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OSTROWSKI TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR MAPPINGS WHOSE DERIVATIVES ARE (α, m)-CONVEX VIA KATUGAMPOLA FRACTIONAL INTEGRALS

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Abstract. In this paper we have established a new identity for Katugampola fractional integrals. By using it we have found some generalizations of Riemann-Liouville fractional integral inequalities of Ostrowski type for (α, m)-convex functions. Also we prove some inequalities by taking particular appropriate values of α and m.

1. INTRODUCTION

The following inequality is known as Ostrowski inequality [9] (see also, [7, page 468]) which gives upper bound for approximation of integral average by the value $f(x)$ at point $x \in [a, b]$. It is proved by Ostrowski in 1938.

Theorem 1.1. *Let $f : I \to \mathbb{R}$ where I is interval in $\mathbb{R}$ be a mapping differentiable in I° the interior of I and $a, b \in I^\circ$, $a < b$. If $\left|f^{'}(t)\right| \le M$ for all $t \in [a, b]$, then we have*

$$\left|f(x) - \frac{1}{(b-a)} \int_a^b f(t)dt\right| \le \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)M,$$

where $x \in [a, b]$.

Ostrowski inequality has aroused the curiosity of many researchers which contributed many generalizations in literature recently Ostrowski type inequalities via Riemann-Liouville fractional integrals are in focus (see, [3, 8] and references their in). As we can find the bounds of different quadrature rules with the help of Ostrowski and Ostrowski type inequalities so Ostrowski and Ostrowski type inequalities have great importance in numerical analysis. Over the years researchers have worked to obtain Ostrowski type inequalities for different kinds of functions [1, 2, 10].

Definition 1.2. A function f is called convex function on the interval $[a, b]$ if for any two points $x, y \in [a, b]$ and any t where, $0 \le t \le 1$

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

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Definition 1.3. [8] A function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be $(\alpha, m)-$convex function where $(\alpha, m) \in [0, 1]^2$ if for any two points $x, y \in [0, b]$ and any t where, $0 \leq t \leq 1$

$$f(tx + (1-t)y) \leq t^{\alpha} f(x) + m(1 - t^{\alpha}) f(y).$$

Remark 1.4. It can be easily seen that.
(i) If $\alpha = 1$ and $m = 1$, then $(\alpha, m)-$convexity reduces to usual convexity defined on $[0, b]$, $b > 0$.
(ii) If $\alpha = 1$, then $(\alpha, m)-$convexity reduces to $m-$convexity defined on $[0, b]$, $b > 0$.
(iii) If $m = 1$, then $(\alpha, m)-$convexity reduces to $\alpha-$convexity defined on $[0, b]$, $b > 0$.

Laurent in [6] provided today's definition of the Riemann-Liouville fractional integral.

Definition 1.5. [6] Let $f \in L_1[a, b]$. The Riemann-Liouville fractional integral $J^{\alpha}_{a+} f$ and $J^{\alpha}_{b-} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t) dt, x > a$$

and

$$J^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha - 1} f(t) dt, x < b,$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha - 1} du$ is the integral representation of Euler gamma function. Here $J^0_{a+} f(x) = J^0_{b-} f(x) = f(x)$. In case of $\alpha = 1$, the Riemann-Liouville fractional integrals reduces to the classical integral.

Definition 1.6. J. Hadamard introduced the Hadamard fractional integral in [4], and is given by

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(log \frac{x}{\tau} \right)^{\alpha - 1} f(\tau) \frac{d\tau}{\tau},$$

for $Re(\alpha) > 0$, $x > a \geq 0$.

Recently Katugampola generalized Riemann-Liouville and Hadamard fractional integrals into a single form called Katugampola fractional integrals.

Definition 1.7. [5] Let $[a, b]$ be a finite interval in $\mathbb{R}$. Then Katugampola fractional integrals of order $\alpha > 0$ for a real valued function f are defined by

$${}^{\rho}I^{\alpha}_{a+} f\left(x\right) = \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)} \int_a^x t^{\rho - 1} \left(x^{\rho} - t^{\rho}\right)^{\alpha - 1} f\left(t\right) dt$$

and

$${}^{\rho}I^{\alpha}_{b-} f\left(x\right) = \frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)} \int_x^b t^{\rho - 1} \left(t^{\rho} - x^{\rho}\right)^{\alpha - 1} f\left(t\right) dt$$

with $a < x < b$ and $\rho > 0$. where $\Gamma\left(\alpha\right)$ is the Euler gamma function. For $\rho = 1$, Katugampola fractional integrals give Riemann-Liouville fractional integrals, while $\rho \to 0^+$ produces the Hadamard fractional integral. For its proof one can check [5].

We organize the paper in such a way that in the following section we prove some Ostrowski type fractional integral inequalities for mappings whose derivatives are (α, m)-convex via Katugampola fractional integrals. We also present some corollaries and some known results by taking particular values of α and m in our results.

2. Ostrowski type fractional inequalities for (α, m)-Convex functions via Katugampola fractional integral

In this section we present some Ostrowski type inequalities for (α, m)-Convex functions via Katugampola fractional integrals. The following lemma is very useful to obtain our results.

Lemma 2.1. *Let I be an open real interval such that $[0, \infty) \subset I$ and $f : I \to \mathbb{R}$ be a differentiable mapping on I such that $f' \in L_1[ma, mb]$, where $ma, mb \in I$ with $a < b$, $m \in (0, 1]$, then for all $x \in (ma, mb)$ and $\rho, \alpha > 0$ we have the following equality*

$$\left(\frac{(x^\rho - m^\rho a^\rho)^\alpha + (m^\rho b^\rho - x^\rho)^\alpha}{b-a}\right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} [{}^\rho I^\alpha_{x^-} f(m^\rho a^\rho)$$
$$+ \ {}^\rho I^\alpha_{x^+} f(m^\rho b^\rho)] = \frac{\rho(x^\rho - m^\rho a^\rho)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + m^\rho(1-t^\rho)a^\rho)dt$$
$$- \frac{\rho(m^\rho b^\rho - x^\rho)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + m^\rho(1-t^\rho)b^\rho)dt. \tag{1}$$

