R}$ is continuous function and $g^{(n)}$ be positive and convex function on $[a, b]$.
Here is the following convolution $f * g$ of functions f and g for Caputo fractional derivatives.

$$(1.6)\qquad {}^{C}D^{\alpha}_{a+}(f*g)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)g^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, x > a$$

and

$$(1.7)\qquad {}^{C}D^{\alpha}_{b-}(f*g)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)g^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, x < b.$$

In [3] following results for m-convex functions via Caputo fractional derivatives hold.

Lemma 1. *Let* $f : [a, mb] \to \mathbb{R}$ *be a differentiable function on* (a, mb) *such that* $f \in C^n[a, mb]$ *with* $a < mb$. *Also let* $f^{(n+1)}$ *be positive and* m*-convex function on* $[a, mb]$. *Then the following equality for Caputo* *fractional derivatives holds*

$$\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}[{}^{C}D^{\alpha}_{a^{+}}f(mb)+{}^{C}D^{\alpha}_{mb-}f(a)] = \frac{mb-a}{2}\int_0^1[(1-t)^{n-\alpha}-t^{n-\alpha}]f^{(n+1)}(ta+m(1-t)b)dt. \tag{1.8}$$

Theorem 2. *Let $f:[a,mb]\to\mathbb{R}$ be a differentiable function on (a,mb) such that $f\in C^{n+1}[a,mb]$ with $0\le a<mb$. If $\left|f^{(n+1)}\right|$ is m-convex on $[a,mb]$, then the following inequality for Caputo fractional derivatives holds*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{a^{+}}f(mb)+{}^{C}D^{\alpha}_{mb-}f(a)\right]\right| \le \frac{mb-a}{2(n-\alpha+1)}\left(1-\frac{1}{2^{n-\alpha}}\right)[f^{(n+1)}(a)+mf^{(n+1)}(b)]. \tag{1.9}$$

There in [3] we remark that for $m=1$ in above results we get the results of [4], and for $\alpha=0$, $n=1$ along with $m=1$ in above results we get results of [2]. In [6] the following results related to Fejér-Hadamard type inequalities via Caputo fractional derivatives are reduced for $m=n=1$ along with $\alpha=0$.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be the function with $a<b$ and $f\in C^n[a,b]$. Also let $f^{(n)}$ be positive and convex functions on $[a,b]$. If $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then following inequalities for Caputo fractional derivatives hold*

$$\begin{aligned} f^{(n)}\left(\frac{a+b}{2}\right)&\left[{}^{C}D^{\alpha}_{a+}g(b)+(-1)^{n C}D^{\alpha}_{b-}g(a)\right]\\ &\le[{}^{C}D^{\alpha}_{a+}(f*g)(b)+(-1)^{n C}D^{\alpha}_{b-}(f*g)(a)]\\ &\le\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[{}^{C}D^{\alpha}_{a+}g(b)+(-1)^{n C}D^{\alpha}_{b-}g(a)\right] \end{aligned} \tag{1.10}$$

In [4] following results for convex functions via Caputo fractional derivatives hold.

Theorem 4. *Let $f:I\to\mathbb{R}$,$0\le a<b$ be a differentiable mapping on I^o such that $f\in C^n[a,b]$. If $|f^{(n+1)}|$ is convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then following inequality for*

Caputo fractional derivatives hold

$$\begin{aligned}&\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}[{}^{C}D^{\alpha}_{a+}g(b)+(-1)^{n C}D^{\alpha}_{b-}g(a)]\right.\\&\left.-[{}^{C}D^{\alpha}_{a+}(f*g)(b)+(-1)^{n C}D^{\alpha}_{b-}(f*g)(a)]\right|\\&\leq\frac{(b-a)^{\alpha+1}\|g^{(n)}\|_{\infty}}{(n-\alpha+1)\Gamma(n-\alpha+1)}\left(1-\frac{1}{2^{n-\alpha}}\right)\left[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|\right]\end{aligned}\tag{1.11}$$

Theorem 5. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I^o with $a < b$. If $|f'|^q$, $q > 1$ is convex on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then following inequality for fractional integrals hold*

$$\begin{aligned}&\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[{}^{C}D^{\alpha}_{a+}g(b)+(-1)^{n C}D^{\alpha}_{b-}g(a)\right]-\right.\\&\left.\left[{}^{C}D^{\alpha}_{a+}(f*g)(b)+(-1)^{n C}D^{\alpha}_{b-}(f*g)(a)\right]\right|\\&\leq\frac{2(b-a)^{1-\alpha}\|g^{(n)}\|_{\infty}\left(1-\frac{1}{2^{\alpha}}\right)}{(n-\alpha+1)\Gamma(n-\alpha+1)}\left(\frac{|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q}{2}\right)^{\frac{1}{q}}\end{aligned}\tag{1.12}$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 6. *Let $: I \to \mathbb{R}, 0 \leq a < b$ be a differential mapping on I^o such that $f \in C^n[a, b]$. Also let $|f^{(n+1)}|^q$, $q > 1$ is convex on $[a, b]$ and $g : [a, b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then following inequalities for Caputo fractional derivatives hold*

