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AN OPERATOR ASSOCIATED TO HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS

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ABSTRACT. In this paper we establish some fundamental properties of the operator

$$D_{a+,b-}f(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right],\ x \in (a,b)$$

for various classes of functions $f : [a,b] \to \mathbb{R}$ including, monotonic, convex and Lipschitzian functions. Various Hermite-Hadamard type inequalities improving some classical results are also provided. Some examples for logarithm are given.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f : [a,b] \to \mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality*.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [7], the recent survey paper [5], the research papers [1]-[2], [8]-[16] and the references therein.

Assume that the function $f : (a,b) \to \mathbb{C}$ is Lebesgue integrable on (a,b). We introduce the following operator

$$D_{a+,b-}f(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right],\ x \in (a,b). \tag{1.2}$$

We observe that if we take $x = \frac{a+b}{2}$, then we have

$$D_{a+,b-}f\left(\frac{a+b}{2}\right) = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Moreover, if $f(a+) := \lim_{x\to a+} f(x)$ exists and is finite, then we have

$$\lim_{x\to a+} D_{a+,b-}f(x) = \frac{1}{2}\left[f(a+) + \frac{1}{b-a}\int_a^b f(t)\,dt\right]$$

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and if $f(b-) := \lim_{x\to b-} f(x)$ exists and is finite, then we have

$$\lim_{x\to b-} D_{a+,b-}f(x) = \frac{1}{2}\left[f(b-) + \frac{1}{b-a}\int_a^b f(t)\,dt\right].$$

So, if $f : [a,b] \to \mathbb{C}$ is Lebesgue integrable on $[a,b]$ and continuous at right in a and at left in b, then we can extend the operator on the whole interval by putting

$$D_{a+,b-}f(a) := \frac{1}{2}\left[f(a) + \frac{1}{b-a}\int_a^b f(t)\,dt\right]$$

and

$$D_{a+,b-}f(b) := \frac{1}{2}\left[f(b) + \frac{1}{b-a}\int_a^b f(t)\,dt\right].$$

If we change the variable $t = (1-s)a + sx$ for $x \in (a,b)$ then we have

$$\frac{1}{x-a}\int_a^x f(t)\,dt = \int_0^1 f((1-s)a + sx)\,ds$$

and if we change the variable $t = (1-s)x + sb$ for $x \in (a,b)$, then we also have

$$\frac{1}{b-x}\int_x^b f(t)\,dt = \int_0^1 f((1-s)x + sb)\,ds,$$

which gives the representation

$$D_{a+,b-}f(x) = \frac{1}{2}\int_0^1 [f((1-s)a + sx) + f((1-s)x + sb)]\,ds,\ x \in (a,b). \tag{1.3}$$

Using the representation (1.3), we observe that the operator $D_{a+,b-}$ is *linear, nonnegative* and *preserves the constant* functions, namely

$$D_{a+,b-}(\alpha f + \beta g) = \alpha D_{a+,b-}(f) + \beta D_{a+,b-}(g)$$

for any complex numbers α, β and integrable functions f, g. If $f \geq 0$ almost everywhere on $[a,b]$ and f is integrable, then $D_{a+,b-}f(x) \geq 0$ for any $x \in (a,b)$. Also, if $f = k$, a constant, then $D_{a+,b-}k(x) = k$ for any $x \in (a,b)$. If we define the function $\mathbf{1}(t) = 1$, $t \in [a,b]$, then, obviously, $D_{a+,b-}\mathbf{1} = \mathbf{1}$.

In this paper we establish some fundamental properties of the operator $D_{a+,b-}f(x)$, $x \in (a,b)$ for various classes of functions $f : [a,b] \to \mathbb{R}$ including, monotonic, convex and Lipschitzian functions. Various Hermite-Hadamard type inequalities improving some classical results are also provided. Some examples for logarithm are given.

2. Some General Properties

The first result collects some of the fundamental properties of the operator $D_{a+,b-}$ as follows:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ and integrable function on $[a,b]$.*

(i) *If f is monotonic nondecreasing on $[a,b]$ then $D_{a+,b-}f$ is monotonic nondecreasing on (a,b).*
(ii) *If f is convex on $[a,b]$ then $D_{a+,b-}f$ is convex on (a,b).*

(iii) *If f is Lipschitzian with the constant L on $[a,b]$, namely*

$$|f(x) - f(y)| \leq L|x-y| \tag{2.1}$$

for any x, $y \in [a,b]$, then $D_{a+,b-}f$ is Lipschitzian with the constant $\frac{1}{2}L$ on (a,b).

Proof. (i) Assume that f is monotonic nondecreasing on $[a,b]$. Let $a < x < y < b$. Then

$$f((1-s)a + sx) \leq f((1-s)a + sy)$$

and

$$f((1-s)x + sb) \leq f((1-s)y + sb)$$

for any $s \in [0,1]$.

