Received 05/01/18

OPERATOR IDENTITIES FOR FUNCTIONS DEFINED BY POWER SERIES WITH APPLICATIONS FOR PERSPECTIVES

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Abstract. In this paper we obtain some identities for functions defined by power series with complex coefficients $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$ or on $\mathbb{C}$, for $R=\infty$. Amongst other we show that if A, B are bounded linear operators on the Hilbert space with A positive, $\alpha\in[0,1]$ and $\|AB\|$, $\left\|A^{\alpha}BA^{1-\alpha}\right\|<R$, then

$$f(AB)A^{1-\alpha}=A^{1-\alpha}f\left(A^{\alpha}BA^{1-\alpha}\right).$$

Applications for the noncommutative perspective

$$\mathcal{P}_{\Phi}(B,A):=A^{1/2}\Phi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},$$

where $\Phi:J\subset\mathbb{R}\to\mathbb{C}$ is continuous, B selfadjoint, A positive and invertible with $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset J$, are also given.

1. Introduction

Let Φ be a continuous function defined on the interval J of real numbers and taking complex values, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset\mathring{J}$, the interior of J. Then by using the *continuous functional calculus* for selfadjoint operators we can define the *perspective* $\mathcal{P}_{\Phi}(B,A)$ by setting

$$\mathcal{P}_{\Phi}(B,A):=A^{1/2}\Phi\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}. \tag{P}$$

It is well known that (see [10] and [9] or [11]), if Φ is an *operator convex (concave) function* defined in the positive half-line $(0,\infty)$, namely

$$\Phi((1-t)C+tD)\leq(\geq)(1-t)\Phi(C)+t\Phi(D)$$

for any $t\in[0,1]$ and positive invertible operators C, D, then the mapping

$$(B,A)\mapsto\mathcal{P}_{\Phi}(B,A)$$

defined in pairs of positive definite operators, is operator convex (concave), namely we have

$$\mathcal{P}_{\Phi}(\lambda B+(1-\lambda)D,\lambda A+(1-\lambda)C)\leq(\geq)\lambda\mathcal{P}_{\Phi}(B,A)+(1-\lambda)\mathcal{P}_{\Phi}(D,C)$$

in the operator order for any positive invertible operators A, B, C, D and $\lambda\in[0,1]$.

In the recent paper [1] we established the following reverse inequality for the perspective $\mathcal{P}_{\Phi}(B,A)$ of a continuous convex function Φ.

1991 *Mathematics Subject Classification.* 47A60, 47A63, 47A64.

Key words and phrases. Functional calculus, Noncommutative perspectives, Power series, Operators on Hilbert spaces.

Let $\Phi : [m, M] \to \mathbb{R}$ be a *convex function* on the real interval $[m, M]$, A a positive invertible operator and B a selfadjoint operator such that

$$mA \leq B \leq MA, \tag{1.1}$$

then we have

$$\begin{aligned} 0 &\leq \frac{1}{M-m}\left[\Phi(m)(MA-B)+\Phi(M)(B-mA)\right]-\mathcal{P}_{\Phi}(B,A) \\ &\leq \frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}\delta_{m,M}(B,A) \\ &\leq \frac{1}{4}(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right]A, \end{aligned} \tag{1.2}$$

where

$$\delta_{m,M}(B,A) := A^{1/2}\left(M1_H - A^{-1/2}BA^{-1/2}\right)\left(A^{-1/2}BA^{-1/2}-m1_H\right)A^{1/2}$$

is the perspective generated by the concave function $\Phi : [m, M] \to \mathbb{R}$, $\Phi(t) = (M-t)(t-m)$.