$$\begin{aligned}(i)\;&\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[{}^{C}D^{\alpha}_{a+}g(b)+(-1)^{n C}D^{\alpha}_{b-}g(a)\right]\right.\\&\left.-\left[{}^{C}D^{\alpha}_{a+}(f*g)(b)+(-1)^{n C}D^{\alpha}_{b-}(f*g)(a)\right]\right|\\&\leq\frac{2^{\frac{1}{p}}(b-a)^{1-\alpha}\|g^{(n)}\|_{\infty}}{(n-\alpha p+1)^{\frac{1}{p}}\Gamma(n-\alpha+1)}(1-2^{\alpha p})^{\frac{1}{p}}\left(\frac{|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q}{2}\right)^{\frac{1}{q}}\end{aligned}\tag{1.13}$$

$$\begin{aligned}(ii)\;&\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}\left[{}^{C}D^{\alpha}_{a+}g(b)+(-1)^{n C}D^{\alpha}_{b-}g(a)\right]\right.\\&\left.-[{}^{C}D^{\alpha}_{a+}(f*g)(b)+(-1)^{n C}D^{\alpha}_{b-}(f*g)(a)]\right|\\&\leq\frac{(b-a)^{1-\alpha}\|g^{(n)}\|_{\infty}}{(n-\alpha+1p)^{\frac{1}{p}}\Gamma(n-\alpha+1)}\left(\frac{|f^{(n+1)}(a)|^q+|f^{(n+1)}(b)|^q}{2}\right)^{\frac{1}{q}}\end{aligned}\tag{1.14}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In this paper we give Fejér-Hadamard type inequalities for m-convex functions via Caputo fractional derivatives and note that results in [6] are special cases of these inequalities. Also we present new results as generalizations of Hadamard inequalities for Caputo fractional derivatives and deduce some results of [3, 11].
In the whole paper $C^n[a,b]$ denotes the space of n-times differentiable functions such that $f^{(n)}$ are continuous on $[a,b]$.

2. Fejér-Hadamard type inequalities for m-convex functions via Caputo fractional derivatives

Following lemma is proved in [4].

Lemma 2. *If $g:[a,b]\to\mathbb{R}$ is an integrable and symmetric to $\frac{a+b}{2}$ with $a<b$, then*

$$^{C}D^{\alpha}_{a+}g(b) = {}^{C}D^{\alpha}_{b-}g(a) = \frac{1}{2}[^{C}D^{\alpha}_{a+}g(b) + (-1)^{n C}D^{\alpha}_{b-}g(a)]$$

Following lemma is given in [16].

Lemma 3. *[16] For $0<\lambda\leq 1$ and $0\leq a<b$, we have*

$$|a^{\lambda}-b^{\lambda}|\leq (b-a)^{\lambda}.$$

Here first we prove following result.

Lemma 4. *If $g:[a,b]\to\mathbb{R}$ is an integrable and symmetric to $\frac{a+mb}{2}$ with $a<b$, then*

$$^{C}D^{\alpha}_{a+}g(mb) = {}^{C}D^{\alpha}_{mb-}g(a) = \frac{1}{2}[^{C}D^{\alpha}_{a+}g(mb) + (-1)^{n C}D^{\alpha}_{mb-}g(a)]$$

Proof. We have

$$^{C}D^{\alpha}_{a+}g(mb) = \frac{1}{\Gamma(n-\alpha)}\int_a^{mb}(mb-x)^{n-\alpha-1}g^{(n)}(x)dx$$

Setting $x=a+mb-x$ in the following integral we have

$$= \frac{1}{\Gamma(n-\alpha)}\int_a^{mb}(x-a)^{n-\alpha-1}g^{(n)}(a+mb-x)dx$$

By symmetricity of $g^{(n)}$ we have $g^{(n)}(a+mb-x)=g^{(n)}(x)$, using it we get

$$= \frac{1}{\Gamma(n-\alpha)}\int_a^{mb}(x-a)^{n-\alpha-1}g^{(n)}(x)dx$$

$$= {}^{C}D^{\alpha}_{mb-}g(a).$$

□

Using above lemma we prove following results.

