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SOME INEQUALITIES OF HERMITE-HADAMARD TYPE FOR CONVEX FUNCTIONS

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Abstract. In this paper we establish some inequalities for the operator

$$D_{a+,b-}f(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right],\ x \in (a,b)$$

in the case of convex functions $f : [a,b] \to \mathbb{R}$. As a consequence, some Hermite-Hadamard type inequalities improving classical results are also provided. Some examples for logarithm are given.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f : [a,b] \to \mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality*.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [7], the recent survey paper [5], the research papers [1]-[2], [9]-[17] and the references therein.

In 1906, Fejér [9], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Lemma 1 (Fejér's Inequality)**.** *Consider the integral $\int_a^b h(x)\,g(x)\,dx$, where h is a convex function in the interval (a,b) and g is a positive function in the same interval such that*

$$g(a+t) = g(b-t),\ \ 0 \le t \le \frac{1}{2}(b-a),$$

i.e., g is symmetric on $[a,b]$. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dt \le \int_a^b h(t)\,g(t)\,dx \le \frac{h(a)+h(b)}{2}\int_a^b g(t)\,dt. \tag{1.2}$$

If h is concave on (a,b), then the inequalities reverse in (2.3).

Clearly, for $g(x) \equiv 1$ on $[a,b]$ we get (1.1).

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We observe that, if we apply Fejér's inequality for the symmetric weight $w(x) = \frac{1}{2}\left[(x-a)^2 + (b-x)^2\right]$ we have for the convex function $f : [a,b] \to \mathbb{R}$ the following weighted integral inequality

$$f\left(\frac{a+b}{2}\right)\frac{(b-a)^3}{3} \le \frac{1}{2}\int_a^b \left[(x-a)^2 + (b-x)^2\right] f(x)\,dx \le \frac{f(a)+f(b)}{2}\frac{(b-a)^3}{3}. \tag{1.3}$$

Assume that the function $f : (a,b) \to \mathbb{C}$ is Lebesgue integrable on (a,b). We introduce the following operator

$$D_{a+,b-}f(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right],\ x \in (a,b).$$

We observe that if we take $x = \frac{a+b}{2}$, then we have

$$D_{a+,b-}f\left(\frac{a+b}{2}\right) = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Moreover, if $f(a+) := \lim_{x\to a+} f(x)$ exists and is finite, then we have

$$\lim_{x\to a+} D_{a+,b-}f(x) = \frac{1}{2}\left[f(a+) + \frac{1}{b-a}\int_a^b f(t)\,dt\right]$$

and if $f(b-) := \lim_{x\to b-} f(x)$ exists and is finite, then we have

$$\lim_{x\to b-} D_{a+,b-}f(x) = \frac{1}{2}\left[f(b-) + \frac{1}{b-a}\int_a^b f(t)\,dt\right].$$

So, if $f : [a,b] \to \mathbb{C}$ is Lebesgue integrable on $[a,b]$ and continuous at right in a and at left in b, then we can extend the operator on the whole interval by putting

$$D_{a+,b-}f(a) := \frac{1}{2}\left[f(a) + \frac{1}{b-a}\int_a^b f(t)\,dt\right]$$

and

$$D_{a+,b-}f(b) = \frac{1}{2}\left[f(b) + \frac{1}{b-a}\int_a^b f(t)\,dt\right].$$

In this paper we establish several inequalities for the operator $D_{a+,b-}f(x)$ for convex functions $f : [a,b] \to \mathbb{R}$ and $x \in (a,b)$. As a consequence, some Hermite-Hadamard type inequalities improving classical results are also provided. Some examples for logarithm are given.