Proof. It is easy to see that

$$\int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + m^\rho m^\rho(1-t^\rho)a^\rho)dt$$
$$= \frac{t^{\alpha\rho+\rho-1} f(t^\rho x^\rho + m^\rho(1-t^\rho)a^\rho)}{\rho t^{\rho-1}(x^\rho - m^\rho a^\rho)}\Bigg|_0^1 - \frac{\alpha\rho + \rho - 1}{\rho(x^\rho - m^\rho a^\rho)} \int_0^1 t^{\alpha\rho-1} f(t^\rho x^\rho$$
$$+ m^\rho(1-t^\rho)a^\rho)dt$$
$$= \frac{f(x^\rho)}{\rho(x^\rho - m^\rho a^\rho)} - \frac{\alpha\rho+\rho-1}{\rho(x^\rho - m^\rho a^\rho)} \int_{ma}^x \left(\frac{y^\rho - m^\rho a^\rho}{x^\rho - m^\rho a^\rho}\right)^{\alpha-1} \frac{y^{\rho-1} f(y^\rho)}{x^\rho - m^\rho a^\rho} dy$$
$$= \frac{f(x^\rho)}{\rho(x^\rho - m^\rho a^\rho)} - \frac{{}^\rho I^\alpha_{x^-} f(m^\rho a^\rho)(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{2-\alpha}(x^\rho - m^\rho a^\rho)^{\alpha+1}} \tag{2}$$

and

$$\int_0^1 t^{\alpha\rho+\rho-1} f'(t^\rho x^\rho + m^\rho(1-t^\rho)b^\rho)dt$$
$$= \frac{t^{\alpha\rho+\rho-1} f(t^\rho x^\rho + m^\rho(1-t^\rho)b^\rho)}{\rho t^{\rho-1}(x^\rho - m^\rho b^\rho)}\Bigg|_0^1 - \frac{\alpha\rho + \rho - 1}{\rho(x^\rho - m^\rho b^\rho)} \int_0^1 t^{\alpha\rho-1} f(t^\rho x^\rho$$
$$+ m^\rho(1-t^\rho)b^\rho)dt$$
$$= \frac{-f(x^\rho)}{\rho(m^\rho b^\rho - x^\rho)} + \frac{\alpha\rho+\rho-1}{\rho(m^\rho b^\rho - x^\rho)} \int_x^{mb} \left(\frac{y^\rho - m^\rho b^\rho}{x^\rho - m^\rho b^\rho}\right)^{\alpha-1} \frac{y^{\rho-1} f(y^\rho)}{x^\rho - m^\rho b^\rho} dy$$
$$= \frac{-f(x^\rho)}{\rho(m^\rho b^\rho - x^\rho)} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{2-\alpha}(m^\rho b^\rho - x^\rho)^{\alpha+1}}. \tag{3}$$

Multiplying (2) by $\frac{\rho(x^\rho - m^\rho a^\rho)^{\alpha+1}}{b-a}$ and (3) by $\frac{\rho(m^\rho b^\rho - x^\rho)^{\alpha+1}}{b-a}$, then adding resulting equations we get (1). □

Theorem 2.2. *Let I be an open real interval such that $[0, \infty) \subset I$ and $f : I \to \mathbb{R}$ be a differentiable mapping on I such that $f' \in L_1[ma, mb]$, where $ma, mb \in I$ with*

$a < b$, $m \in (0,1]$. If $\left|f'\right|$ is $(\alpha, m)-$convex on $[ma, mb]$ and $\left|f'(x^\rho)\right| \leq M$, then the following inequality for Katugampola fractional integrals holds

$$\begin{aligned}(4)\quad &\left|\left(\frac{(x^\rho - m^\rho a^\rho)^\alpha + (m^\rho b^\rho - x^\rho)^\alpha}{b-a}\right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \left[{}^\rho I^\alpha_{x^-} f(m^\rho a^\rho) + \right.\right.\\ &\left.\left. {}^\rho I^\alpha_{x^+} f(m^\rho b^\rho)\right]\right| \leq \frac{M\left[(x^\rho - m^\rho a^\rho)^{\alpha+1} + (m^\rho b^\rho - x^\rho)^{\alpha+1}\right]}{b-a}\left[\frac{1+m^\rho\alpha}{1+2\alpha}\right],\end{aligned}$$

with $\alpha, \rho > 0$ and $x \in [ma, mb]$.

Proof. Using Lemma 2.1, $(\alpha, m)-$convexity of $\left|f'\right|$, and upper bound of $\left|f'(x^\rho)\right|$ we have

$$\begin{aligned}&\left|\left(\frac{(x^\rho - m^\rho a^\rho)^\alpha + (m^\rho b^\rho - x^\rho)^\alpha}{b-a}\right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \left[{}^\rho I^\alpha_{x^-} f(m^\rho a^\rho) + \right.\right.\\ &\left.\left. {}^\rho I^\alpha_{x^+} f(m^\rho b^\rho)\right]\right| \leq \frac{\rho(x^\rho - m^\rho a^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho + m^\rho(1-t^\rho)a^\rho)\right|dt\\ &+ \frac{\rho(m^\rho b^\rho - x^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho + m^\rho(1-t^\rho)b^\rho)\right|dt\\ &\leq \frac{\rho(x^\rho - m^\rho a^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left[t^{\alpha\rho}\left|f'(x^\rho)\right| + m^\rho(1-t^{\alpha\rho})\left|f'(a^\rho)\right|\right]dt\\ &+ \frac{\rho(m^\rho b^\rho - x^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left[t^{\alpha\rho}\left|f'(x^\rho)\right| + m^\rho(1-t^{\alpha\rho})\left|f'(b^\rho)\right|\right]dt\\ &\leq \frac{M\rho(x^\rho - m^\rho a^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left[t^{\alpha\rho} + m^\rho(1-t^{\alpha\rho})\right]dt\\ &+ \frac{M\rho(m^\rho b^\rho - x^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left[t^{\alpha\rho} + m^\rho(1-t^{\alpha\rho})\right]dt\\ &= \frac{M\rho\left[(x^\rho - m^\rho a^\rho)^{\alpha+1} + (m^\rho b^\rho - x^\rho)^{\alpha+1}\right]}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left[t^{\alpha\rho} + m^\rho(1-t^{\alpha\rho})\right]dt\\ &\leq \frac{M\left[(x^\rho - m^\rho a^\rho)^{\alpha+1} + (m^\rho b^\rho - x^\rho)^{\alpha+1}\right]}{b-a}\left[\frac{1+m^\rho\alpha}{1+2\alpha}\right]\end{aligned}$$

This completes the proof. □

Corollary 2.3. *In Theorem 2.2, if we take $\alpha = 1$ and $m = 1$ which means that $(\alpha, m)-$convexity reduces to usual convexity, then (4) becomes the following inequality*

$$\begin{aligned}&\left|\left(\frac{b^\rho - a^\rho}{b-a}\right) f(x^\rho) - \frac{2\rho - 1}{b-a}\int_a^b t^{\rho-1} f(t^\rho)dt\right|\\ (5)\quad &\leq \frac{2M\left[(x^\rho - a^\rho)^2 + (b^\rho - x^\rho)^2\right]}{3(b-a)};\ x \in [a,b],\end{aligned}$$

with $\alpha, \rho > 0$.