Theorem 7. *Let $f : [a,b] \to \mathbb{R}$ be the function with $a < b$ and $f \in C^n[a,b]$. Also let $f^{(n)}$ be positive and m-convex functions on $[a,b]$. If $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+mb}{2}$, then following inequalities for Caputo fractional derivatives hold*

$$
\begin{aligned}
(2.1)\qquad & 2f^{(n)}\left(\frac{a+mb}{2}\right) {}^{C}D^{\alpha}_{b-}g\left(\frac{a}{m}\right) \le (1+m)^{C}D^{\alpha}_{b-}(f*g)\left(\frac{a}{m}\right)\\
&\le \frac{\left[f^{(n)}(a) - m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right]}{b-\frac{a}{m}} {}^{C}D^{\alpha-1}_{b-}g\left(\frac{a}{m}\right)\\
&+ m\left[f^{(n)}(b) + mf^{(n)}\left(\frac{a}{m^2}\right)\right] {}^{C}D^{\alpha}_{b-}g\left(\frac{a}{m}\right).
\end{aligned}
$$

Proof. Using m-convexity of $f^{(n)}$ we have

$$
\begin{aligned}
(2.2)\quad f^{(n)}\left(\frac{a+mb}{2}\right) &= f^{(n)}\left(\frac{ta+m(1-t)b+m\left(tb+(1-t)\frac{a}{m}\right)}{2}\right)\\
&\le \frac{f^{(n)}(ta+m(1-t)b) + mf^{(n)}(tb+(1-t)\frac{a}{m})}{2},
\end{aligned}
$$

where $t \in [0,1]$.
Multiplying both sides of above inequality with $2t^{n-\alpha-1}g^{(n)}(tb+(1-t)\frac{a}{m})$ and integrating the resulting inequality over $[0,1]$ we have,

$$
\begin{aligned}
&2f^{(n)}\left(\frac{a+mb}{2}\right)\int_0^1 t^{n-\alpha-1}g^{(n)}\left(tb+(1-t)\frac{a}{m}\right)dt\\
&\le \int_0^1 t^{n-\alpha-1}f^{(n)}(ta+m(1-t)b)g^{(n)}\left(tb+(1-t)\frac{a}{m}\right)dt\\
&+ m\int_0^1 t^{n-\alpha-1}f^{(n)}\left(tb+(1-t)\frac{a}{m}\right)g^{(n)}\left(tb+(1-t)\frac{a}{m}\right)dt.
\end{aligned}
$$

Putting $x = tb + (1-t)\frac{a}{m}$ and using $f^{(n)}(a + mb - mx) = f^{(n)}(x)$ we get

$$\frac{2}{(b-\frac{a}{m})^{n-\alpha}} f^{(n)}\left(\frac{a+mb}{2}\right) \int_{\frac{a}{m}}^{b} \left(x - \frac{a}{m}\right)^{n-\alpha+1} g^{(n)}(x)dx$$
$$\leq \frac{1}{(b-\frac{a}{m})^{n-\alpha}} \left[\int_{\frac{a}{m}}^{b} \left(x - \frac{a}{m}\right)^{n-\alpha-1} f^{(n)}(x)g^{(n)}(x)dx\right.$$
$$\left. + m\int_{\frac{a}{m}}^{b} \left(x - \frac{a}{m}\right)^{n-\alpha-1} f^{(n)}(x)g^{(n)}(x)dx\right]$$
$$= \frac{1}{(b-\frac{a}{m})^{n-\alpha}} (1+m) \int_{\frac{a}{m}}^{b} \left(x - \frac{a}{m}\right)^{n-\alpha-1} f^{(n)}(x)g^{(n)}(x)dx$$
$$2f^{(n)}\left(\frac{a+mb}{2}\right) {}^{C}D_{b-}^{\alpha} g\left(\frac{a}{m}\right) \leq (1+m) {}^{C}D_{b-}^{\alpha}(f * g)\left(\frac{a}{m}\right).$$

For second inequality of (2.1) m-convexity of $f^{(n)}$ gives

$$f^{(n)}(ta + m(1-t)b) + mf^{(n)}\left(tb + (1-t)\frac{a}{m}\right)$$
$$\leq m\left[f^{(n)}(b) + mf^{(n)}\left(\frac{a}{m^2}\right)\right] + t\left[f^{(n)}(a) - m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right],$$

where $t \in [0,1]$.
Multiplying both sides of above inequality with $t^{n-\alpha-1}g^{(n)}\left(tb + (1-t)\frac{a}{m}\right)$ and integrating the resulting inequality over $[0,1]$ we get,

$$\int_0^1 t^{n-\alpha-1} f^{(n)}(ta + m(1-t)b) g^{(n)}\left(tb + (1-t)\frac{a}{m}\right) dt$$
$$+ m\int_0^1 t^{n-\alpha-1} f^{(n)}\left(tb + (1-t)\frac{a}{m}\right) g^{(n)}\left(tb + (1-t)\frac{a}{m}\right) dt$$
$$\leq m\left[f^{(n)}(b) + mf^{(n)}\left(\frac{a}{m^2}\right)\right] \int_0^1 t^{n-\alpha-1} g^{(n)}\left(tb + (1-t)\frac{a}{m}\right) dt$$
$$+ \left[f^{(n)}(a) - m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right] \int_0^1 t^{n-\alpha} g^{(n)}\left(tb + (1-t)\frac{a}{m}\right) dt$$

from which one can get second inequality of (2.1). □

Remark 1. *In Theorem 7, if we take $m = 1$ and use Lemma 2, we get Theorem 3.*

Next we need following lemma.