2. A Refinement of the Second HH-Inequality

We have the following lower and upper bounds for $D_{a+,b-}f$:

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in(a,b)$ we have*

$$
\begin{aligned}
f\left(\frac{x+\frac{a+b}{2}}{2}\right) &\le \frac{1}{2}\left[f\left(\frac{a+x}{2}\right)+f\left(\frac{x+b}{2}\right)\right]\le D_{a+,b-}f(x) \\
&\le \frac{1}{2}\left[f(x)+\frac{f(a)+f(b)}{2}\right] \\
&\le \frac{1}{2}\left[\frac{(b-x)f(a)+(x-a)f(b)}{b-a}+\frac{f(a)+f(b)}{2}\right] \\
&\left(=\frac{3b-2x-a}{4(b-a)}f(a)+\frac{b+2x-3a}{4(b-a)}f(b)\right).
\end{aligned} \tag{2.1}
$$

Proof. The first inequality in (2.1) follows by the convexity of f on $[a,b]$. By Hermite-Hadamard inequality we have

$$f\left(\frac{a+x}{2}\right)\le\frac{1}{x-a}\int_a^x f(t)\,dt\le\frac{f(a)+f(x)}{2}$$

and

$$f\left(\frac{x+b}{2}\right)\le\frac{1}{b-x}\int_x^b f(t)\,dt\le\frac{f(x)+f(a)}{2}$$

for any $x\in(a,b)$.

If we add these two inequalities and divide by 2 we get

$$
\begin{aligned}
\frac{1}{2}\left[f\left(\frac{a+x}{2}\right)+f\left(\frac{x+b}{2}\right)\right] &\le \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right] \\
&\le \frac{1}{2}\left[f(x)+\frac{f(a)+f(b)}{2}\right],
\end{aligned}
$$

which proves the second and the third inequalities in (2.1).

By the convexity of f we also have

$$f(x)=f\left(\frac{(b-x)a+(x-a)b}{b-a}\right)\le\frac{(b-x)f(a)+(x-a)f(b)}{b-a}$$

and then

$$
\begin{aligned}
&\frac{1}{2}\left[f(x)+\frac{f(a)+f(b)}{2}\right] \\
&\le \frac{1}{2}\left[\frac{(b-x)f(a)+(x-a)f(b)}{b-a}+\frac{f(a)+f(b)}{2}\right],
\end{aligned}
$$

which proves the last inequality in (2.1). □

The following lemma is of interest in itself:

Lemma 3. *Assume that the function $f:(a,b)\to\mathbb{C}$ is Lebesgue integrable on (a,b) and $f(a+)$, $f(b-)$ exists and are finite. Then we have*

$$\int_a^b D_{a+,b-}f(x)\,dx=\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx. \tag{2.2}$$

Proof. We have

$$\int_a^b D_{a+,b-}f(x)\,dx := \frac{1}{2}\left[\int_a^b \left(\frac{1}{x-a}\int_a^x f(t)\,dt\right)dx + \int_a^b \left(\frac{1}{b-x}\int_x^b f(t)\,dt\right)dx\right].$$

Observe that, integrating by parts, we have

$$\begin{aligned}
\int_a^b \left(\frac{1}{x-a}\int_a^x f(t)\,dt\right)dx &= \int_a^b \left(\int_a^x f(t)\,dt\right)d(\ln(x-a)) \\
&= \ln(x-a)\left(\int_a^x f(t)\,dt\right)\Big|_{a+}^{b} - \int_a^b \ln(x-a)\,f(x)\,dx \\
&= \ln(b-a)\left(\int_a^b f(t)\,dt\right) - \lim_{x\to a+}\left[\ln(x-a)\left(\int_a^x f(t)\,dt\right)\right] - \int_a^b \ln(x-a)\,f(x)\,dx.
\end{aligned}$$

Since

$$\begin{aligned}
\lim_{x\to a+}\left[\ln(x-a)\left(\int_a^x f(t)\,dt\right)\right] &= \lim_{x\to a+}\left[(x-a)\ln(x-a)\left(\frac{1}{x-a}\int_a^x f(t)\,dt\right)\right] \\
&= \lim_{x\to a+}\left[(x-a)\ln(x-a)\right]\lim_{x\to a+}\left(\frac{1}{x-a}\int_a^x f(t)\,dt\right) = 0f(a+) = 0,
\end{aligned}$$

hence

$$\begin{aligned}
\int_a^b \left(\frac{1}{x-a}\int_a^x f(t)\,dt\right)dx &= \ln(b-a)\left(\int_a^b f(t)\,dt\right) - \int_a^b \ln(x-a)\,f(x)\,dx \\
&= \int_a^b \left[\ln(b-a) - \ln(x-a)\right]f(x)\,dx = \int_a^b \ln\left(\frac{b-a}{x-a}\right)f(x)\,dx
\end{aligned}$$