If we sum these inequalities and divide by 2 we get

$$\frac{1}{2}[f((1-s)a+sx) + f((1-s)x+sb)] \leq \frac{1}{2}[f((1-s)a+sy) + f((1-s)y+sb)]$$

for any $s \in [0,1]$.

By integrating this inequality on $[0,1]$ and using the representation (1.3) we get $D_{a+,b-}f(x) \leq D_{a+,b-}f(y)$.

(ii) Now, assume that f is convex on $[a,b]$. Then for x, $y \in (a,b)$ and α, $\beta > 0$ with $\alpha + \beta = 1$, we have

$$\begin{aligned} f((1-s)a + s(\alpha x + \beta y)) &= f(\alpha[(1-s)a+sx] + \beta[(1-s)a+sy]) \\ &\leq \alpha f((1-s)a+sx) + \beta f((1-s)a+sy) \end{aligned}$$

and

$$\begin{aligned} f((1-s)(\alpha x + \beta y) + sb) &= f(\alpha[(1-s)x+sb] + \beta[(1-s)y+sb]) \\ &\leq \alpha f((1-s)x+sb) + \beta f((1-s)y+sb) \end{aligned}$$

for any $s \in [0,1]$.

If we add these two inequalities and divide by 2 we get

$$\begin{aligned} &\frac{1}{2}[f((1-s)a + s(\alpha x+\beta y)) + f((1-s)(\alpha x+\beta y)+sb)] \\ &\leq \alpha\frac{1}{2}[f((1-s)a+sx) + f((1-s)x+sb)] \\ &+ \beta\frac{1}{2}[f((1-s)a+sy) + f((1-s)y+sb)] \end{aligned}$$

for any $s \in [0,1]$.

If we integrate this inequality and use the representation (1.3) we get

$$\begin{aligned} &D_{a+,b-}f(\alpha x+\beta y) \\ &= \frac{1}{2}\int_0^1 [f((1-s)a + s(\alpha x+\beta y)) + f((1-s)(\alpha x+\beta y)+sb)]\,ds \\ &\leq \alpha\frac{1}{2}\int_0^1 [f((1-s)a+sx) + f((1-s)x+sb)]\,ds \\ &+ \beta\frac{1}{2}\int_0^1 [f((1-s)a+sy) + f((1-s)y+sb)]\,ds \\ &= \alpha D_{a+,b-}f(x) + \beta D_{a+,b-}f(y), \end{aligned}$$

which proves the convexity of $D_{a+,b-}f$.

(iii) Let $x,\ y\in(a,b)$. Then

$$\begin{aligned}
&\left|D_{a+,b-}f(x)-D_{a+,b-}f(y)\right|\\
&=\frac{1}{2}\left|\int_0^1\left[f((1-s)a+sx)+f((1-s)x+sb)\right]ds\right.\\
&\left.-\int_0^1\left[f((1-s)a+sy)+f((1-s)y+sb)\right]ds\right|\\
&=\frac{1}{2}\left|\int_0^1\left[f((1-s)a+sx)-f((1-s)a+sy)\right]ds\right.\\
&\left.+\int_0^1\left[f((1-s)x+sb)-f((1-s)y+sb)\right]ds\right|\\
&\leq\frac{1}{2}\left|\int_0^1\left[f((1-s)a+sx)-f((1-s)a+sy)\right]ds\right|\\
&+\frac{1}{2}\left|\int_0^1\left[f((1-s)x+sb)-f((1-s)y+sb)\right]ds\right|\\
&\leq\frac{1}{2}\int_0^1\left|f((1-s)a+sx)-f((1-s)a+sy)\right|ds\\
&+\frac{1}{2}\int_0^1\left|f((1-s)x+sb)-f((1-s)y+sb)\right|ds\\
&=:K(x,y).
\end{aligned}$$

Since f is Lipschitzian with the constant L on $[a,b]$, then

$$\begin{aligned}
&\int_0^1\left|f((1-s)a+sx)-f((1-s)a+sy)\right|ds\\
&\leq L\int_0^1\left|(1-s)a+sx-(1-s)a-sy\right|ds=\frac{1}{2}L|x-y|
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\left|f((1-s)x+sb)-f((1-s)y+sb)\right|ds\\
&\leq L\int_0^1\left|(1-s)x+sb-(1-s)y-sb\right|ds=\frac{1}{2}L|x-y|
\end{aligned}$$

for any $x,\ y\in(a,b)$.