Lemma 5. *Let $f : [a,b] \to \mathbb{R}$ with $0 \leq a < b$ be a positive and m-convex function on $[a,b]$ such that $f \in C^{n+1}[a,b]$. If $g : [a,b] \to \mathbb{R}$*

is integrable and symmetric to $\frac{a+mb}{2}$. *If* $g \in C^{n+1}[a,b]$ *then following equality for Caputo fractional derivatives holds*

$$\begin{aligned}
&\left(\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}\right)\left[({}^{C}D^{\alpha}_{a+}g)(mb)+(-1)^{n}({}^{C}D^{\alpha}_{mb-}g)(a)\right] \\
&-\left[({}^{C}D^{\alpha}_{a+}(f*g))(mb)+(-1)^{n}({}^{C}D^{\alpha}_{mb-}(f*g))(a)\right] \\
&=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{mb}\left[\int_{a}^{t}(mb-s)^{n-\alpha-1}g^{(n)}(s)ds\right. \\
&\left.-\int_{t}^{mb}(s-a)^{n-\alpha-1}g^{(n)}(s)ds\right]f^{(n+1)}(t)dt. \qquad (2.3)
\end{aligned}$$

Proof. One can note that

$$\begin{aligned}
&\frac{1}{\Gamma(n-\alpha)}\int_{a}^{mb}\left[\int_{a}^{t}(mb-s)^{n-\alpha-1}g^{(n)}(s)ds\right. \\
&\left.-\int_{t}^{mb}(s-a)^{n-\alpha-1}g^{(n)}(s)ds\right]f^{(n+1)}(t)dt \\
&=\frac{1}{\Gamma(n-\alpha)}\left[\int_{a}^{mb}\left(\int_{a}^{t}(mb-s)^{n-\alpha-1}g^{(n)}(s)ds\right)f^{(n+1)}(t)dt\right. \qquad (2.4) \\
&\left.+\int_{a}^{mb}\left(-\int_{t}^{mb}(s-a)^{n-\alpha-1}g^{(n)}(s)ds\right)f^{(n+1)}(t)dt\right],
\end{aligned}$$

By simple calculation one can get

$$\begin{aligned}
&\int_{a}^{mb}\left(\int_{a}^{t}(mb-s)^{n-\alpha-1}g^{(n)}(s)ds\right)f^{(n+1)}(t)dt \\
&=\left[\left(\int_{a}^{mb}(mb-s)^{n-\alpha-1}g^{(n)}(s)ds\right)f^{(n)}(mb)\right. \\
&\left.-\int_{a}^{mb}(mb-t)^{n-\alpha-1}f^{(n)}(t)g^{(n)}(t)dt\right] \\
&=\Gamma(n-\alpha)\left[f^{(n)}(mb)({}^{C}D^{\alpha}_{a+}g)(mb)-({}^{C}D^{\alpha}_{a+}(f*g))(mb)\right] \\
&=\Gamma(n-\alpha)\left[\frac{f^{(n)}(mb)}{2}\left[({}^{C}D^{\alpha}_{a+}g)(mb)+(-1)^{n}({}^{C}D^{\alpha}_{mb-}g)(a)\right]\right. \\
&\left.-({}^{C}D^{\alpha}_{a+}(f*g))(mb)\right],
\end{aligned}$$

and

$$\int_a^{mb}\left(-\int_t^{mb}(s-a)^{n-\alpha-1}g(s)ds\right)f^{(n+1)}(t)dt$$
$$=\left(\int_a^{mb}(s-a)^{n-\alpha-1}g(s)ds\right)f(a)-\int_a^{mb}(t-a)^{n-\alpha-1}f^{(n)}(t)g^{(n)}(t)dt$$
$$=\Gamma(n-\alpha)\left[\frac{f^{(n)}(a)}{2}[D^{\alpha}_{a+}g(mb)+(-1)^{nC}D^{\alpha}_{mb-}g(a)]-(-1)^{nC}D^{\alpha}_{mb-}(f*g)(a)\right].$$

Hence (2.3) is established. □

Remark 2. *In Lemma 5,*

(i) If we take g(x) = 1, then equality (2.3) becomes equality (1.8) of Lemma 1.

(ii) If we take $g(x)=1$ along with $m=1$ in above lemma we get [4, Lemma 3].

(iii) If we take $m=1$ in above lemma we get [6, Lemma 4].

(iv) If we take $\alpha=0$, $n=m=1$ along with $g(x)=1$ in above lemma we get [2, Lemma 2.1].