Also, integrating by parts, we have

$$\begin{aligned}
\int_a^b \left(\frac{1}{b-x}\int_x^b f(t)\,dt\right)dx &= -\int_a^b \left(\int_x^b f(t)\,dt\right)d(\ln(b-x)) \\
&= -\ln(b-x)\left(\int_x^b f(t)\,dt\right)\Big|_a^{b-} + \int_a^b \ln(b-x)\,d\left(\int_x^b f(t)\,dt\right) \\
&= -\lim_{x\to b-}\left[\ln(b-x)\left(\int_x^b f(t)\,dt\right)\right] + \ln(b-a)\left(\int_a^b f(t)\,dt\right) - \int_a^b \ln(b-x)\,f(x)\,dx \\
&= \ln(b-a)\left(\int_a^b f(t)\,dt\right) - \int_a^b \ln(b-x)\,f(x)\,dx = \int_a^b \ln\left(\frac{b-a}{b-x}\right)f(x)\,dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_a^b D_{a+,b-}f(x)\,dx &= \frac{1}{2}\left[\int_a^b \ln\left(\frac{b-a}{x-a}\right)f(x)\,dx + \int_a^b \ln\left(\frac{b-a}{b-x}\right)f(x)\,dx\right] \\
&= \frac{1}{2}\int_a^b \ln\left[\frac{(b-a)^2}{(x-a)(b-x)}\right]f(x)\,dx = \int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx
\end{aligned}$$

and the equality (2.2) is obtained. □

We can state now the second main result that provides a refinement of the second Hermite-Hadamard inequality:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &\le \frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx \\ &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx+\frac{f(a)+f(b)}{2}\right] \\ &\le \frac{f(a)+f(b)}{2}.\end{aligned}\tag{2.3}$$

Proof. By taking the integral in (2.1) we get

$$\begin{aligned}\frac{1}{2}\int_a^b\left[f\left(\frac{a+x}{2}\right)+f\left(\frac{x+b}{2}\right)\right]dx &\le \int_a^b D_{a+,b-}f(x)\,dx \\ &\le \frac{1}{2}\left[\int_a^b f(x)\,dx+\frac{f(a)+f(b)}{2}(b-a)\right].\end{aligned}\tag{2.4}$$

Using the change of variable we have

$$\int_a^b f\left(\frac{a+x}{2}\right)dx = 2\int_a^{\frac{a+b}{2}} f(y)\,dy \text{ and } \int_a^b f\left(\frac{x+b}{2}\right)dx = 2\int_{\frac{a+b}{2}}^b f(y)\,dy$$

and then

$$\frac{1}{2}\int_a^b\left[f\left(\frac{a+x}{2}\right)+f\left(\frac{x+b}{2}\right)\right]dx = \int_a^{\frac{a+b}{2}} f(y)\,dy+\int_{\frac{a+b}{2}}^b f(y)\,dy = \int_a^b f(y)\,dy.$$

Then by (2.4) and Lemma 3 we get the desired result (2.3). □

3. Some Reverse Inequalities

We have the following reverse inequalities as well:

Lemma 4. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $x\in(a,b)$*

$$\begin{aligned}0 &\le D_{a+,b-}f(x)-\frac{1}{2}\left[f\left(\frac{a+x}{2}\right)+f\left(\frac{x+b}{2}\right)\right] \\ &\le \frac{1}{16}\left\{\left[f'_-(x)-f'_+(a)\right](x-a)+\left[f'_-(b)-f'_+(x)\right](b-x)\right\}\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}0 &\le \frac{1}{2}\left[f(x)+\frac{f(a)+f(b)}{2}\right]-D_{a+,b-}f(x) \\ &\le \frac{1}{16}\left\{\left[f'_-(x)-f'_+(a)\right](x-a)+\left[f'_-(b)-f'_+(x)\right](b-x)\right\}.\end{aligned}\tag{3.2}$$

Proof. We use the following refinement-reverse inequality of the first Hermite-Hadamard inequality obtained in [3]

$$0\le\frac{1}{d-c}\int_c^d f(s)\,ds-f\left(\frac{c+d}{2}\right)\le\frac{1}{8}\left[f'_-(d)-f'_+(c)\right](d-c)\tag{3.3}$$

that holds for the convex function f on $[c,d]$.