Therefore

$$K(x,y)\leq\frac{1}{4}L|x-y|+\frac{1}{4}L|x-y|=\frac{1}{2}L|x-y|,$$

which shows that $D_{a+,b-}f$ is Lipschitzian with the constant $\frac{1}{2}L$. □

Now, for $\phi,\ \Phi\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions, see for instance [6]

$$\begin{aligned}
&\bar{U}_{[a,b]}(\phi,\Phi)\\
&:=\left\{f:[a,b]\to\mathbb{C}\,|\operatorname{Re}\left[(\Phi-f(t))\left(\overline{f(t)}-\overline{\phi}\right)\right]\geq0\text{ for almost every }t\in[a,b]\right\}
\end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\phi,\Phi) := \left\{ f : [a,b] \to \mathbb{C} | \ \left| f(t) - \frac{\phi+\Phi}{2} \right| \le \frac{1}{2} |\Phi - \phi| \ \text{ for a.e. } t \in [a,b] \right\}.$$

The following representation result may be stated.

Proposition 1. *For any $\phi,\ \Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that $\bar{U}_{[a,b]}(\phi,\Phi)$ and $\bar{\Delta}_{[a,b]}(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\phi,\Phi) = \bar{\Delta}_{[a,b]}(\phi,\Phi). \tag{2.2}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\phi+\Phi}{2} \right| \le \frac{1}{2} |\Phi - \phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi - z)(\bar{z} - \phi)\right] \ge 0.$$

This follows by the equality

$$\frac{1}{4} |\Phi - \phi|^2 - \left| z - \frac{\phi+\Phi}{2} \right|^2 = \operatorname{Re}\left[(\Phi - z)(\bar{z} - \phi)\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.2) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any $\phi,\ \Phi \in \mathbb{C}$, $\phi \neq \Phi$, we have that*

$$\begin{aligned} \bar{U}_{[a,b]}(\phi,\Phi) = \{ f : [a,b] \to \mathbb{C} \mid \ & (\operatorname{Re}\Phi - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\phi) \\ & + (\operatorname{Im}\Phi - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\phi) \ge 0 \textit{ for a.e. } t \in [a,b] \}. \end{aligned}$$

Now, if we assume that $\operatorname{Re}(\Phi) \ge \operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi) \ge \operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\phi,\Phi) := \{ f : [a,b] \to \mathbb{C} \mid \ & \operatorname{Re}(\Phi) \ge \operatorname{Re} f(t) \ge \operatorname{Re}(\phi) \\ & \text{and } \operatorname{Im}(\Phi) \ge \operatorname{Im} f(t) \ge \operatorname{Im}(\phi) \ \text{ for a.e. } t \in [a,b] \}. \end{aligned}$$

One can easily observe that $\bar{S}_{[a,b]}(\phi,\Phi)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\phi,\Phi) \subseteq \bar{U}_{[a,b]}(\phi,\Phi).$$

Theorem 2. *Let $f : [a,b] \to \mathbb{C}$ is Lebesgue integrable on $[a,b]$ and there exists the constants $\phi,\ \Phi \in \mathbb{C}$, $\phi \neq \Phi$, such that $f \in \bar{\Delta}_{[a,b]}(\phi,\Phi)$, then also $D_{a+,b-}f \in \bar{\Delta}_{[a,b]}(\phi,\Phi)$.*

Proof. Let $x \in (a,b)$. Then

$$\begin{aligned} D_{a+,b-}f(x) - \frac{\phi+\Phi}{2} &= \frac{1}{2}\int_0^1 \left[f((1-s)a + sx) - \frac{\phi+\Phi}{2} \right] \\ &\quad + \frac{1}{2}\int_0^1 \left[f((1-s)x + sb) - \frac{\phi+\Phi}{2} \right] ds. \end{aligned} \tag{2.3}$$

Since $f \in \bar{\Delta}_{[a,b]}(\phi,\Phi)$, then

$$\left| f((1-s)a + sx) - \frac{\phi+\Phi}{2} \right| \le \frac{1}{2} |\Phi - \phi|$$

and

$$\left| f\left(\left(1-s\right) x+sb\right) -\frac{\phi +\Phi }{2}\right| \leq \frac{1}{2}\left| \Phi -\phi \right|$$

for almost every $s\in \left[0,1\right]$.

By taking the modulus in (2.3), we get

$$\begin{aligned} \left| D_{a+,b-}f\left(x\right) -\frac{\phi +\Phi }{2}\right| &\leq \frac{1}{2}\int_{0}^{1}\left| f\left(\left(1-s\right) a+sx\right) -\frac{\phi +\Phi }{2}\right| \\ &\quad +\frac{1}{2}\int_{0}^{1}\left| f\left(\left(1-s\right) x+sb\right) -\frac{\phi +\Phi }{2}\right| ds \\ &\leq \frac{1}{4}\left| \Phi -\phi \right| +\frac{1}{4}\left| \Phi -\phi \right| =\frac{1}{2}\left| \Phi -\phi \right| , \end{aligned}$$

which proves the statement. □

3. Some Inequalities for Convex Functions

We have the following lower and upper bounds for $D_{a+,b-}f$:

Theorem 3. *Let $f:\left[a,b\right] \rightarrow \mathbb{R}$ be a convex function on $\left[a,b\right]$. Then for any $x\in \left(a,b\right)$ we have*

$$\begin{aligned} f\left(\frac{x+\frac{a+b}{2}}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{a+x}{2}\right) +f\left(\frac{x+b}{2}\right) \right] \leq D_{a+,b-}f\left(x\right) \\ &\leq \frac{1}{2}\left[f\left(x\right) +\frac{f\left(a\right) +f\left(b\right) }{2}\right] \\ &\leq \frac{1}{2}\left[\frac{\left(b-x\right) f\left(a\right) +\left(x-a\right) f\left(b\right) }{b-a}+\frac{f\left(a\right) +f\left(b\right) }{2}\right] . \end{aligned} \tag{3.1}$$

Proof. The first inequality in (3.1) follows by the convexity of f on $\left[a,b\right]$. By Hermite-Hadamard inequality (1.1) we have

$$f\left(\frac{a+x}{2}\right) \leq \frac{1}{x-a}\int_{a}^{x}f\left(t\right) dt\leq \frac{f\left(a\right) +f\left(x\right) }{2}$$

and

$$f\left(\frac{x+b}{2}\right) \leq \frac{1}{b-x}\int_{x}^{b}f\left(t\right) dt\leq \frac{f\left(x\right) +f\left(a\right) }{2}$$

for any $x\in \left(a,b\right)$.

If we add these two inequalities and divide by 2 we get

$$\begin{aligned} \frac{1}{2}\left[f\left(\frac{a+x}{2}\right) +f\left(\frac{x+b}{2}\right) \right] &\leq \frac{1}{2}\left[\frac{1}{x-a}\int_{a}^{x}f\left(t\right) dt+\frac{1}{b-x}\int_{x}^{b}f\left(t\right) dt\right] \\ &\leq \frac{1}{2}\left[f\left(x\right) +\frac{f\left(a\right) +f\left(b\right) }{2}\right] , \end{aligned}$$

which proves the second and the third inequalities in (3.1).

By the convexity of f we also have

$$f\left(x\right) =f\left(\frac{\left(b-x\right) a+\left(x-a\right) b}{b-a}\right) \leq \frac{\left(b-x\right) f\left(a\right) +\left(x-a\right) f\left(b\right) }{b-a}$$

and then

$$\frac{1}{2}\left[f(x)+\frac{f(a)+f(b)}{2}\right] \leq \frac{1}{2}\left[\frac{(b-x)f(a)+(x-a)f(b)}{b-a}+\frac{f(a)+f(b)}{2}\right],$$

for any $x \in (a,b)$, which proves the last inequality in (3.1). □

We have the following reverse inequalities as well:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in(a,b)$*

$$\begin{aligned}(3.2)\qquad 0 &\leq \frac{1}{16}\left\{\left[f'_{+}\left(\frac{a+x}{2}\right)-f'_{-}\left(\frac{a+x}{2}\right)\right](x-a)\right.\\ &\quad\left.+\left[f'_{+}\left(\frac{x+b}{2}\right)-f'_{-}\left(\frac{x+b}{2}\right)\right](b-x)\right\}\\ &\leq D_{a+,b-}f(x)-\frac{1}{2}\left[f\left(\frac{a+x}{2}\right)+f\left(\frac{x+b}{2}\right)\right]\\ &\leq \frac{1}{16}\left\{\left[f'_{-}(x)-f'_{+}(a)\right](x-a)+\left[f'_{-}(b)-f'_{+}(x)\right](b-x)\right\}\end{aligned}$$

and

$$\begin{aligned}(3.3)\qquad 0 &\leq \frac{1}{16}\left\{\left[f'_{+}\left(\frac{a+x}{2}\right)-f'_{-}\left(\frac{a+x}{2}\right)\right](x-a)\right.\\ &\quad\left.+\left[f'_{+}\left(\frac{x+b}{2}\right)-f'_{-}\left(\frac{x+b}{2}\right)\right](b-x)\right\}\\ &\leq \frac{1}{2}\left[f(x)+\frac{f(a)+f(b)}{2}\right]-D_{a+,b-}f(x)\\ &\leq \frac{1}{16}\left\{\left[f'_{-}(x)-f'_{+}(a)\right](x-a)+\left[f'_{-}(b)-f'_{+}(x)\right](b-x)\right\}.\end{aligned}$$

Proof. We use the following refinement-reverse inequality of the first Hermite-Hadamard inequality obtained in [3]

$$\begin{aligned}(3.4)\qquad 0 &\leq \frac{1}{8}\left[f'_{+}\left(\frac{c+d}{2}\right)-f'_{-}\left(\frac{c+d}{2}\right)\right](d-c)\\ &\leq \frac{1}{d-c}\int_{c}^{d}f(s)\,ds-f\left(\frac{c+d}{2}\right)\leq\frac{1}{8}\left[f'_{-}(d)-f'_{+}(c)\right](d-c)\end{aligned}$$

that holds for the convex function f on $[c,d]$.