Theorem 8. *Let $f:[a,b]\to\mathbb{R}$ with $0\le a<b$ be a positive such that $f\in C^{n+1}[a,b]$. Also let $|f^{(n+1)}|$ is m-convex function on $[a,b]$. If $g:[a,b]\to\mathbb{R}$ is integrable and symmetric to $\frac{a+mb}{2}$. Also $g\in C^{n+1}[a,b]$ then following equality for Caputo fractional derivatives holds*

$$\Bigg|\left(\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}\right)[({}^CD^{\alpha}_{a+}g)(mb)+(-1)^n({}^CD^{\alpha}_{mb-}g)(a)]$$
$$-[({}^CD^{\alpha}_{a+}(f*g))(mb)+(-1)^n({}^CD^{\alpha}_{mb-}(f*g))(a)]\Bigg|$$
$$\le\frac{(mb-a)^{n-\alpha+1}\|g^{(n)}\|_\infty}{(n-\alpha+1)\Gamma(n-\alpha+1)}\left(1-\frac{1}{2^{n-\alpha}}\right)\left[|f^{(n+1)}(a)|+m|f^{(n+1)}(b)|\right],$$

where $\|g^{(n)}\|_\infty=sup_{x\in[a,b]}|g^{(n)}(x)|$.

Proof. Using Lemma 5 we have

$$\Bigg|\left(\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}\right)[({}^CD^{\alpha}_{a+}g)(mb)+(-1)^n({}^CD^{\alpha}_{mb-}g)(a)]$$
$$-[({}^CD^{\alpha}_{a+}(f*g))(mb)+(-1)^n({}^CD^{\alpha}_{mb-}(f*g))(a)]\Bigg|$$
$$\le\frac{1}{\Gamma(n-\alpha)}\int_a^{mb}\Bigg|\int_a^t(mb-s)^{n-\alpha-1}g^{(n)}(s)ds$$
$$-\int_t^{mb}(s-a)^{n-\alpha-1}g^{(n)}(s)ds\Bigg||f^{(n+1)}(t)|dt. \tag{2.5}$$

Using m-convexity of $|f^{(n+1)}|$ we have

$$|f^{(n+1)}(t)| \leq \frac{mb-t}{mb-a}|f^{(n+1)}(a)| + m\frac{t-a}{mb-a}|f^{(n+1)}(b)|, \tag{2.6}$$

where $t \in [a,b]$.

One can have by symmetricity of $g^{(n)}$

$$\begin{aligned}\int_t^{mb}(s-a)^{n-\alpha-1}g^{(n)}(s)ds &= \int_a^{a+mb-t}(mb-s)^{n-\alpha-1}g^{(n)}(a+mb-s)ds\\ &= \int_a^{a+mb-t}(mb-s)^{n-\alpha-1}g^{(n)}(s)ds.\end{aligned}$$

This gives

$$\begin{aligned}&\left|\int_a^t(mb-s)^{n-\alpha-1}g^{(n)}(s)ds - \int_t^{mb}(s-a)^{n-\alpha-1}g^{(n)}(s)ds\right|\\ &= \left|\int_t^{a+mb-t}(mb-s)^{n-\alpha-1}g^{(n)}(s)ds\right|\\ &\leq \begin{cases}\int_t^{a+mb-t}|(mb-s)^{n-\alpha-1}g^{(n)}(s)|ds, & t \in [a, \frac{a+mb}{2}]\\ \int_{a+mb-t}^{t}|(mb-s)^{n-\alpha-1}g^{(n)}(s)|ds, & t \in [\frac{a+mb}{2}, mb].\end{cases}\end{aligned} \tag{2.7}$$

By virtue of (2.5), (2.6), (2.7), we have

$$\begin{aligned}
&\left|\left(\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}\right)\left[({}^CD^{\alpha}_{a+}g)(mb)+(-1)^n({}^CD^{\alpha}_{mb-}g)(a)\right]\right.\\
&\left.-\left[({}^CD^{\alpha}_{a+}(f*g))(mb)+(-1)^n({}^CD^{\alpha}_{mb-}(f*g))(a)\right]\right|\\
&\leq \frac{1}{\Gamma(n-\alpha)}\left[\int_a^{\frac{a+mb}{2}}\left(\int_t^{a+mb-t}\left|(mb-s)^{n-\alpha-1}g^{(n)}(s)\right|ds\right)\right.\\
&\left(\frac{mb-t}{mb-a}|f^{(n+1)}(a)|+m\frac{t-a}{mb-a}|f^{(n+1)}(b)|\right)dt\\
&+\int_{\frac{a+mb}{2}}^{mb}\left(\int_{a+mb-t}^{t}\left|(mb-s)^{n-\alpha-1}g^{(n)}(s)ds\right|\right)\\
&\left.\left(\frac{mb-t}{mb-a}|f^{(n+1)}(a)|+m\frac{t-a}{mb-a}|f^{(n+1)}(b)|\right)dt\right]\\
&\leq \frac{\|g^{(n)}\|_\infty}{\Gamma(n-\alpha+1)(mb-a)}\left[\int_a^{\frac{a+mb}{2}}\left((mb-t)^{n-\alpha}-(t-a)^{n-\alpha}\right)\right.\\
&\left((mb-t)|f^{(n+1)}(a)|+m(t-a)|f^{(n+1)}(b)|\right)dt\\
&+\int_{\frac{a+mb}{2}}^{mb}\left((t-a)^{n-\alpha}-(mb-t)^{n-\alpha}\right)
\end{aligned}$$

$$\left.\left((mb-t)|f^{(n+1)}(a)|+m(t-a)|f^{(n+1)}(b)|\right)dt\right]. \tag{2.8}$$