Let $x \in (a,b)$. Then by (3.3) we get

$$0 \leq \frac{1}{x-a}\int_a^x f(s)\,ds - f\left(\frac{a+x}{2}\right) \leq \frac{1}{8}\left[f'_-(x) - f'_+(a)\right](x-a) \tag{3.4}$$

and

$$0 \leq \frac{1}{b-x}\int_x^b f(s)\,ds - f\left(\frac{x+b}{2}\right) \leq \frac{1}{8}\left[f'_-(b) - f'_+(x)\right](b-x). \tag{3.5}$$

If we add (3.4) and (3.5) and divide by 2 we get (3.1).

Further on, by using the following refinement-reverse inequality of the second Hermite-Hadamard inequality obtained in [4]

$$0 \leq \frac{f(c)+f(d)}{2} - \frac{1}{d-c}\int_c^d f(s)\,ds \leq \frac{1}{8}\left[f'_-(d) - f'_+(c)\right](d-c) \tag{3.6}$$

that holds for the convex function f on $[c,d]$, we have

$$0 \leq \frac{f(a)+f(x)}{2} - \frac{1}{x-a}\int_a^x f(s)\,ds \leq \frac{1}{8}\left[f'_-(x) - f'_+(a)\right](x-a) \tag{3.7}$$

and

$$0 \leq \frac{f(x)+f(b)}{2} - \frac{1}{b-x}\int_x^b f(s)\,ds \leq \frac{1}{8}\left[f'_-(b) - f'_+(x)\right](x-a) \tag{3.8}$$

for any $x \in (a,b)$.

If we add (3.7) and (3.8) and divide by 2 we get (3.2). $\square$

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ which is differentiable on (a,b). Then for any $x \in (a,b)$ we have*

$$\begin{aligned} 0 &\leq D_{a+,b-}f(x) - \frac{1}{2}\left[f\left(\frac{a+x}{2}\right) + f\left(\frac{x+b}{2}\right)\right] \\ &\leq \frac{1}{16}\left\{\left[f'(x) - f'_+(a)\right](x-a) + \left[f'_-(b) - f'(x)\right](b-x)\right\} \\ &\leq \frac{1}{16}\begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left[f'_-(b) - f'_+(a)\right] \\ \\ \left[\frac{1}{2}\left(f'_-(b) - f'_+(a)\right) + \left|f'(x) - \frac{f'_-(b)+f'_+(a)}{2}\right|\right](b-a) \end{cases} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[f(x) + \frac{f(a)+f(b)}{2}\right] - D_{a+,b-}f(x) \\ &\leq \frac{1}{16}\left\{\left[f'(x) - f'_+(a)\right](x-a) + \left[f'_-(b) - f'(x)\right](b-x)\right\} \\ &\leq \frac{1}{16}\begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]\left[f'_-(b) - f'_+(a)\right] \\ \\ \left[\frac{1}{2}\left(f'_-(b) - f'_+(a)\right) + \left|f'(x) - \frac{f'_-(b)+f'_+(a)}{2}\right|\right](b-a). \end{cases} \end{aligned} \tag{3.10}$$

Using the above lemma we can state the following result as well:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

$$\begin{aligned}(3.11)\quad 0 &\le \frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\le \frac{1}{8}\left(\frac{f(b)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right) + \frac{1}{32}\left[f'_-(b) - f'_+(a)\right](b-a) \\ &\le \frac{3}{64}\left[f'_-(b) - f'_+(a)\right](b-a)\end{aligned}$$

and

$$\begin{aligned}(3.12)\quad 0 &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2}\right] \\ &- \frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx \\ &\le \frac{1}{8}\left(\frac{f(b)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right) + \frac{1}{32}\left[f'_-(b) - f'_+(a)\right](b-a) \\ &\le \frac{3}{64}\left[f'_-(b) - f'_+(a)\right](b-a).\end{aligned}$$