Corollary 2.4. *In Theorem 2.2, if we take $\alpha = 1$ which means that $(\alpha, m)-$convexity reduces to $m-$convexity, then (4) becomes the following inequality*

$$\begin{aligned}&\left|\left(\frac{m^\rho b^\rho - m^\rho a^\rho}{b-a}\right) f(x^\rho) - \frac{2\rho - 1}{b-a}\int_{ma}^{mb} t^{\rho-1} f(t^\rho)dt\right| \\ &\leq \frac{M\left[(x^\rho - m^\rho a^\rho)^2 + (m^\rho b^\rho - x^\rho)^2\right]}{b-a}\left[\frac{1+m^\rho}{3}\right]; \, x \in [ma, mb], \end{aligned} \tag{6}$$

with $\alpha, \rho > 0$.

Corollary 2.5. *In Theorem 2.2, if we take $m = 1$ which means that $(\alpha, m)-$convexity reduces to $\alpha-$convexity, then (4) becomes the following inequality*

$$\begin{aligned}&\left|\left(\frac{(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha}{b-a}\right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\left[{}^\rho I^\alpha_{x^-} f(a^\rho) + {}^\rho I^\alpha_{x^+} f(b^\rho)\right]\right| \\ &\leq \frac{M\left[(x^\rho - a^\rho)^{\alpha+1} + (b^\rho - x^\rho)^{\alpha+1}\right]}{b-a}\left[\frac{1+\alpha}{1+2\alpha}\right]; \, x \in [a, b], \end{aligned} \tag{7}$$

with $\alpha, \rho > 0$.

Remark 2.6. (i) If we put $\rho = 1$ in (4) we get [8, Theorem 4].

Theorem 2.7. *Let I be an open real interval such that $[0, \infty) \subset I$ and $f : I \to \mathbb{R}$ be a differentiable mapping on I such that $f^{'} \in L_1[ma, mb]$, where $ma, mb \in I$ with $a < b$, $m \in (0, 1]$. If $\left|f^{'}\right|^q, q > 1$, is $(\alpha, m)-$convex on $[ma, mb]$ and $\left|f^{'}(x^\rho)\right| \leq M$, then the following inequality for Katugampola fractional integrals holds*

$$\begin{aligned}&\left|\left(\frac{(x^\rho - m^\rho a^\rho)^\alpha + (m^\rho b^\rho - x^\rho)^\alpha}{b-a}\right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\left[{}^\rho I^\alpha_{x^-} f(m^\rho a^\rho) + \right.\right. \\ &\left.\left.{}^\rho I^\alpha_{x^+} f(m^\rho b^\rho)\right]\right| \leq \frac{M\rho\left[(x^\rho - m^\rho a^\rho)^{\alpha+1} + (m^\rho b^\rho - x^\rho)^{\alpha+1}\right]}{(b-a)\left(p(\alpha\rho + \rho - 1) + 1\right)^{\frac{1}{p}}}\left(\frac{1 + m^\rho \alpha\rho}{\alpha\rho + 1}\right)^{\frac{1}{q}}, \end{aligned} \tag{8}$$

with $\alpha, \rho > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in [ma, mb]$.

Proof. Using Lemma 2.1 and Hölder's inequality we have

$$\Bigg|\left(\frac{(x^\rho-m^\rho a^\rho)^\alpha+(m^\rho b^\rho-x^\rho)^\alpha}{b-a}\right)f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\left[{}^\rho I^\alpha_{x^-}f(m^\rho a^\rho)+\right.$$
$$\left.{}^\rho I^\alpha_{x^+}f(m^\rho b^\rho)\right]\Bigg|\leq\frac{\rho(x^\rho-m^\rho a^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+m^\rho(1-t^\rho)a^\rho)\right|dt$$
$$+\frac{\rho(m^\rho b^\rho-x^\rho)^{\alpha+1}}{b-a}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho+m^\rho(1-t^\rho)b^\rho)\right|dt$$
$$\leq\frac{\rho(x^\rho-m^\rho a^\rho)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p(\alpha\rho+\rho-1)}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(t^\rho x^\rho+m^\rho(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{\rho(m^\rho b^\rho-x^\rho)^{\alpha+1}}{b-a}\left(\int_0^1 t^{p(\alpha\rho+\rho-1)}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(t^\rho x^\rho+m^\rho(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}. \tag{9}$$

Since $\left|f'\right|^q$ is (α,m)-Convex and $\left|f'(x^\rho)\right|\leq M$, $x\in[a,b]$, there for we have

$$\left(\int_0^1\left|f'(t^\rho x^\rho+m^\rho(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}\leq M\left(\frac{1+m^\rho\alpha\rho}{\alpha\rho+1}\right)^{\frac{1}{q}} \tag{10}$$

similarly

$$\left(\int_0^1\left|f'(t^\rho x^\rho+m^\rho(1-t^\rho)b^\rho)\right|^q dt\right)^{\frac{1}{q}}\leq M\left(\frac{1+m^\rho\alpha\rho}{\alpha\rho+1}\right)^{\frac{1}{q}} \tag{11}$$

We also have

$$\int_0^1 t^{p(\alpha\rho+\rho-1)}dt=\frac{1}{1+p(\alpha\rho+\rho-1)}. \tag{12}$$

Using (10), (11) and (12) in (9) we can get (8).
This completes the proof. □

Corollary 2.8. *In Theorem 2.7, if we take $\alpha=1$ and $m=1$ which means that $(\alpha,m)-$convexity reduces to usual convexity, then (8) becomes the following inequality*

$$\left|\left(\frac{b^\rho-a^\rho}{b-a}\right)f(x^\rho)-\frac{2\rho-1}{b-a}\int_a^b t^{\rho-1}f(t^\rho)dt\right|$$
$$\leq\frac{M\rho\left[(x^\rho-a^\rho)^2+(b^\rho-x^\rho)^2\right]}{(b-a)\left(p(2\rho-1)+1\right)^{\frac{1}{p}}};\ x\in[a,b], \tag{13}$$

with $\alpha,\rho>0$.