Let $\Phi : J \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{J}$, the interior of J. Suppose that there exists the constants d, D such that

$$d \leq \Phi''(t) \leq D \text{ for any } t \in \mathring{J}. \tag{1.3}$$

If A is a positive invertible operator and B a selfadjoint operator such that the condition (1.1) is valid with $[m, M] \subset \mathring{J}$, then we have the following result as well [2]

$$\begin{aligned} &\frac{1}{2}d\delta_{m,M}(B,A) \\ &\leq \frac{1}{M-m}\left[\Phi(m)(MA-B)+\Phi(M)(B-mA)\right]-\mathcal{P}_{\Phi}(B,A) \\ &\leq \frac{1}{2}D\delta_{m,M}(B,A). \end{aligned} \tag{1.4}$$

If $d > 0$, then the first inequality in (1.4) is better than the same inequality in (1.2).

For other recent results for perspectives, see [3]-[8] and [12]-[15].

Further, let B be a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Let Φ be a continuous function defined on the interval J and assume that $\mathrm{Sp}\left(BA^{-1}\right) \subset \mathring{J}$, then by using the continuous functional calculus for selfadjoint operators we can also define the *quasi-perspective* $\mathcal{Q}_{\Phi}(B,A)$ by setting

$$\mathcal{Q}_{\Phi}(B,A) := A\Phi\left(A^{-1}B\right). \tag{1.5}$$

We observe that if A and B are commutative with $\mathrm{Sp}\left(BA^{-1}\right) \subset \mathring{J}$, then

$$\mathcal{P}_{\Phi}(B,A) = \mathcal{Q}_{\Phi}(B,A). \tag{1.6}$$

It is then natural to ask whether or not *the equality (1.6) holds for some subclasses of continuous functions Φ defined on the interval J and non-commutative selfadjoint operators A and B with A a positive invertible operator and*

$$\mathrm{Sp}\left(A^{-1/2}BA^{-1/2}\right),\ \mathrm{Sp}\left(BA^{-1}\right) \subset \mathring{J}.$$

An answer to this question will be provided in the last section by the use of some identities of interest established below for functions defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R>0$ or on $\mathbb{C}$, for $R=\infty$. Some other operator equalities of interest are established.

2. Main Results

We have the following identity:

Theorem 1. *Let $f(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$ or on $\mathbb{C}$, for $R=\infty$. If A, $B\in\mathcal{B}(H)$ with A positive, $\alpha\in[0,1]$ and $\|AB\|$, $\left\|A^{\alpha}BA^{1-\alpha}\right\|<R$, then*

$$f(AB)A^{1-\alpha}=A^{1-\alpha}f\left(A^{\alpha}BA^{1-\alpha}\right). \tag{2.1}$$

Proof. We claim that for any natural number $n\geq 0$ and $\alpha\in[0,1]$ we have

$$(AB)^n A^{1-\alpha}=A^{1-\alpha}\left(A^{\alpha}BA^{1-\alpha}\right)^n. \tag{2.2}$$

For $n=0$, the identity reduces to $A^{1-\alpha}=A^{1-\alpha}$ while for $n=1$ it becomes $ABA^{1-\alpha}=ABA^{1-\alpha}$.

Assume that it holds for "n" with $n\geq 2$ and let us prove it for "$n+1$".

We have

$$\begin{aligned} A^{1-\alpha}\left(A^{\alpha}BA^{1-\alpha}\right)^{n+1} &= A^{1-\alpha}\left(A^{\alpha}BA^{1-\alpha}\right)^{n}A^{\alpha}BA^{1-\alpha} \\ &= (AB)^n A^{1-\alpha}A^{\alpha}BA^{1-\alpha} \text{ (by induction hypothesis)} \\ &= (AB)^n ABA^{1-\alpha}=(AB)^{n+1}A^{1-\alpha} \end{aligned}$$

and the identity (2.2) is thus proved.