Proof. By taking the integral mean in (3.1) we get

$$\begin{aligned}(3.13)\quad 0 &\le \frac{1}{b-a}\int_a^b D_{a+,b-}f(x)\,dx - \frac{1}{2}\frac{1}{b-a}\int_a^b \left[f\left(\frac{a+x}{2}\right) + f\left(\frac{x+b}{2}\right)\right]dx \\ &\le \frac{1}{16}\frac{1}{b-a}\int_a^b \left\{\left[f'_-(x) - f'_+(a)\right](x-a) + \left[f'_-(b) - f'_+(x)\right](b-x)\right\}dx.\end{aligned}$$

Observe that, by (2.2) we have

$$\int_a^b D_{a+,b-}f(x)\,dx = \int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx$$

and

$$\frac{1}{2}\frac{1}{b-a}\int_a^b \left[f\left(\frac{a+x}{2}\right) + f\left(\frac{x+b}{2}\right)\right]dx = \int_a^b f(y)\,dy.$$

Since $f'_-(x) = f'_+(x)$ for almost every $x\in(a,b)$, we can write $f'(x)$ everywhere in the integral. By using the integration by parts formula, we have

$$\begin{aligned}\int_a^b \left[f'(x) - f'_+(a)\right](x-a)\,dx &= \int_a^b f'(x)(x-a)\,dx - \int_a^b f'_+(a)(x-a)\,dx \\ &= f(b)(b-a) - \int_a^b f(x)\,dx - \frac{1}{2}f'_+(a)(b-a)^2\end{aligned}$$

and

$$\begin{aligned}\int_a^b \left[f'_-(b)-f'(x)\right](b-x)\,dx &= \int_a^b f'_-(b)(b-x)\,dx-\int_a^b f'(x)(b-x)\,dx\\ &=\frac{1}{2}f'_-(b)(b-a)^2-\left[f(x)(b-x)|_a^b+\int_a^b f(x)\,dx\right]\\ &=\frac{1}{2}f'_-(b)(b-a)^2-\left[-f(a)(b-a)+\int_a^b f(x)\,dx\right]\\ &=\frac{1}{2}f'_-(b)(b-a)^2+f(a)(b-a)-\int_a^b f(x)\,dx,\end{aligned}$$

which gives

$$\begin{aligned}&\int_a^b \left\{\left[f'_-(x)-f'_+(a)\right](x-a)+\left[f'_-(b)-f'_+(x)\right](b-x)\right\}dx\\ &=f(b)(b-a)-\int_a^b f(x)\,dx-\frac{1}{2}f'_+(a)(b-a)^2\\ &+\frac{1}{2}f'_-(b)(b-a)^2+f(a)(b-a)-\int_a^b f(x)\,dx\\ &=[f(b)+f(a)](b-a)+\frac{1}{2}\left[f'_-(b)-f'_+(a)\right](b-a)^2-2\int_a^b f(x)\,dx.\end{aligned}$$

By (3.13) we get

$$\begin{aligned}(3.14)\quad 0&\le\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\\ &\le\frac{1}{16}\frac{1}{b-a}\left\{[f(b)+f(a)](b-a)+\frac{1}{2}\left[f'_-(b)-f'_+(a)\right](b-a)^2-2\int_a^b f(x)\,dx\right\}\\ &=\frac{f(b)+f(a)}{16}+\frac{1}{32}\left[f'_-(b)-f'_+(a)\right](b-a)-\frac{1}{8}\frac{1}{b-a}\int_a^b f(x)\,dx\end{aligned}$$

and the second inequality in (3.11) is proved.