One can have

$$\begin{aligned}
&\int_a^{\frac{a+mb}{2}}((mb-t)^{n-\alpha}-(t-a)^{n-\alpha})(mb-t)dt\\
&=\int_{\frac{a+mb}{2}}^{mb}((t-a)^{n-\alpha}-(mb-t)^{n-\alpha})(t-a)dt
\end{aligned}$$

$$=\frac{(mb-a)^{n-\alpha+2}}{n-\alpha+1}\left(\frac{n-\alpha+1}{n-\alpha+2}-\frac{1}{2^{n-\alpha+1}}\right) \tag{2.9}$$

and

$$\begin{aligned}
&\int_a^{\frac{a+mb}{2}}((mb-t)^{n-\alpha}-(t-a)^{n-\alpha})(t-a)dt\\
&=\int_{\frac{a+mb}{2}}^{mb}((t-a)^{n-\alpha}-(mb-t)^{n-\alpha})(mb-t)dt
\end{aligned}$$

$$=\frac{(mb-a)^{n-\alpha+2}}{(n-\alpha+1)}\left(\frac{1}{n-\alpha+2}-\frac{1}{2^{n-\alpha+1}}\right). \tag{2.10}$$

Using (2.9), (2.10) in (2.8) we get required result. □

Remark 3. *In Theorem 8,*

(i) if we take $m = 1$ *we get Theorem 4.*

(ii) if we take $g(x) = 1$ *in above Theorem we get inequality (1.9) of Theorem 2.*

(iii) if we take $g(x) = 1$ *along with* $m = 1$ *in above Theorem we get [4, Theorem 3].*

(iv) if we take $\alpha = 0$ *along with* $g(x) = n = m = 1$ *in above Theorem we get [2, Theorem 2.2].*

Theorem 9. *Let* $f : [a, b] \to \mathbb{R}, 0 \leq a < b$ *be a mapping such that* $f \in C^{n+1}[a, b]$*. If* $|f^{(n+1)}|^q$*,* $q > 1$ *is m-convex on* $[a, b]$ *and* $g : [a, b] \to \mathbb{R}$ *is continuous and symmetric to* $\frac{a+mb}{2}$*. Also* $g \in C^{n+1}[a, b]$*, then following inequality for Caputo fractional derivatives hold*

$$\begin{aligned}
&\left|\left(\frac{f^{(n)}(a) + f^{(n)}(mb)}{2}\right)\left[({}^C D^{\alpha}_{a+}g)(mb) + (-1)^n({}^C D^{\alpha}_{mb-}g)(a)\right]\right. \\
&\left. - \left[({}^C D^{\alpha}_{a+}(f * g))(mb) + (-1)^n({}^C D^{\alpha}_{mb-}(f * g))(a)\right]\right| \\
&\leq \frac{2(mb-a)^{n-\alpha-1}\|g^{(n)}\|_\infty}{(n-\alpha+1)\Gamma(n-\alpha+1)(mb-a)^{\frac{1}{q}}}\left(1 - \frac{1}{2^{n-\alpha}}\right) \\
&\left(\frac{|f^{(n+1)}(a)|^q + m|f^{(n+1)}(b)|^q}{2}\right)^{\frac{1}{q}} \qquad (2.11)
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$*.*

Proof. By Using Lemma 5, Hölder inequality, inequality (2.7) and m-convexity of $|f^{(n+1)}|^q$ respectively we have