Now, let $m\geq 1$, multiply (2.2) by α_n and sum from 0 to m to get

$$\left(\sum_{n=0}^{m}\alpha_n (AB)^n\right)A^{1-\alpha}=A^{1-\alpha}\sum_{n=0}^{m}\alpha_n\left(A^{\alpha}BA^{1-\alpha}\right)^n. \tag{2.3}$$

Since $\|AB\|$, $\left\|A^{\alpha}BA^{1-\alpha}\right\|<R$ then the series

$$\sum_{n=0}^{\infty}\alpha_n (AB)^n \text{ and } \sum_{n=0}^{\infty}\alpha_n\left(A^{\alpha}BA^{1-\alpha}\right)^n$$

are convergent in strong topology of $\mathcal{B}(H)$ and

$$\sum_{n=0}^{\infty}\alpha_n (AB)^n=f(AB) \text{ and } \sum_{n=0}^{\infty}\alpha_n\left(A^{\alpha}BA^{1-\alpha}\right)^n=f\left(A^{\alpha}BA^{1-\alpha}\right),$$

then by letting $m\to\infty$ in (2.3) we deduce the desired result (2.1). $\square$

Corollary 1. *With the assumptions of Theorem 1 and if A is invertible, then we have the equality*

$$f(AB)=A^{1-\alpha}f\left(A^{\alpha}BA^{1-\alpha}\right)A^{\alpha-1} \tag{2.4}$$

provided $\|AB\|$, $\left\|A^{\alpha}BA^{1-\alpha}\right\|<R$.

In particular,

$$f(AB)=A^{1/2}f\left(A^{1/2}BA^{1/2}\right)A^{-1/2}, \tag{2.5}$$

provided $\|AB\|$, $\left\|A^{1/2}BA^{1/2}\right\| < R$ *and*

$$f(AB) = Af(BA)A^{-1} \tag{2.6}$$

if $\|AB\|$, $\|BA\| < R$.

We also have:

Corollary 2. *With the assumptions of Theorem 1 and if A is invertible with* $\left\|A^{-1}B\right\|$, $\left\|A^{-1/2}BA^{-1/2}\right\| < R$, *then*

$$Af\left(A^{-1}B\right) = A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}. \tag{2.7}$$

Remark 1. *Since* $\|AB\| \le \|A\|\|B\|$ *and*

$$\left\|A^{\alpha}BA^{1-\alpha}\right\| \le \|A\|^{\alpha}\|B\|\|A\|^{1-\alpha} = \|A\|\|B\|,$$

then by assuming $\|A\|\|B\| < R$ *it follows that* $\|AB\| < R$ *and* $\left\|A^{\alpha}BA^{1-\alpha}\right\| < R$ *for any* $\alpha \in [0,1]$. *Therefore, if we assume that* $\|A\|\|B\| < R$ *in Theorem 1 then we get the equality (2.1). In particular, if* $R = 1$ *and* $\|A\|, \|B\| < 1$, *then the conclusion of Theorem 1 remains valid. This fact provides many examples since numerous fundamental functions defined as power series are convergent on the open disk* $D(0,1)$. *Some instances of interest are provided below.*

If we consider the exponential function $f(z) = \sum_{n=0}^{\infty}\frac{1}{n!}z^n = \exp(z)$, $z \in \mathbb{C}$, then from (2.4)-(2.6) we get

$$\begin{aligned} \exp(AB) &= A^{1-\alpha}\exp\left(A^{\alpha}BA^{1-\alpha}\right)A^{\alpha-1} \\ &= A^{1/2}\exp\left(A^{1/2}BA^{1/2}\right)A^{-1/2} = A\exp(BA)A^{-1} \end{aligned} \tag{2.8}$$

for any A, $B \in \mathcal{B}(H)$ with A positive and invertible and $\alpha \in [0,1]$.

Similar equalities hold for the trigonometric functions

$$\sin z = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}, \quad \cos z = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}$$

or for hyperbolic functions

$$\sinh z = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}, \quad \cosh z = \sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}.$$

If we consider $f(z) = \sum_{n=0}^{\infty} z^n = (1-z)^{-1}$, $z \in D(0,1)$, then for any A, $B \in \mathcal{B}(H)$ with A positive and invertible, $\|A\|$, $\|B\| < 1$ and $\alpha \in [0,1]$, we have

$$\begin{aligned} (1_H - AB)^{-1} &= A^{1-\alpha}\left(1_H - A^{\alpha}BA^{1-\alpha}\right)^{-1}A^{\alpha-1} \\ &= A^{1/2}\left(1_H - A^{1/2}BA^{1/2}\right)^{-1}A^{-1/2} = A(1_H - BA)^{-1}A^{-1}. \end{aligned} \tag{2.9}$$