Let $x\in(a,b)$. Then by (3.4) we get

$$\begin{aligned}(3.5)\qquad 0 &\leq \frac{1}{8}\left[f'_{+}\left(\frac{a+x}{2}\right)-f'_{-}\left(\frac{a+x}{2}\right)\right](x-a)\\ &\leq \frac{1}{x-a}\int_{a}^{x}f(s)\,ds-f\left(\frac{a+x}{2}\right)\leq\frac{1}{8}\left[f'_{-}(x)-f'_{+}(a)\right](x-a)\end{aligned}$$

and

$$(3.6)\qquad 0 \leq \frac{1}{8}\left[f'_{+}\left(\frac{x+b}{2}\right) - f'_{-}\left(\frac{x+b}{2}\right)\right](b-x)$$
$$\leq \frac{1}{b-x}\int_{x}^{b} f(s)\,ds - f\left(\frac{x+b}{2}\right) \leq \frac{1}{8}\left[f'_{-}(b) - f'_{+}(x)\right](b-x).$$

If we add (3.5) and (3.6) and divide by 2 we get (3.2).

Further on, by using the following refinement-reverse inequality of the second Hermite-Hadamard inequality obtained in [4]

$$0 \leq \frac{1}{8}\left[f'_{+}\left(\frac{c+d}{2}\right) - f'_{-}\left(\frac{c+d}{2}\right)\right](d-c)$$
$$\leq \frac{f(c)+f(d)}{2} - \frac{1}{d-c}\int_{c}^{d} f(s)\,ds \leq \frac{1}{8}\left[f'_{-}(d) - f'_{+}(c)\right](d-c)$$

that holds for the convex function f on $[c,d]$, we have

$$(3.7)\qquad 0 \leq \frac{1}{8}\left[f'_{+}\left(\frac{a+x}{2}\right) - f'_{-}\left(\frac{a+x}{2}\right)\right](x-a)$$
$$\leq \frac{f(a)+f(x)}{2} - \frac{1}{x-a}\int_{a}^{x} f(s)\,ds \leq \frac{1}{8}\left[f'_{-}(x) - f'_{+}(a)\right](x-a)$$

and

$$(3.8)\qquad 0 \leq \frac{1}{8}\left[f'_{+}\left(\frac{x+b}{2}\right) - f'_{-}\left(\frac{x+b}{2}\right)\right](b-x)$$
$$\leq \frac{f(x)+f(b)}{2} - \frac{1}{b-x}\int_{x}^{b} f(s)\,ds \leq \frac{1}{8}\left[f'_{-}(b) - f'_{+}(x)\right](x-a)$$

for any $x \in (a,b)$.

If we add (3.7) and (3.8) and divide by 2 we get (3.3). □

The case of differentiable convex functions that is important for applications provides the following upper bounds:

Corollary 2. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ which is differentiable on (a,b). Then for any $x \in (a,b)$ we have*

$$(3.9)\qquad 0 \leq D_{a+,b-}f(x) - \frac{1}{2}\left[f\left(\frac{a+x}{2}\right) + f\left(\frac{x+b}{2}\right)\right]$$
$$\leq \frac{1}{16}\left\{\left[f'(x) - f'_{+}(a)\right](x-a) + \left[f'_{-}(b) - f'(x)\right](b-x)\right\}$$
$$\leq \frac{1}{16}\begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left[f'_{-}(b) - f'_{+}(a)\right] \\ \\ \left[(x-a)^{p} + (b-x)^{p}\right]^{1/p} \\ \times\left[\left[f'(x) - f'_{+}(a)\right]^{q} + \left[f'_{-}(b) - f'(x)\right]^{q}\right]^{1/q} \\ \textit{if } p,q > 1 \textit{ with } \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \left[\frac{1}{2}\left(f'_{-}(b) - f'_{+}(a)\right) + \left|f'(x) - \frac{f'_{-}(b) + f'_{+}(a)}{2}\right|\right](b-a) \end{cases}$$

and

$$
\begin{aligned}
(3.10)\qquad 0 &\leq \frac{1}{2}\left[f(x)+\frac{f(a)+f(b)}{2}\right]-D_{a+,b-}f(x)\\
&\leq \frac{1}{16}\left\{\left[f'(x)-f'_{+}(a)\right](x-a)+\left[f'_{-}(b)-f'(x)\right](b-x)\right\}\\
&\leq \frac{1}{16}\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[f'_{-}(b)-f'_{+}(a)\right]\\ \\ \left[(x-a)^{p}+(b-x)^{p}\right]^{1/p}\\ \times\left[\left[f'(x)-f'_{+}(a)\right]^{q}+\left[f'_{-}(b)-f'(x)\right]^{q}\right]^{1/q}\\ \text{if } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[\frac{1}{2}\left(f'_{-}(b)-f'_{+}(a)\right)+\left|f'(x)-\frac{f'_{-}(b)+f'_{+}(a)}{2}\right|\right](b-a).\end{cases}
\end{aligned}
$$