Corollary 2.9. *In Theorem 2.7, if we take $\alpha=1$ which means that $(\alpha,m)-$convexity reduces to $m-$convexity, then (8) becomes the following inequality*

$$\left|\left(\frac{m^\rho b^\rho-m^\rho a^\rho}{b-a}\right)f(x^\rho)-\frac{2\rho-1}{b-a}\int_{ma}^{mb} t^{\rho-1}f(t^\rho)dt\right|$$
$$\leq\frac{M\rho\left[(x^\rho-a^\rho)^2+(b^\rho-x^\rho)^2\right]}{(b-a)\left(p(2\rho-1)+1\right)^{\frac{1}{p}}}\left(\frac{1+\rho m^\rho}{\rho+1}\right)^{\frac{1}{q}};\ x\in[ma,mb], \tag{14}$$

with $\alpha, \rho > 0$.

Corollary 2.10. *In Theorem 2.7, if we take* $m = 1$ *which means that* $(\alpha, m)-$*convexity reduces to* $\alpha-$*convexity, then (8) becomes the following inequality*

$$\left| \left(\frac{(x^\rho - a^\rho)^\alpha + (b^\rho - x^\rho)^\alpha}{b-a} \right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \left[{}^\rho I^\alpha_{x^-} f(a^\rho) + {}^\rho I^\alpha_{x^+} f(b^\rho) \right] \right|$$
$$\leq \frac{M\left[(x^\rho - a^\rho)^{\alpha+1} + (b^\rho - x^\rho)^{\alpha+1}\right]}{(b-a)\left(p(\alpha\rho + \rho - 1) + 1\right)^{\frac{1}{p}}}; \; x \in [a, b], \tag{15}$$

with $\alpha, \rho > 0$.

Remark 2.11. (i) If we put $\rho = 1$ in (8) we get [8, Theorem 5].
(ii) If we put $\rho = 1$ and $\alpha = 1$ in (8) we get [8, Theorem 2].

Theorem 2.12. *Let* I *be an open real interval such that* $[0, \infty) \subset I$ *and* $f : I \to \mathbb{R}$ *be a differentiable mapping on* I *such that* $f^{'} \in L_1[ma, mb]$, *where* $ma, mb \in I$ *with* $a < b$, $m \in (0, 1]$. *If* $\left|f^{'}\right|^q, q > 1$ *is* $(\alpha, m)-$*convex on* $[ma, mb]$ *and* $\left|f^{'}(x^\rho)\right| \leq M$, *then the following inequality for Katugampola fractional integrals holds*

$$\left| \left(\frac{(x^\rho - m^\rho a^\rho)^\alpha + (m^\rho b^\rho - x^\rho)^\alpha}{b-a} \right) f(x^\rho) \right.$$
$$\left. - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \left[{}^\rho I^\alpha_{x^-} f(m^\rho a^\rho) + {}^\rho I^\alpha_{x^+} f(m^\rho b^\rho) \right] \right|$$
$$\leq \frac{M\rho\left[(x^\rho - m^\rho a^\rho)^{\alpha+1} + (m^\rho b^\rho - x^\rho)^{\alpha+1}\right]}{(b-a)\left(\rho(\alpha+1)\right)^{1-\frac{1}{q}}} \left(\frac{1 + m^\rho\alpha}{\rho(2\alpha+1)}) \right)^{\frac{1}{q}}, \tag{16}$$

with $\alpha, \rho > 0$ *and* $x \in [ma, mb]$.

Proof. Using Lemma 2.1 and power mean inequality we have

$$\left| \left(\frac{(x^\rho - m^\rho a^\rho)^\alpha + (m^\rho b^\rho - x^\rho)^\alpha}{b-a} \right) f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)} \left[{}^\rho I^\alpha_{x^-} f(m^\rho a^\rho) + \right.\right.$$
$$\left.\left. {}^\rho I^\alpha_{x^+} f(m^\rho b^\rho) \right] \right|$$
$$\leq \frac{\rho(x^\rho - m^\rho a^\rho)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha\rho+\rho-1} \left|f^{'}(t^\rho x^\rho + m^\rho(1-t^\rho)a^\rho)\right| dt$$
$$+ \frac{\rho(m^\rho b^\rho - x^\rho)^{\alpha+1}}{b-a} \int_0^1 t^{\alpha\rho+\rho-1} \left|f^{'}(t^\rho x^\rho + m^\rho(1-t^\rho)b^\rho)\right| dt$$
$$\leq \frac{\rho(x^\rho - m^\rho a^\rho)^{\alpha+1}}{b-a} \left(\int_0^1 t^{\alpha\rho+\rho-1} dt \right)^{1-\frac{1}{q}} \left(t^{\alpha\rho+\rho-1} \int_0^1 \left|f^{'}(t^\rho x^\rho + m^\rho(1-t^\rho)a^\rho)\right|^q dt \right)^{\frac{1}{q}}$$
$$+ \frac{\rho(m^\rho b^\rho - x^\rho)^{\alpha+1}}{b-a} \left(\int_0^1 t^{\alpha\rho+\rho-1} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{\alpha\rho+\rho-1} \left|f^{'}(t^\rho x^\rho + m^\rho(1-t^\rho)a^\rho)\right|^q dt \right)^{\frac{1}{q}}. \tag{17}$$

Since $\left|f^{'}\right|^{q}$ is (α,m)-Convex and $\left|f^{'}(x^{\rho})\right| \leq M$, $x \in [a,b]$, there for we have

$$\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^{\rho}x^{\rho}+m^{\rho}(1-t^{\rho})a^{\rho})\right|^{q}dt\right)^{\frac{1}{q}} \leq M\left(\frac{1+m^{\rho}\alpha}{\rho(2\alpha+1)})\right)^{\frac{1}{q}} \tag{18}$$

similarly

$$\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^{\rho}x^{\rho}+m^{\rho}(1-t^{\rho})b^{\rho})\right|^{q}dt\right)^{\frac{1}{q}} \leq M\left(\frac{1+m^{\rho}\alpha}{\rho(2\alpha+1)})\right)^{\frac{1}{q}} \tag{19}$$