If we consider $f(z) = \sum_{n=1}^{\infty}\frac{1}{n}z^n = \ln(1-z)^{-1}$, $z \in D(0,1)$, then for any A, $B \in \mathcal{B}(H)$ with A positive and invertible, $\|A\|$, $\|B\| < 1$ and $\alpha \in [0,1]$, we have

$$\begin{aligned} \ln(1-AB)^{-1} &= A^{1-\alpha}\left[\ln\left(1_H - A^{\alpha}BA^{1-\alpha}\right)^{-1}\right]A^{\alpha-1} \\ &= A^{1/2}\left[\ln\left(1_H - A^{1/2}BA^{1/2}\right)^{-1}\right]A^{-1/2} \\ &= A\left[\ln(1_H - BA)^{-1}\right]A^{-1}. \end{aligned} \tag{2.10}$$

We have:

Theorem 2. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$ or on $\mathbb{C}$ for $R = \infty$. If A, $B \in \mathcal{B}(H)$ with A positive, $\alpha \in [0,1]$ and $\|BAB^*\|$, $\left\|A^\alpha B^* BA^{1-\alpha}\right\| < R$, then*

$$f(BAB^*) BA^{1-\alpha} = BA^{1-\alpha} f\left(A^\alpha B^* BA^{1-\alpha}\right). \tag{2.11}$$

Proof. We claim that for any natural number $n \geq 0$ and $\alpha \in [0,1]$ we have

$$(BAB^*)^n BA^{1-\alpha} = BA^{1-\alpha}\left(A^\alpha B^* BA^{1-\alpha}\right)^n. \tag{2.12}$$

For $n = 0$ the equality reduces to $BA^{1-\alpha} = BA^{1-\alpha}$ while for $n = 1$ it becomes $BAB^*BA^{1-\alpha} = BAB^*BA^{1-\alpha}$.

Assume that it holds for "n" with $n \geq 2$ and let us prove it for "$n+1$".

We have

$$\begin{aligned} &BA^{1-\alpha}\left(A^\alpha B^* BA^{1-\alpha}\right)^{n+1} \\ &= BA^{1-\alpha}\left(A^\alpha B^* BA^{1-\alpha}\right)^n A^\alpha B^* BA^{1-\alpha} \\ &= (BAB^*)^n BA^{1-\alpha} A^\alpha B^* BA^{1-\alpha} \text{ (by induction hypothesis)} \\ &= (BAB^*)^n BAB^* BA^{1-\alpha} = (BAB^*)^{n+1} BA^{1-\alpha} \end{aligned} \tag{2.13}$$

and the identity (2.13) is thus proved.

Now, let $m \geq 1$, multiply (2.13) by α_n and sum from 0 to m to get

$$\left(\sum_{n=0}^{m} \alpha_n (BAB^*)^n\right) BA^{1-\alpha} = BA^{1-\alpha} \sum_{n=0}^{m} \alpha_n \left(A^\alpha B^* BA^{1-\alpha}\right)^n. \tag{2.14}$$

Since $\|BAB^*\|$, $\left\|A^\alpha B^* BA^{1-\alpha}\right\| < R$ then the series

$$\sum_{n=0}^{\infty} \alpha_n (BAB^*)^n \text{ and } \sum_{n=0}^{\infty} \alpha_n \left(A^\alpha B^* BA^{1-\alpha}\right)^n$$

are convergent in strong topology of $\mathcal{B}(H)$ and

$$\sum_{n=0}^{\infty} \alpha_n (BAB^*)^n = f(BAB^*) \text{ and } \sum_{n=0}^{\infty} \alpha_n \left(A^\alpha B^* BA^{1-\alpha}\right)^n = f\left(A^\alpha B^* BA^{1-\alpha}\right),$$

then by letting $m \to \infty$ in (2.14) we deduce the desired result (2.11). □

Corollary 3. *With the assumptions of Theorem 2 and if A and B are invertible, then we have the equality*

$$f(BAB^*) = BA^{1-\alpha} f\left(A^\alpha B^* BA^{1-\alpha}\right) A^{\alpha-1} B^{-1}, \tag{2.15}$$

provided $\|BAB^*\|$, $\left\|A^\alpha B^* BA^{1-\alpha}\right\| < R$.