Since

$$\begin{aligned}\frac{f(b)+f(a)}{16}-\frac{1}{8}\frac{1}{b-a}\int_a^b f(x)\,dx&=\frac{1}{8}\left(\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right)\\ &\le\frac{1}{64}\left[f'_-(b)-f'_+(a)\right](b-a)\end{aligned}$$

the last part of (3.11) thus follows.

The inequality (3.12) follows in a similar way by integrating the inequality (3.2). We omit the details. □

4. A Refinement of a Weighted Integral Inequality

We need the following equality as well:

Lemma 5. *Assume that the function $f:(a,b)\to\mathbb{C}$ is Lebesgue integrable on (a,b). Then we have*

$$\int_a^b (x-a)(b-x)\,D_{a+,b-}f(x)\,dx = \frac{1}{4}\int_a^b \left[(x-a)^2+(b-x)^2\right] f(x)\,dx. \tag{4.1}$$

Proof. We have

$$\begin{aligned}&\int_a^b (x-a)(b-x)\,D_{a+,b-}f(x)\,dx \\ &= \frac{1}{2}\left[\int_a^b (b-x)\left(\int_a^x f(t)\,dt\right)dx + \int_a^b (x-a)\left(\int_x^b f(t)\,dt\right)dx\right].\end{aligned} \tag{4.2}$$

Using the integration by parts formula, we have

$$\begin{aligned}&\int_a^b (b-x)\left(\int_a^x f(t)\,dt\right)dx \\ &= -\int_a^b \left(\int_a^x f(t)\,dt\right) d\left(\frac{(b-x)^2}{2}\right) \\ &= -\left(\int_a^x f(t)\,dt\right)\frac{(b-x)^2}{2}\Bigg|_{a+}^{b} + \int_a^b \frac{(b-x)^2}{2} d\left(\int_a^x f(t)\,dt\right) \\ &= \frac{1}{2}\int_a^b (b-x)^2 f(x)\,dx\end{aligned}$$

and

$$\begin{aligned}&\int_a^b (x-a)\left(\int_x^b f(t)\,dt\right)dx \\ &= \int_a^b \left(\int_x^b f(t)\,dt\right) d\left(\frac{(x-a)^2}{2}\right) \\ &= \frac{(x-a)^2}{2}\int_x^b f(t)\,dt\Bigg|_{a}^{b-} - \int_a^b \frac{(x-a)^2}{2} d\left(\int_x^b f(t)\,dt\right) \\ &= \frac{1}{2}\int_a^b (x-a)^2 f(x)\,dx,\end{aligned}$$

which, by (4.2) produces the desired result (4.1). □

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\frac{(b-a)^3}{3} &\le \left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\frac{(b-a)^3}{6} \\ \le 8\left[\int_a^{\frac{a+b}{2}} (x-a)\left(\frac{a+b}{2}-x\right) f(x)\,dx \right. &+ \left. \int_{\frac{a+b}{2}}^b \left(x-\frac{a+b}{2}\right)(b-x) f(x)\,dx\right] \\ &\le \frac{1}{2}\int_a^b \left[(x-a)^2+(b-x)^2\right] f(x)\,dx \\ \le \int_a^b (x-a)(b-x) f(x)\,dx + \frac{f(a)+f(b)}{12}(b-a)^3 &\le \frac{f(a)+f(b)}{2}\frac{(b-a)^3}{3}.\end{aligned} \tag{4.3}$$

Proof. From (2.1) we get by multiplying with $(x-a)(b-x)$ and integration, that

$$\begin{aligned}
&\frac{1}{2}\int_a^b (x-a)(b-x)\left[f\left(\frac{a+x}{2}\right)+f\left(\frac{x+b}{2}\right)\right]dx \\
&\leq \int_a^b (x-a)(b-x)\,D_{a+,b-}f(x)\,dx \\
&= \frac{1}{4}\int_a^b \left[(x-a)^2+(b-x)^2\right]f(x)\,dx \\
&\leq \frac{1}{2}\int_a^b (x-a)(b-x)\left[f(x)+\frac{f(a)+f(b)}{2}\right]dx,
\end{aligned}$$

namely

$$\begin{aligned}
&\int_a^b (x-a)(b-x)\left[f\left(\frac{a+x}{2}\right)+f\left(\frac{x+b}{2}\right)\right]dx \\
&= \frac{1}{2}\int_a^b \left[(x-a)^2+(b-x)^2\right]f(x)\,dx \\
&\leq \int_a^b (x-a)(b-x)\left[f(x)+\frac{f(a)+f(b)}{2}\right]dx,
\end{aligned} \tag{4.4}$$