We also have

$$\int_0^1 t^{\alpha\rho+\rho-1}dt = \frac{1}{\rho(\alpha+1)}. \tag{20}$$

Using (18), (19) and (20) in (17) we can get (16).
This completes the proof. □

Corollary 2.13. *In Theorem* 2.12*, if we take $\alpha = 1$ and $m = 1$ which means that $(\alpha,m)-$convexity reduces to usual convexity, then (16) becomes the following inequality*

$$\begin{aligned}&\left|\left(\frac{b^{\rho}-a^{\rho}}{b-a}\right)f(x^{\rho}) - \frac{2\rho-1}{b-a}\int_a^b t^{\rho-1}f(t^{\rho})dt\right| \\ &\leq \frac{M\rho\left[(x^{\rho}-a^{\rho})^{2}+(b^{\rho}-x^{\rho})^{2}\right]}{(b-a)}\left(\frac{1}{2\rho}\right)^{1-\frac{1}{q}}\left(\frac{2}{3\rho}\right)^{\frac{1}{q}};\ x\in[a,b],\end{aligned} \tag{21}$$

with $\alpha,\rho > 0$.

Corollary 2.14. *In Theorem* 2.12*, if we take $\alpha = 1$ which means that $(\alpha,m)-$convexity reduces to $m-$convexity, then (16) becomes the following inequality*

$$\begin{aligned}&\left|\left(\frac{m^{\rho}b^{\rho}-m^{\rho}a^{\rho}}{b-a}\right)f(x^{\rho}) - \frac{2\rho-1}{b-a}\int_{ma}^{mb} t^{\rho-1}f(t^{\rho})dt\right| \\ &\leq \frac{M\rho\left[(x^{\rho}-m^{\rho}a^{\rho})^{2}+(m^{\rho}b^{\rho}-x^{\rho})^{2}\right]}{b-a}\left(\frac{1}{2\rho}\right)^{1-\frac{1}{q}}\left(\frac{1+m^{\rho}}{3\rho}\right)^{\frac{1}{q}};\ x\in[ma,mb],\end{aligned} \tag{22}$$

with $\alpha,\rho > 0$.

Corollary 2.15. *In Theorem* 2.12*, if we take $m = 1$ which means that $(\alpha,m)-$convexity reduces to $\alpha-$convexity, then (16) becomes the following inequality*

$$\begin{aligned}&\left|\left(\frac{(x^{\rho}-a^{\rho})^{\alpha}+(b^{\rho}-x^{\rho})^{\alpha}}{b-a}\right)f(x^{\rho}) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}(b-a)}\left[{}^{\rho}I^{\alpha}_{x^-}f(a^{\rho}) + {}^{\rho}I^{\alpha}_{x^+}f(b^{\rho})\right]\right| \\ &\leq \frac{M\rho\left[(x^{\rho}-a^{\rho})^{\alpha+1}+(b^{\rho}-x^{\rho})^{\alpha+1}\right]}{b-a}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\frac{\alpha+1}{\rho(2\alpha+1)})\right)^{\frac{1}{q}};\ x\in[a,b],\end{aligned} \tag{23}$$

with $\alpha,\rho > 0$.

Remark 2.16. (i) If we put $\rho = 1$ in (16) we get [8, Theorem 6].
(ii) If we put $\rho = 1$ and $\alpha = 1$ in (8) we get [8, Theorem 3].

To give more results we need the following lemma which we will use in squeal.

Lemma 2.17. *Let I be an open real interval such that $[0,\infty)\subset I$ and $f: I\to\mathbb{R}$ be a differentiable mapping on I such that $f^{'}\in L_1[ma,mb]$, where $ma,mb\in I$ with $a<b$, $m\in(0,1]$. Then for all $x\in(ma,mb)$ we have the following equality*

$$\begin{aligned} & f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-}f(m^\rho a^\rho)}{2(x^\rho-m^\rho a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x^+}f(m^\rho b^\rho)}{2(m^\rho b^\rho-x^\rho)^\alpha}\right] \\ & =\frac{\rho(x^\rho-m^\rho a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}f^{'}(t^\rho x^\rho+m^\rho(1-t^\rho)a^\rho)dt \\ & -\frac{\rho(m^\rho b^\rho-x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}f^{'}(t^\rho x^\rho+m^\rho(1-t^\rho)b^\rho)dt \end{aligned} \tag{24}$$

with $\alpha,\rho>0$.

Proof. It is easy to see that

$$\begin{aligned} & \int_0^1 t^{\alpha\rho+\rho-1}f^{'}(t^\rho x^\rho+m^\rho m^\rho(1-t^\rho)a^\rho)dt \\ & =\left.\frac{t^{\alpha\rho+\rho-1}f(t^\rho x^\rho+m^\rho(1-t^\rho)a^\rho)}{\rho t^{\rho-1}(x^\rho-m^\rho a^\rho)}\right|_0^1-\frac{\alpha\rho+\rho-1}{\rho(x^\rho-m^\rho a^\rho)}\int_0^1 t^{\alpha\rho-1}f(t^\rho x^\rho \\ & +m^\rho(1-t^\rho)a^\rho)dt \\ & =\frac{f(x^\rho)}{\rho(x^\rho-m^\rho a^\rho)}-\frac{\alpha\rho+\rho-1}{\rho(x^\rho-m^\rho a^\rho)}\int_{ma}^x\left(\frac{y^\rho-m^\rho a^\rho}{x^\rho-m^\rho a^\rho}\right)^{\alpha-1}\frac{y^{\rho-1}f(y^\rho)}{x^\rho-m^\rho a^\rho}dy \\ & =\frac{f(x^\rho)}{\rho(x^\rho-m^\rho a^\rho)}-\frac{{}^\rho I^\alpha_{x^-}f(m^\rho a^\rho)(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{2-\alpha}(x^\rho-m^\rho a^\rho)^{\alpha+1}} \end{aligned} \tag{25}$$