The proof follows from Theorem 3.2 by using the Hölder's elementary inequality

$$
mr+ns\leq\begin{cases}\max\{m,n\}(r+s);\\ \\ (m^{p}+n^{p})^{1/p}(r^{q}+s^{q})^{1/q},\ p,q>1,\frac{1}{p}+\frac{1}{q}=1\end{cases}
$$

for $m,\ r,\ n,\ s\geq 0$.

Remark 1. *With the assumptions of Corollary 2 we have the following reverses of Hermite-Hadamard type inequalities*

$$
\begin{aligned}
(3.11)\qquad 0 &\leq \frac{1}{b-a}\int_{a}^{b}f(t)\,dt-\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\\
&\leq \frac{1}{32}\left[f'_{-}(b)-f'_{+}(a)\right](b-a)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.12)\qquad 0 &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\\
&\leq \frac{1}{32}\left[f'_{-}(b)-f'_{+}(a)\right](b-a).
\end{aligned}
$$

4. Inequalities for Hölder Continuous Functions

We say that the function $f:[a,b]\to\mathbb{C}$ is of *H-r-Hölder type* if

$$
|f(t)-f(s)|\leq H|t-s|^{r}
$$

for any $t,\ s\in[a,b]$, where $H>0$ and $r\in(0,1]$. If $r=1$ and we put $H=L$, then we call the function of *L-Lipschitz type.*

We have:

Theorem 5. *If f is of H-r-Hölder type on $[a,b]$ with $H>0$ and $r\in(0,1]$, then for any $x\in(a,b)$ we have*

$$
(4.1)\qquad \left|D_{a+,b-}f(x)-f(x)\right|\leq\frac{1}{2(r+1)}H\left[(x-a)^{r}+(b-x)^{r}\right]
$$

and

$$
(4.2)\qquad \left|D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{2(r+1)}H\left[(x-a)^{r}+(b-x)^{r}\right].
$$

In particular, if f is of L-Lipschitz type, then

$$|D_{a+,b-}f(x)-f(x)|\leq\frac{1}{4}L(b-a) \tag{4.3}$$

and

$$\left|D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{4}L(b-a) \tag{4.4}$$

for any $x\in(a,b)$.

Proof. We observe that, for any $x\in(a,b)$ we have that

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{x-a}\int_a^x[f(t)-f(x)]\,dt+\frac{1}{b-x}\int_x^b[f(t)-f(x)]\,dt\right]\\
&=\frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt-f(x)+\frac{1}{b-x}\int_x^b f(t)\,dt-f(x)\right]\\
&=D_{a+,b-}f(x)-f(x),
\end{aligned}$$

and by taking the modulus, we get

$$\begin{aligned}
&|D_{a+,b-}f(x)-f(x)|\\
&\leq\frac{1}{2}\left[\frac{1}{x-a}\left|\int_a^x[f(t)-f(x)]\,dt\right|+\frac{1}{b-x}\left|\int_x^b[f(t)-f(x)]\right|dt\right]\\
&\leq\frac{1}{2}\left[\frac{1}{x-a}\int_a^x|f(t)-f(x)|\,dt+\frac{1}{b-x}\int_x^b|f(t)-f(x)|\,dt\right]\\
&\leq\frac{1}{2}\left[\frac{H}{x-a}\int_a^x|t-x|^r\,dt+\frac{H}{b-x}\int_x^b|t-x|^r\,dt\right]\\
&=\frac{1}{2}\left[\frac{H(x-a)^r}{r+1}+\frac{H(b-x)^r}{r+1}\right]=\frac{1}{2(r+1)}H\left[(x-a)^r+(b-x)^r\right]
\end{aligned}$$

that proves (4.1).