$$\begin{aligned}
&\left|\left(\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}\right)\left[({}^C D^{\alpha}_{a+}g)(mb)+(-1)^n({}^C D^{\alpha}_{mb-}g)(a)\right]\right.\\
&\left.-\left[({}^C D^{\alpha}_{a+}(f*g))(mb)+(-1)^n({}^C D^{\alpha}_{mb-}(f*g))(a)\right]\right|\\
&\leq \frac{1}{\Gamma(n-\alpha)}\left[\int_a^{mb}\left|\int_t^{a+mb-t}(mb-s)^{n-\alpha-1}g^{(n)}(s)ds\right|dt\right]^{1-\frac{1}{q}}\\
&\left[\int_a^{mb}\left|\int_t^{a+mb-t}(mb-s)^{n-\alpha-1}g^{(n)}(s)ds\right||f^{(n+1)}(t)|^q dt\right]^{\frac{1}{q}}\\
&\leq \frac{1}{\Gamma(n-\alpha)}\left[\int_a^{\frac{a+mb}{2}}\left(\int_t^{a+mb-t}|(mb-s)^{n-\alpha-1}g^{(n)}(s)|ds\right)dt\right.\\
&\left.+\int_{\frac{a+mb}{2}}^{mb}\left(\int_{a+mb-t}^{t}|(mb-s)^{n-\alpha-1}g^{(n)}(s)|ds\right)dt\right]^{1-\frac{1}{q}}\\
&\left[\int_a^{\frac{a+mb}{2}}\left(\int_t^{a+mb-t}\left|(mb-s)^{n-\alpha-1}g(s)\right|ds\right)|f^{(n+1)}(t)|^q dt\right.\\
&\left.+\int_{\frac{a+mb}{2}}^{mb}\left(\int_{a+mb-t}^{t}|(mb-s)^{n-\alpha-1}g^{(n)}(s)|ds\right)|f^{(n+1)}(t)|^q dt\right]^{\frac{1}{q}}\\
&\leq \frac{\|g\|_\infty}{\Gamma(n-\alpha)}\left[\left(\frac{2(mb-a)^{n-\alpha-1}}{(n-\alpha)(n-\alpha-1)}\left(1-\frac{1}{2^{n-\alpha}}\right)\right)^{1-\frac{1}{q}}\right.\\
&\left.\left(\frac{(|f^{(n+1)}(a)|^q+m|f^{(n+1)}(b)|^q)(mb-a)^{n-\alpha-1}}{(n-\alpha)(n-\alpha+1)(mb-a)}\left(1-\frac{1}{2^{n-\alpha}}\right)\right)^{\frac{1}{q}}\right].
\end{aligned}$$

From which after a little computation one can have required result. □

Remark 4. *If we take $m=1$ in Theorem 9 then we get Theorem 5.*

By using Lemma 3 , we prove the following results.

Theorem 10. *Let : $[a,b]\to\mathbb{R}, 0\leq a<b$ be a mapping such that $f\in C^{n+1}[a,b]$. Also let $|f^{(n+1)}|^q$, $q>1$ is m-convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous and symmetric to $\frac{a+mb}{2}$. Also $g\in C^{n+1}[a,b]$,* *then following inequalities for Caputo fractional derivatives hold*

$$(i)\left|\left(\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}\right)\left[({}^{C}D^{\alpha}_{a+}g)(mb)+(-1)^{n}({}^{C}D^{\alpha}_{mb-}g)(a)\right]\right.$$
$$\left.-\left[({}^{C}D^{\alpha}_{a+}(f*g))(mb)+(-1)^{n}({}^{C}D^{\alpha}_{mb-}(f*g))(a)\right]\right|$$
$$\leq\frac{2^{\frac{1}{p}}(mb-a)^{n-\alpha-1}\|g^{(n)}\|_{\infty}}{(np-\alpha p+1)^{\frac{1}{p}}\Gamma(n-\alpha+1)}\left(1-\frac{1}{2^{np-\alpha p}}\right)^{\frac{1}{p}}$$
$$\left(\frac{|f^{(n+1)}(a)|^{q}+m|f^{(n+1)}(b)|^{q}}{2}\right)^{\frac{1}{q}}. \tag{2.12}$$

$$(ii)\left|\left(\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}\right)\left[({}^{C}D^{\alpha}_{a+}g)(mb)+(-1)^{n}({}^{C}D^{\alpha}_{mb-}g)(a)\right]\right.$$
$$\left.-[({}^{C}D^{\alpha}_{a+}(f*g))(mb)+(-1)^{n}({}^{C}D^{\alpha}_{mb-}(f*g))(a)]\right|$$
$$\leq\frac{(mb-a)^{n-\alpha-1}\|g^{(n)}\|_{\infty}}{(np-\alpha p+1)^{\frac{1}{p}}\Gamma(n-\alpha+1)}\left(\frac{|f^{(n+1)}(a)|^{q}+m|f^{(n+1)}(b)|^{q}}{2}\right)^{\frac{1}{q}}, \tag{2.13}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By Using Lemma 5, Hölder inequality, inequality (2.7) and m-convexity of $|f'|^q$ we have