In particular,

$$f(BAB^*) = BA^{1/2} f\left(A^{1/2} B^* BA^{1/2}\right) A^{-1/2} B^{-1}, \tag{2.16}$$

provided $\|BAB^*\|$, $\left\|A^{1/2} B^* BA^{1/2}\right\| < R$,

$$f(BAB^*) = BAf(B^*BA) A^{-1} B^{-1}, \tag{2.17}$$

provided $\|BAB^*\|$, $\|B^*BA\| < R$ and

$$f(BAB^*) = Bf(AB^*B) B^{-1}, \tag{2.18}$$

provided $\|BAB^*\|, \|AB^*B\| < R$.

Remark 2. *We also have, by (2.16) that*

$$f(BAB^*)B = BAA^{-1/2}f\left(A^{1/2}B^*BA^{1/2}\right)A^{-1/2}$$

or

$$(BA)^{-1}f(BAB^*)B = A^{-1/2}f\left(A^{1/2}B^*BA^{1/2}\right)A^{-1/2}$$

namely

$$A^{-1}B^{-1}f(BAB^*)B = A^{-1/2}f\left(A^{1/2}B^*BA^{1/2}\right)A^{-1/2}.$$

If in this equality we replace A *by* A^{-1}, *then we get*

$$AB^{-1}f\left(BA^{-1}B^*\right)B = A^{1/2}f\left(A^{-1/2}B^*BA^{-1/2}\right)A^{1/2}, \tag{2.19}$$

provided that A *and* B *are invertible,* A *is positive and*

$$\left\|BA^{-1}B^*\right\|, \left\|A^{-1/2}B^*BA^{-1/2}\right\| < R.$$

We also observe that, if $A, B \in \mathcal{B}(H)$ *with* A *positive and* $\|A\|\|B\|^2 < R$, *then* $\|BAB^*\|, \left\|A^\alpha B^*BA^{1-\alpha}\right\| < R$ *and the equality (2.15) holds true. In particular, if* $R = 1$ *and* $\|A\|, \|B\| < 1$, *then the conclusion of Theorem 2 remains valid. This fact provides many examples since numerous fundamental functions defined as power series are convergent on the open disk* $D(0,1)$. *Some instances of interest are provided below.*

If we consider the exponential function $f(z) = \sum_{n=0}^{\infty}\frac{1}{n!}z^n = \exp(z)$, $z \in \mathbb{C}$, then from (2.15)-(2.18) we get

$$\begin{aligned} \exp(BAB^*) &= BA^{1-\alpha}\exp\left(A^\alpha B^*BA^{1-\alpha}\right)A^{\alpha-1}B^{-1} \\ &= BA^{1/2}\exp\left(A^{1/2}B^*BA^{1/2}\right)A^{-1/2}B^{-1} \\ &= BAf\exp(B^*BA)A^{-1}B^{-1} = B\exp(AB^*B)B^{-1} \end{aligned} \tag{2.20}$$

for any invertible $A, B \in \mathcal{B}(H)$ with A positive and $\alpha \in [0,1]$.

If we consider $f(z) = \sum_{n=0}^{\infty} z^n = (1-z)^{-1}$, $z \in D(0,1)$, then for any invertible $A, B \in \mathcal{B}(H)$ with A positive, $\|A\|, \|B\| < 1$ and $\alpha \in [0,1]$, we have

$$\begin{aligned} (1_H - BAB^*)^{-1} &= BA^{1-\alpha}\left(1_H - A^\alpha B^*BA^{1-\alpha}\right)^{-1}A^{\alpha-1}B^{-1} \\ &= BA^{1/2}\left(1_H - A^{1/2}B^*BA^{1/2}\right)^{-1}A^{-1/2}B^{-1} \\ &= BA(1_H - B^*BA)^{-1}A^{-1}B^{-1} \\ &= B(1_H - AB^*B)^{-1}B^{-1} \end{aligned} \tag{2.21}$$