and

$$\begin{aligned} & \int_0^1 t^{\alpha\rho+\rho-1}f^{'}(t^\rho x^\rho+m^\rho(1-t^\rho)b^\rho)dt \\ & =\left.\frac{t^{\alpha\rho+\rho-1}f(t^\rho x^\rho+m^\rho(1-t^\rho)b^\rho)}{\rho t^{\rho-1}(x^\rho-m^\rho b^\rho)}\right|_0^1-\frac{\alpha\rho+\rho-1}{\rho(x^\rho-m^\rho b^\rho)}\int_0^1 t^{\alpha\rho-1}f(t^\rho x^\rho \\ & +m^\rho(1-t^\rho)b^\rho)dt \\ & =\frac{-f(x^\rho)}{\rho(m^\rho b^\rho-x^\rho)}+\frac{\alpha\rho+\rho-1}{\rho(m^\rho b^\rho-x^\rho)}\int_x^{mb}\left(\frac{y^\rho-m^\rho b^\rho}{x^\rho-m^\rho b^\rho}\right)^{\alpha-1}\frac{y^{\rho-1}f(y^\rho)}{x^\rho-m^\rho b^\rho}dy \\ & =\frac{-f(x^\rho)}{\rho(m^\rho b^\rho-x^\rho)}+\frac{{}^\rho I^\alpha_{x^+}f(b^\rho)(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{2-\alpha}(m^\rho b^\rho-x^\rho)^{\alpha+1}}. \end{aligned} \tag{26}$$

Multiplying (25) by $\frac{\rho(x^\rho-m^\rho a^\rho)}{2}$ and (26) by $\frac{\rho(m^\rho b^\rho-x^\rho)}{2}$, then adding resulting equations we get (24). □

Theorem 2.18. *Let I be an open real interval such that $[0,\infty)\subset I$ and $f: I\to\mathbb{R}$ be a differentiable mapping on I such that $f^{'}\in L_1[ma,mb]$, where $ma,mb\in I$ with $a<b$, $m\in(0,1]$. If $\left|f^{'}\right|$ is $(\alpha,m)-$convex on $[ma,mb]$ and $\left|f^{'}(x^\rho)\right|\le M$, then the*

following inequality for Katugampola fractional integrals holds

$$\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x-} f(m^\rho a^\rho)}{2(x^\rho - m^\rho a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(m^\rho b^\rho)}{2(m^\rho b^\rho - x^\rho)^\alpha}\right]\right| \leq \frac{Mm^\rho\left[b^\rho - a^\rho\right]}{2}\left[\frac{1+m^\rho\alpha}{1+2\alpha}\right]; x \in [ma, mb], \tag{27}$$

with $\alpha, \rho > 0$.

Proof. Using Lemma 2.17, $(\alpha, m)-$convexity of $\left|f'\right|$, and upper bound of $\left|f'(x^\rho)\right|$ we have

$$\begin{aligned}
&\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x-} f(m^\rho a^\rho)}{2(x^\rho - m^\rho a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x+} f(m^\rho b^\rho)}{2(m^\rho b^\rho - x^\rho)^\alpha}\right]\right| \\
&\leq \frac{\rho(x^\rho - m^\rho a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho + m^\rho(1-t^\rho)a^\rho)\right|dt \\
&+ \frac{\rho(m^\rho b^\rho - x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho + m^\rho(1-t^\rho)b^\rho)\right|dt \\
&\leq \frac{\rho(x^\rho - m^\rho a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left[t^{\alpha\rho}\left|f'(x^\rho)\right| + m^\rho(1-t^{\alpha\rho})\left|f'(a^\rho)\right|\right]dt \\
&+ \frac{\rho(m^\rho b^\rho - x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left[t^{\alpha\rho}\left|f'(x^\rho)\right| + m^\rho(1-t^{\alpha\rho})\left|f'(b^\rho)\right|\right]dt \\
&\leq \frac{M\rho(x^\rho - m^\rho a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left[t^{\alpha\rho} + m^\rho(1-t^{\alpha\rho})\right]dt \\
&+ \frac{M\rho(m^\rho b^\rho - x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left[t^{\alpha\rho} + m^\rho(1-t^{\alpha\rho})\right]dt \\
&= \frac{M\rho\left[m^\rho b^\rho - m^\rho a^\rho\right]}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left[t^{\alpha\rho} + m^\rho(1-t^{\alpha\rho})\right]dt \\
&\leq \frac{Mm^\rho\left[b^\rho - a^\rho\right]}{2}\left[\frac{1+m^\rho\alpha}{1+2\alpha}\right].
\end{aligned}$$

This completes the proof. □

Corollary 2.19. *In Theorem 2.18, if we take $\alpha = 1$ and $m = 1$ which means that $(\alpha, m)-$convexity reduces to usual convexity, then (27) becomes the following inequality*

$$\left|f(x^\rho) - \frac{2\rho-1}{2}\left[\frac{\int_a^x t^{\rho-1}f(t^\rho)dt}{x^\rho - a^\rho} + \frac{\int_x^b t^{\rho-1}f(t^\rho)dt}{b^\rho - x^\rho}\right]\right| \leq \frac{M(b^\rho - a^\rho)}{3}; x \in [a, b], \tag{28}$$

with $\alpha, \rho > 0$.

Corollary 2.20. *In Theorem 2.18, if we take $\alpha = 1$ which means that $(\alpha, m)-$convexity reduces to $m-$convexity, then (27) becomes the following inequality*

$$\left|f(x^\rho) - \frac{2\rho-1}{2}\left[\frac{\int_{ma}^{x} t^{\rho-1}f(t^\rho)dt}{x^\rho - m^\rho a^\rho} + \frac{\int_{x}^{mb} t^{\rho-1}f(t^\rho)dt}{m^\rho b^\rho - x^\rho}\right]\right|$$
$$\leq \frac{Mm^\rho\left[b^\rho - a^\rho\right]}{2}\left[\frac{1+m^\rho}{3}\right];\ x \in [ma, mb], \tag{29}$$

with $\alpha, \rho > 0$.

Corollary 2.21. *In Theorem 2.18, if we take $m = 1$ which means that $(\alpha, m)-$convexity reduces to $\alpha-$convexity, then (27) becomes the following inequality*

$$\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha}\right]\right|$$
$$\leq \frac{M\left[b^\rho - a^\rho\right]}{2}\left[\frac{1+\alpha}{1+2\alpha}\right];\ x \in [a, b], \tag{30}$$

with $\alpha, \rho > 0$.