We observe that, for any $x\in(a,b)$ we also have that

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{x-a}\int_a^x[f(t)-f(a)]\,dt+\frac{1}{b-x}\int_x^b[f(t)-f(b)]\,dt\right]\\
&=\frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt-f(a)+\frac{1}{b-x}\int_x^b f(t)\,dt-f(b)\right]\\
&=D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2},
\end{aligned}$$

and by taking the modulus, we get

$$\begin{aligned}
&\left|D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2}\right| \\
&\leq \frac{1}{2}\left[\frac{1}{x-a}\left|\int_a^x [f(t)-f(a)]\,dt\right|+\frac{1}{b-x}\left|\int_x^b [f(t)-f(b)]\right|dt\right] \\
&\leq \frac{1}{2}\left[\frac{1}{x-a}\int_a^x |f(t)-f(a)|\,dt+\frac{1}{b-x}\int_x^b |f(t)-f(b)|\,dt\right] \\
&\leq \frac{1}{2}\left[\frac{H}{x-a}\int_a^x |t-a|^r\,dt+\frac{H}{b-x}\int_x^b |t-b|^r\,dt\right] \\
&= \frac{1}{2}\left[\frac{H(x-a)^r}{r+1}+\frac{H(b-x)^r}{r+1}\right]=\frac{1}{2(r+1)}H\left[(x-a)^r+(b-x)^r\right],
\end{aligned}$$

that proves (4.2). □

We also have:

Theorem 6. *If f is of H-r-Hölder type on $[a,b]$ with $H>0$ and $r\in(0,1]$, then for any $x\in(a,b)$ we have*

$$\begin{aligned}
&\left|D_{a+,b-}f(x)-\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]\right| \\
&\leq \frac{1}{2^{r+1}(r+1)}H\left[(x-a)^r+(b-x)^r\right].
\end{aligned} \tag{4.5}$$

In particular, if f is of L-Lipschitz type, then

$$\left|D_{a+,b-}f(x)-\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]\right|\leq\frac{1}{8}L(b-a) \tag{4.6}$$

for any $x\in(a,b)$.

Proof. We observe that, for any $x\in(a,b)$ we have that

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{x-a}\int_a^x\left[f(t)-f\left(\frac{x+a}{2}\right)\right]dt+\frac{1}{b-x}\int_x^b\left[f(t)-f\left(\frac{x+b}{2}\right)\right]dt\right] \\
&=\frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt-f\left(\frac{x+a}{2}\right)+\frac{1}{b-x}\int_x^b f(t)\,dt-f\left(\frac{x+b}{2}\right)\right] \\
&=D_{a+,b-}f(x)-\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right].
\end{aligned}$$

By taking the modulus we have

$$\begin{aligned}
&\left|D_{a+,b-}f(x)-\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]\right| \\
&\leq \frac{1}{2}\left[\frac{1}{x-a}\int_a^x\left|f(t)-f\left(\frac{x+a}{2}\right)\right|dt+\frac{1}{b-x}\int_x^b\left|f(t)-f\left(\frac{x+b}{2}\right)\right|dt\right] \\
&\leq \frac{1}{2}\left[\frac{H}{x-a}\int_a^x\left|t-\frac{x+a}{2}\right|^r dt+\frac{H}{b-x}\int_x^b\left|t-\frac{x+b}{2}\right|^r dt\right] \\
&= \frac{1}{2}\left[\frac{H}{x-a}\frac{(x-a)^{r+1}}{2^r(r+1)}+\frac{H}{b-x}\frac{(b-x)^{r+1}}{2^r(r+1)}\right] \\
&= \frac{H}{2^{r+1}(r+1)}\left[(x-a)^r+(b-x)^r\right]
\end{aligned}$$

for any $x\in(a,b)$, which proves the inequality (4.5). □

Remark 2. *If we take in Theorem 6* $x=\frac{a+b}{2}$, *then we get*

$$\begin{aligned}
0 &\leq \left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right| \\
&\leq \frac{1}{4^r(r+1)}H(b-a)^r
\end{aligned} \tag{4.7}$$

and

$$0\leq\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right|\leq\frac{1}{8}L(b-a).$$

5. Applications

We define the *logarithmic mean* $L(x,y)$, given by

$$L(x,y):=\frac{y-x}{\ln y-\ln x}$$

and *identric mean* $I(x,y)$, given by

$$I(x,y):=\frac{1}{e}\left(\frac{y^y}{x^x}\right)^{1/(y-x)},$$

for x, $y>0$ and $x\neq y$. In each case we define the mean as x when $y=x$.