Since

$$\int_a^b (x-a)(b-x)\,dx = \frac{(b-a)^3}{6},$$

then

$$\begin{aligned}
&\int_a^b (x-a)(b-x)\left[f(x)+\frac{f(a)+f(b)}{2}\right]dx \\
&= \int_a^b (x-a)(b-x)\,f(x)\,dx + \frac{f(a)+f(b)}{12}(b-a)^3.
\end{aligned}$$

By Fejér 's inequality for the positive symmetric weight $w(x)=(x-a)(b-x)$ we also have

$$\begin{aligned}
\int_a^b (x-a)(b-x)\,f(x)\,dx &\leq \frac{f(a)+f(b)}{2}\int_a^b (x-a)(b-x)\,dx \\
&= \frac{f(a)+f(b)}{12}(b-a)^3,
\end{aligned}$$

which proves the last inequality in (4.3).

Using the appropriate change of variable we have

$$\int_a^b (x-a)(b-x)\,f\left(\frac{a+x}{2}\right)dx = 8\int_a^{\frac{a+b}{2}} (y-a)\left(\frac{a+b}{2}-y\right)f(y)\,dy$$

and

$$\int_a^b (x-a)(b-x)\,f\left(\frac{x+b}{2}\right)dx = 8\int_{\frac{a+b}{2}}^b \left(y-\frac{a+b}{2}\right)(b-y)\,f(y)\,dy$$

and by (4.4) we get the fourth inequality in (4.3).

By Fejér 's inequality we also have

$$f\left(\frac{3a+b}{4}\right)\int_a^{\frac{a+b}{2}} (y-a)\left(\frac{a+b}{2}-y\right)dy \leq \int_a^{\frac{a+b}{2}} (y-a)\left(\frac{a+b}{2}-y\right)f(y)\,dy$$

and

$$f\left(\frac{a+3b}{4}\right)\int_{\frac{a+b}{2}}^{b}\left(y-\frac{a+b}{2}\right)(b-y)\le\int_{\frac{a+b}{2}}^{b}\left(y-\frac{a+b}{2}\right)(b-y)\,f(y)\,dy$$

and since

$$\int_{a}^{\frac{a+b}{2}}(y-a)\left(\frac{a+b}{2}-y\right)dy=\int_{\frac{a+b}{2}}^{b}\left(y-\frac{a+b}{2}\right)(b-y)=\frac{(b-a)^{3}}{48},$$

then

$$\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\frac{(b-a)^{3}}{6}$$
$$\le 8\left[\int_{a}^{\frac{a+b}{2}}(y-a)\left(\frac{a+b}{2}-y\right)f(y)\,dy+\int_{\frac{a+b}{2}}^{b}\left(y-\frac{a+b}{2}\right)(b-y)\,f(y)\,dy\right],$$

which proves the second inequality in (4.3). □

Remark 1. *We observe that the approach in Theorem 3 provides a better result than Fejér's inequality in (1.3).*

5. Applications

We define the *logarithmic mean* $L(x,y)$, given by

$$L(x,y):=\frac{y-x}{\ln y-\ln x}$$

and *identric mean* $I(x,y)$, given by

$$I(x,y):=\frac{1}{e}\left(\frac{y^{y}}{x^{x}}\right)^{1/(y-x)},$$

for x, $y>0$ and $x\neq y$. In each case we define the mean as x when $y=x$.