Remark 2.22. (i) If we put $\rho = 1$ in (27), then we get the result for Riemann-Liouville fractional integrals

Theorem 2.23. *Let I be an open real interval such that $[0, \infty) \subset I$ and $f : I \to \mathbb{R}$ be a differentiable mapping on I such that $f' \in L_1[ma, mb]$, where $ma, mb \in I$ with $a < b$, $m \in (0, 1]$. If $\left|f'\right|^q, q > 1$, is $(\alpha, m)-$convex on $[ma, mb]$ and $\left|f'(x^\rho)\right| \leq M$, then the following inequality for Katugampola fractional integrals holds*

$$\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-} f(m^\rho a^\rho)}{2(x^\rho - m^\rho a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(m^\rho b^\rho)}{2(m^\rho b^\rho - x^\rho)^\alpha}\right]\right|$$
$$\leq \frac{M\rho\left[m^\rho b^\rho - m^\rho a^\rho\right]}{2\left(p(\alpha\rho+\rho-1)+1\right)^{\frac{1}{p}}}\left(\frac{1+m^\rho\alpha\rho}{\alpha\rho+1}\right)^{\frac{1}{q}}, \tag{31}$$

with $\alpha, \rho > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in [ma, mb]$.

Proof. Using Lemma 2.17 and Hölder's inequality we have

$$\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-} f(m^\rho a^\rho)}{2(x^\rho - m^\rho a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(m^\rho b^\rho)}{2(m^\rho b^\rho - x^\rho)^\alpha}\right]\right|$$
$$\leq \frac{\rho(x^\rho - m^\rho a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho + m^\rho(1-t^\rho)a^\rho)\right|dt$$
$$+ \frac{\rho(m^\rho b^\rho - x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f'(t^\rho x^\rho + m^\rho(1-t^\rho)b^\rho)\right|dt$$
$$\leq \frac{\rho(x^\rho - m^\rho a^\rho)}{2}\left(\int_0^1 t^{p(\alpha\rho+\rho-1)}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(t^\rho x^\rho + m^\rho(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{\rho(m^\rho b^\rho - x^\rho)}{2}\left(\int_0^1 t^{p(\alpha\rho+\rho-1)}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(t^\rho x^\rho + m^\rho(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}. \tag{32}$$

Since $\left|f^{'}\right|^q$ is (α,m)-convex and $\left|f^{'}(x^\rho)\right| \leq M$, $x\in[a,b]$, there for we have

$$\left(\int_0^1 \left|f^{'}(t^\rho x^\rho + m^\rho(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}} \leq M\left(\frac{1+m^\rho\alpha\rho}{\alpha\rho+1}\right)^{\frac{1}{q}} \tag{33}$$

similarly

$$\left(\int_0^1 \left|f^{'}(t^\rho x^\rho + m^\rho(1-t^\rho)b^\rho)\right|^q dt\right)^{\frac{1}{q}} \leq M\left(\frac{1+m^\rho\alpha\rho}{\alpha\rho+1}\right)^{\frac{1}{q}} \tag{34}$$

We also have

$$\int_0^1 t^{p(\alpha\rho+\rho-1)}dt = \frac{1}{1+p(\alpha\rho+\rho-1)}. \tag{35}$$

Using (33), (34) and (35) in (32) we can get (31).
This completes the proof. □

Corollary 2.24. *In Theorem 2.22, if we take $\alpha = 1$ and $m = 1$ which means that $(\alpha,m)-$convexity reduces to usual convexity, then (31) becomes the following inequality*

$$\begin{aligned}&\left|f(x^\rho) - \frac{2\rho-1}{2}\left[\frac{\int_a^x t^{\rho-1}f(t^\rho)dt}{x^\rho-a^\rho} + \frac{\int_x^b t^{\rho-1}f(t^\rho)dt}{b^\rho-x^\rho}\right]\right|\\ &\leq \frac{M\rho\left[b^\rho-a^\rho\right]}{2\left(p(2\rho-1)+1\right)^{\frac{1}{p}}};\ x\in[a,b],\end{aligned} \tag{36}$$

with $\alpha,\rho>0$.

Corollary 2.25. *In Theorem 2.22, if we take $\alpha = 1$ which means that $(\alpha,m)-$convexity reduces to $m-$convexity, then (31) becomes the following inequality*

$$\begin{aligned}&\left|f(x^\rho) - \frac{2\rho-1}{2}\left[\frac{\int_{ma}^x t^{\rho-1}f(t^\rho)dt}{x^\rho-m^\rho a^\rho} + \frac{\int_x^{mb} t^{\rho-1}f(t^\rho)dt}{m^\rho b^\rho-x^\rho}\right]\right|\\ &\leq \frac{Mm^\rho\rho\left[b^\rho-a^\rho\right]}{2\left(p(2\rho-1)+1\right)^{\frac{1}{p}}}\left(\frac{1+\rho m^\rho}{\rho+1}\right)^{\frac{1}{q}};\ x\in[ma,mb],\end{aligned} \tag{37}$$

with $\alpha,\rho>0$.

Corollary 2.26. *In Theorem 2.22, if we take $m = 1$ which means that $(\alpha,m)-$convexity reduces to $\alpha-$convexity, then (31) becomes the following inequality*

$$\begin{aligned}&\left|f(x^\rho) - \frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-}f(a^\rho)}{2(x^\rho-a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+}f(b^\rho)}{2(b^\rho-x^\rho)^\alpha}\right]\right|\\ &\leq \frac{M\rho\left[b^\rho-a^\rho\right]}{2\left(p(\alpha\rho+\rho-1)+1\right)^{\frac{1}{p}}};\ x\in[a,b],\end{aligned} \tag{38}$$

with $\alpha,\rho>0$.