$$\Bigg|\left(\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}\right)\left[({}^{C}D^{\alpha}_{a+}g)(mb)+(-1)^{n}({}^{C}D^{\alpha}_{mb-}g)(a)\right]$$
$$-[({}^{C}D^{\alpha}_{a+}(f*g))(mb)+(-1)^{n}({}^{C}D^{\alpha}_{mb-}(f*g))(a)]\Bigg|$$
$$\leq \frac{1}{\Gamma(n-\alpha)}\left(\int_a^{mb}\left|\int_t^{a+mb-t}(mb-s)^{n-\alpha-1}g^{(n)}(s)ds\right|^p dt\right)^{\frac{1}{p}}$$
$$\left(\int_a^{mb}|f^{(n+1)}(t)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(n-\alpha)}\left[\int_a^{\frac{a+mb}{2}}\left(\int_t^{a+mb-t}|(mb-s)^{n-\alpha-1}g^{(n)}(s)|^p ds\right)dt\right.$$
$$\left.+\int_{\frac{a+mb}{2}}^{mb}\left(\int_{a+mb-t}^{t}|(mb-s)^{n-\alpha-1}g^{(n)}(s)|^p ds\right)dt\right]^{\frac{1}{p}}$$
$$\left[\int_a^{mb}\left(\frac{mb-t}{mb-a}|f^{(n+1)}(a)|^q+m\frac{t-a}{mb-a}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}$$
$$\leq \frac{\|g^{(n)}\|_\infty}{\Gamma(n-\alpha+1)}\left[\int_a^{\frac{a+mb}{2}}\left((mb-t)^{n-\alpha}-(t-a)^{n-\alpha}\right)^p dt\right.$$
$$\left.+\int_{\frac{a+mb}{2}}^{mb}\left((t-a)^{n-\alpha}-(mb-t)^{n-\alpha}\right)^p dt\right]^{\frac{1}{p}}$$
$$\left[\int_a^{mb}\left(\frac{mb-t}{mb-a}|f^{(n+1)}(a)|^q+m\frac{t-a}{mb-a}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}. \tag{2.14}$$

Now

$$(A-B)^q \leq A^q - B^q, \quad A \geq B \geq 0$$

gives

$$[(mb-t)^{n-\alpha}-(t-a)^{n-\alpha}]^p \leq (mb-t)^{(n-\alpha)p}-(t-a)^{(n-\alpha)p} \tag{2.15}$$

for $t \in [a, \frac{a+mb}{2}]$,
and

$$[(t-a)^{n-\alpha}-(mb-t)^{n-\alpha}]^p \leq (t-a)^{(n-\alpha)p}-(mb-t)^{(n-\alpha)p} \tag{2.16}$$

for $t \in [\frac{a+mb}{2}, mb]$.

Using (2.15) and (2.16) in inequality (2.14) and solving we get required result.

For (2.13) use (2.14) and Lemma 3. □

Remark 5. *In Theorem 10, if we take $m = 1$ we get Theorem 6.*

References

[1] M. Bombardelli, S. Varo sanec, *Properties of h-convex functions related to the Hermite Hadamard Fejer inequalities*, Comput. Math. Appl., 58 (2009), 1869–1877.

[2] S. S. Dragmoir, R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. 11(5) (1998), 91-95.

[3] G. Farid, S. Naqvi, B. Tariq, *Caputo fractional integral inequalities via m-convex functions*, (Submitted).

[4] G. Farid, S. Naqvi, A. U. Rehman, *Hadamard and Fejér-Hadamard Inequalities And Related Results Via Caputo fractional derivatives*, (Submitted).

[5] L. Fejer, Uberdie Fourierreihen, II, Math. Naturwise. Anz Ungar. Akad., Wiss, 24 (1906), 369-390, (in Hungarian).

[6] İ. İşcan, *Hermite Hadamard Féjer type inequalities for convex functions via fractional integrals*, *Stud. Univ. Babeş-Bolyai Math.* 60(3) (2015), 355-366.

[7] İ. İşcan, *New estimates on generalization of some integral inequalities for (α, m)-convex functions*, Contemp. Anal. Appl. Math., 1(2) (2013), 253-264.

[8] A. A Kilbas, M. Saigo, R. K. Saxena, *Generalized Mittag-Leffler function and generalized fractional calculus operators*, Integral Transform. Spec. Funct. 15 (2004), 31-49.

[9] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional derivatial equations*, North-Holland Math. Stud. Elsevier New York-London, 2006.

[10] M.Z. Sarikaya, *On new Hermite Hadamard Fejér type integral inequalities*, Stud. Univ. Babe s– Bolyai Math. 57(3) (2012), 377386.

[11] M.Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, J. Math. Comput. Model. 57 (2013), 2403-2407.

[12] H. M. Srivastava, and Z.Tomovski, *Fractional Calculas with an integral operator containing generalized Mittag-Leffler function in the kerna*, Appl. Math. Comput., 211 (2009), 198-210.

[13] K.–L. Tseng, G.-S. Yang and K.-C. Hsu, *Some inequalities for differentiable mappings and applications to Fej er inequality and*

weighted trapezoidal formula, Taiwanese j. Mathematics, 15(4) (2011), 1737-1747.
[14] Z. Tomovski, R. Garra, *Analytic solutions of fractional integro-differential equations of volterra type with variable coefficients*, Fractional Calc. Appl. Anal 17(1) (2014), 38-60.
[15] Z.Tomovski, R. Hilfer, H. M. Srivastava, *Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler functions*, Integral Transform Spec. Funct. Vol. 21(11) (2010), 797-814.
[16] J. Wang, C. Zhu and Y. Zhou, *New generalized Hermite-Hadamard type inequalities and applications to special means*, J. Inequal. Appl., 2013(325) (2013), 15 pages.

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