We observe that if $f_{-1}(t)=\frac{1}{t}$, $t>0$, then

$$\frac{1}{y-x}\int_{x}^{y}f_{-1}(t)\,dt=\frac{1}{y-x}\int_{x}^{y}\frac{1}{t}dt=\frac{1}{L(x,y)}$$

and if $f_{0}(t)=\ln t$, $t>0$, then

$$\frac{1}{y-x}\int_{x}^{y}f_{0}(t)\,dt=\frac{1}{y-x}\int_{x}^{y}\ln t dt=\ln I(x,y).$$

Therefore we have

$$\begin{aligned}D_{a+,b-}f_{-1}(x)&:=\frac{1}{2}\left[\frac{1}{L(a,x)}+\frac{1}{L(x,b)}\right]\\&=\frac{L(a,x)+L(x,b)}{2L(a,x)L(x,b)}=H^{-1}\left(L(a,x),L(x,b)\right),\ x\in(a,b)\end{aligned}$$

and

$$\begin{aligned}D_{a+,b-}f_{0}(x)&:=\frac{1}{2}\left[\ln I(a,x)+\ln I(x,b)\right]=\ln\left(\sqrt{I(a,x)I(x,b)}\right)\\&=\ln G\left(I(a,x),I(x,b)\right),\ x\in(a,b)\end{aligned}$$

where $H(\alpha,\beta):=\frac{2\alpha\beta}{\alpha+\beta}$ is the *harmonic mean* and $G(\alpha,\beta):=\sqrt{\alpha\beta}$ is the *geometric mean* of the positive numbers $\alpha,\beta>0$.

From (2.1) we then have the inequalities

$$
\begin{aligned}
A^{-1}(x, A(a,b)) &\le H^{-1}(A(a,x), A(x,b)) \le H^{-1}(L(a,x), L(x,b)) \\
&\le H^{-1}\left(x, H^{-1}(a,b)\right) \\
&\le \frac{1}{2}\left[\frac{(b-x)\,a^{-1} + (x-a)\,b^{-1}}{b-a} + H^{-1}(a,b)\right] \text{ for } x \in (a,b)
\end{aligned} \tag{5.1}
$$

where $A(\alpha, \beta) := \frac{\alpha+\beta}{2}$ is the arithmetic mean.

We also have for the convex function $f(t) = -\ln t$, $t > 0$ that

$$
\begin{aligned}
\ln(A(x, A(a,b))) &\ge \ln G(A(a,x), A(x,b)) \ge \ln G(I(a,x), I(x,b)) \\
&\ge \ln G(x, G(a,b)) \ge \ln G\left(a^{\frac{b-x}{b-a}} b^{\frac{x-a}{b-a}}, G(a,b)\right),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
A(x, A(a,b)) &\ge G(A(a,x), A(x,b)) \ge G(I(a,x), I(x,b)) \\
&\ge G(x, G(a,b)) \ge G\left(a^{\frac{b-x}{b-a}} b^{\frac{x-a}{b-a}}, G(a,b)\right), \; x \in (a,b).
\end{aligned} \tag{5.2}
$$

From (2.3) we have the inequalities

$$
\begin{aligned}
L^{-1}(a,b) &\le \frac{1}{b-a}\int_a^b H^{-1}(L(a,x), L(x,b))\,dx \le H^{-1}(L(a,b), H(a,b)) \\
&\le H^{-1}(a,b)
\end{aligned} \tag{5.3}
$$

and

$$
\begin{aligned}
\ln I(a,b) &\ge \frac{1}{b-a}\int_a^b \ln G(I(a,x), I(x,b))\,dx \\
&\ge \ln G(I(a,b), G(a,b)) \ge \ln G(a,b).
\end{aligned} \tag{5.4}
$$

These are equivalent to

$$
\begin{aligned}
L(a,b) &\ge \left[\frac{1}{b-a}\int_a^b H^{-1}(L(a,x), L(x,b))\,dx\right]^{-1} \ge H(L(a,b), H(a,b)) \\
&\ge H(a,b)
\end{aligned} \tag{5.5}
$$

and

$$
\begin{aligned}
I(a,b) &\ge \exp\left[\frac{1}{b-a}\int_a^b \ln G(I(a,x), I(x,b))\,dx\right] \\
&\ge G(I(a,b), G(a,b)) \ge G(a,b).
\end{aligned} \tag{5.6}
$$

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