We observe that if $f_{-1}(t)=\frac{1}{t}$, $t>0$, then

$$\frac{1}{y-x}\int_x^y f_{-1}(t)\,dt=\frac{1}{y-x}\int_x^y\frac{1}{t}dt=\frac{1}{L(x,y)}$$

and if $f_0(t)=\ln t$, $t>0$, then

$$\frac{1}{y-x}\int_x^y f_0(t)\,dt=\frac{1}{y-x}\int_x^y\ln t\,dt=\ln I(x,y).$$

Therefore we have

$$D_{a+,b-}f_{-1}(x) := \frac{1}{2}\left[\frac{1}{L(a,x)} + \frac{1}{L(x,b)}\right] = \frac{L(a,x)+L(x,b)}{2L(a,x)L(x,b)} = H^{-1}(L(a,x),L(x,b)),\ x \in (a,b)$$

and

$$D_{a+,b-}f_0(x) := \frac{1}{2}\left[\ln I(a,x) + \ln I(x,b)\right] = \ln\left(\sqrt{I(a,x)I(x,b)}\right) = \ln G(I(a,x),I(x,b)),\ x \in (a,b),$$

where $H(\alpha,\beta) := \frac{2\alpha\beta}{\alpha+\beta}$ is the *harmonic mean* and $G(\alpha,\beta) := \sqrt{\alpha\beta}$ is the *geometric mean* of the positive numbers $\alpha,\beta > 0$.

Writing the inequality (3.9) for the functions f_{-1} and $-f_0$ we get

$$0 \le H^{-1}(L(a,x),L(x,b)) - H^{-1}(A(a,x),A(x,b)) \le \frac{1}{16}\left[\frac{a+x}{a^2x^2}(x-a)^2 + \frac{x+b}{x^2b^2}(b-x)^2\right] \tag{5.1}$$

and

$$0 \le \ln G(A(a,x),A(x,b)) - \ln G(I(a,x),I(x,b)) \le \frac{1}{16}\left[\frac{(x-a)^2}{ax} + \frac{(b-x)^2}{xb}\right] \tag{5.2}$$

for $x \in (a,b) \subset (0,\infty)$. Here $A(\alpha,\beta) := \frac{\alpha+\beta}{2}$ denoted the *arithmetic mean*.

Writing the inequality (3.10) for the functions f_{-1} and $-f_0$ we get

$$0 \le H^{-1}\left(x, H^{-1}(a,b)\right) - H^{-1}(L(a,x),L(x,b)) \le \frac{1}{16}\left[\frac{a+x}{a^2x^2}(x-a)^2 + \frac{x+b}{x^2b^2}(b-x)^2\right] \tag{5.3}$$

and

$$0 \le \ln G(I(a,x),I(x,b)) - \ln G(x,G(a,b)) \le \frac{1}{16}\left[\frac{(x-a)^2}{ax} + \frac{(b-x)^2}{xb}\right] \tag{5.4}$$

for $x \in (a,b) \subset (0,\infty)$.

References

[1] M. Alomari and M. Darus, The Hadamard's inequality for s-convex function. *Int. J. Math. Anal.* (Ruse) **2** (2008), no. 13-16, 639–646.

[2] E. F. Beckenbach, Convex functions, *Bull. Amer. Math. Soc.* **54**(1948), 439–460.

[3] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), No. 2, Article 31, 8 pp. [Online http://www.emis.de/journals/JIPAM/article183.html?sid=183].

[4] S. S. Dragomir, An Inequality Improving the Second Hermite-Hadamard Inequality for Convex Functions Defined on Linear Spaces and Applications for Semi-Inner Products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 3, Article 35, 8 pp. [Online https://www.emis.de/journals/JIPAM/article187.html?sid=187].

[5] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results, *Australian J. Math. Anal. Appl.*, Volume **14**, Issue 1, Article 1, pp. 1-287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[6] S. S. Dragomir, M. S. Moslehian and Y. J. Cho, Some reverses of the Cauchy-Schwarz inequality for complex functions of self-adjoint operators in Hilbert spaces. *Math. Inequal. Appl.* **17** (2014), no. 4, 1365–1373.

[7] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].

[8] A. El Farissi, Simple proof and refinement of Hermite-Hadamard inequality, *J. Math. Ineq.* **4** (2010), No. 3, 365–369.

[9] E. Kikianty and S. S. Dragomir, Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space, *Math. Inequal. Appl.* **13** (2010), no. 1, 1–32.

[10] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, *Aequationes Math.* **28** (1985), 229–232.

[11] C. E. M. Pearce and A. M. Rubinov, P-functions, quasi-convex functions, and Hadamard-type inequalities. *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.

[12] M. Z. Sarikaya, A. Saglam, and H. Yildirim, On some Hadamard-type inequalities for h-convex functions. *J. Math. Inequal.* **2** (2008), no. 3, 335–341.

[13] E. Set, M. E. Özdemir and M. Z. Sarıkaya, New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. *Facta Univ. Ser. Math. Inform.* **27** (2012), no. 1, 67–82.

[14] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions. *Acta Math. Univ. Comenian.* (N.S.) **79** (2010), no. 2, 265–272.

[15] M. Tunç, Ostrowski-type inequalities via h-convex functions with applications to special means. *J. Inequal. Appl.* **2013**, 2013:326.

[16] S. Varošanec, On h-convexity. *J. Math. Anal. Appl.* **326** (2007), no. 1, 303–311.

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