Remark 2.27. (i) If we put $\rho = 1$ in (31), then we get the result for Riemann-Liouville fractional integrals

Theorem 2.28. *Let I be an open real interval such that $[0,\infty)\subset I$ and $f: I\to\mathbb{R}$ be a differentiable mapping on I such that $f^{'}\in L_1[ma,mb]$, where $ma,mb\in I$ with $a<b$, $m\in(0,1]$. If $\left|f^{'}\right|^q, q>1$ is $(\alpha,m)-$convex on $[ma,mb]$ and $\left|f^{'}(x^\rho)\right|\le M$, then the following inequality for Katugampola fractional integrals holds*

$$\left|f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-}f(m^\rho a^\rho)}{2(x^\rho-m^\rho a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x^+}f(m^\rho b^\rho)}{2(m^\rho b^\rho-x^\rho)^\alpha}\right]\right|$$
$$\le\frac{M\rho m^\rho\left[b^\rho-a^\rho\right]}{2\left(\rho(\alpha+1)\right)^{1-\frac{1}{q}}}\left(\frac{1+m^\rho\alpha}{\rho(2\alpha+1)})\right)^{\frac{1}{q}};x\in[ma,mb], \tag{39}$$

with $\alpha,\rho>0$.

Proof. Using Lemma 2.17 and power mean inequality we have

$$\left|f(x^\rho)-\frac{(\alpha\rho+\rho-1)\Gamma(\alpha)}{\rho^{1-\alpha}}\left[\frac{{}^\rho I^\alpha_{x^-}f(m^\rho a^\rho)}{2(x^\rho-m^\rho a^\rho)^\alpha}+\frac{{}^\rho I^\alpha_{x^+}f(m^\rho b^\rho)}{2(m^\rho b^\rho-x^\rho)^\alpha}\right]\right|$$
$$\le\frac{\rho(x^\rho-m^\rho a^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+m^\rho(1-t^\rho)a^\rho)\right|dt$$
$$+\frac{\rho(m^\rho b^\rho-x^\rho)}{2}\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+m^\rho(1-t^\rho)b^\rho)\right|dt$$
$$\le\frac{\rho(x^\rho-m^\rho a^\rho)}{2}\left(\int_0^1 t^{\alpha\rho+\rho-1}dt\right)^{1-\frac{1}{q}}\left(t^{\alpha\rho+\rho-1}\int_0^1\left|f^{'}(t^\rho x^\rho+m^\rho(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{\rho(m^\rho b^\rho-x^\rho)}{2}\left(\int_0^1 t^{\alpha\rho+\rho-1}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+m^\rho(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}. \tag{40}$$

Since $\left|f^{'}\right|^q$ is (α,m)-Convex and $\left|f^{'}(x^\rho)\right|\le M$, $x\in[a,b]$, there for we have

$$\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+m^\rho(1-t^\rho)a^\rho)\right|^q dt\right)^{\frac{1}{q}}\le M\left(\frac{1+m^\rho\alpha}{\rho(2\alpha+1)})\right)^{\frac{1}{q}} \tag{41}$$

similarly

$$\left(\int_0^1 t^{\alpha\rho+\rho-1}\left|f^{'}(t^\rho x^\rho+m^\rho(1-t^\rho)b^\rho)\right|^q dt\right)^{\frac{1}{q}}\le M\left(\frac{1+m^\rho\alpha}{\rho(2\alpha+1)})\right)^{\frac{1}{q}} \tag{42}$$

We also have

$$\int_0^1 t^{\alpha\rho+\rho-1}dt=\frac{1}{\rho(\alpha+1)}. \tag{43}$$

Using (41), (42) and (43) in (40) we can get (39).
This completes the proof. □

Corollary 2.29. *In Theorem 2.26, if we take $\alpha=1$ and $m=1$ which means that $(\alpha,m)-$convexity reduces to usual convexity, then (39) becomes the following* *inequality*

$$\left| f(x^\rho) - \frac{2\rho - 1}{2} \left[\frac{\int_a^x t^{\rho-1} f(t^\rho) dt}{x^\rho - a^\rho} + \frac{\int_x^b t^{\rho-1} f(t^\rho) dt}{b^\rho - x^\rho} \right] \right|$$

$$\leq \frac{M\rho\,[b^\rho - a^\rho]}{2} \left(\frac{1}{2\rho} \right)^{1-\frac{1}{q}} \left(\frac{2}{3\rho} \right)^{\frac{1}{q}}; \; x \in [a, b], \tag{44}$$

with $\alpha, \rho > 0$.

Corollary 2.30. *In Theorem* 2.26*, if we take* $\alpha = 1$ *which means that* $(\alpha, m)-$*convexity reduces to* $m-$*convexity, then (39) becomes the following inequality*

$$\left| f(x^\rho) - \frac{2\rho - 1}{2} \left[\frac{\int_{ma}^x t^{\rho-1} f(t^\rho) dt}{x^\rho - m^\rho a^\rho} + \frac{\int_x^{mb} t^{\rho-1} f(t^\rho) dt}{m^\rho b^\rho - x^\rho} \right] \right|$$

$$\leq \frac{Mm^\rho \rho\,[b^\rho - a^\rho]}{2} \left(\frac{1}{2\rho} \right)^{1-\frac{1}{q}} \left(\frac{1 + m^\rho}{3\rho} \right)^{\frac{1}{q}}; \; x \in [ma, mb], \tag{45}$$

with $\alpha, \rho > 0$.

Corollary 2.31. *In Theorem* 2.26*, if we take* $m = 1$ *which means that* $(\alpha, m)-$*convexity reduces to* $\alpha-$*convexity, then (39) becomes the following inequality*

$$\left| f(x^\rho) - \frac{(\alpha\rho + \rho - 1)\Gamma(\alpha)}{\rho^{1-\alpha}} \left[\frac{{}^\rho I^\alpha_{x^-} f(a^\rho)}{2(x^\rho - a^\rho)^\alpha} + \frac{{}^\rho I^\alpha_{x^+} f(b^\rho)}{2(b^\rho - x^\rho)^\alpha} \right] \right|$$

$$\leq \frac{M\rho\,[b^\rho - a^\rho]}{2} \left(\frac{1}{\rho(\alpha + 1)} \right)^{1-\frac{1}{q}} \left(\frac{\alpha + 1}{\rho(2\alpha + 1)}) \right)^{\frac{1}{q}}; \; x \in [a, b], \tag{46}$$

with $\alpha, \rho > 0$.

Remark 2.32. (i) If we put $\rho = 1$ in (39), then we get the result for Riemann-Liouville fractional integrals

Conclusion. All results proved in this research paper can also be deduced for Hadamard fractional integrals just by taking limits when parameter $\rho \to 0^+$.

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