and by using the function $f(z) = \sum_{n=1}^{\infty}\frac{1}{n}z^n = \ln(1-z)^{-1}$, $z \in D(0,1)$, then for any invertible $A, B \in \mathcal{B}(H)$ with A positive, $\|A\|, \|B\| < 1$ and $\alpha \in [0,1]$, we have

$$\begin{aligned} \ln(1_H - BAB^*)^{-1} &= BA^{1-\alpha}\ln\left(1_H - A^\alpha B^*BA^{1-\alpha}\right)^{-1}A^{\alpha-1}B^{-1} \\ &= BA^{1/2}\ln\left(1_H - A^{1/2}B^*BA^{1/2}\right)^{-1}A^{-1/2}B^{-1} \\ &= BA\ln(1_H - B^*BA)^{-1}A^{-1}B^{-1} \\ &= B\ln(1_H - AB^*B)^{-1}B^{-1}. \end{aligned} \tag{2.22}$$

3. Applications

We have:

Proposition 1. *Let $\Phi(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$ or on $\mathbb{C}$, for $R=\infty$. Assume that A, $B\in\mathcal{B}(H)$ with A positive and invertible, B selfadjoint and $\left\|A^{-1}B\right\|$, $\left\|A^{-1/2}BA^{-1/2}\right\|<R$, then*

$$\mathcal{P}_{\Phi}(B,A)=\mathcal{Q}_{\Phi}(B,A). \tag{3.1}$$

The proof follows by Corollary 2 for $f=\Phi$.

Proposition 2. *Let $\Phi(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$ or on $\mathbb{C}$, for $R=\infty$. Assume that A, $B\in\mathcal{B}(H)$ are invertible with A positive and $\left\|A^{-1/2}B^*\right\|$, $\left\|BA^{-1/2}\right\|<\sqrt{R}$, then*

$$\mathcal{P}_{\Phi}\left(|B|^2,A\right)=AB^{-1}f\left(\left|A^{-1/2}B^*\right|^2\right)B. \tag{3.2}$$

The proof follows by the identity (2.19) for $f=\Phi$.

If we take in (3.2) $B=C^{1/2}$ where C is positive and invertible, then we get

$$\begin{aligned}\mathcal{P}_{\Phi}(C,A)&=AC^{-1/2}f\left(C^{1/2}A^{-1}C^{1/2}\right)C^{1/2}\\&=AC^{-1/2}f\left(C^{1/2}A^{-1}C^{1/2}\right)C^{-1/2}C=A\mathcal{P}_{\Phi}\left(A^{-1},C^{-1}\right)C.\end{aligned}$$

Therefore we can state the following result of interest:

Corollary 4. *Let $\Phi(z)=\sum_{n=0}^{\infty}\alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$ or on $\mathbb{C}$, for $R=\infty$. Assume that B, $A\in\mathcal{B}(H)$ are invertible, positive and $\left\|A^{-1}B\right\|$, $\left\|BA^{-1}\right\|<R$, then*

$$\mathcal{P}_{\Phi}\left(A^{-1},B^{-1}\right)=A^{-1}\mathcal{P}_{\Phi}(B,A)B^{-1}. \tag{3.3}$$

If, for instance we consider the function $f(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n=\exp(z)$, $z\in\mathbb{C}$, then for A positive and invertible and B selfadjoint, we have

$$\mathcal{P}_{\exp}(B,A):=A^{1/2}\exp\left(A^{-1/2}BA^{-1/2}\right)A^{1/2},\ \mathcal{Q}_{\exp}(B,A):=A\exp\left(A^{-1}B\right)$$

and by (3.1) we obtain

$$\mathcal{P}_{\exp}(B,A)=\mathcal{Q}_{\exp}(B,A). \tag{3.4}$$

Moreover, if A and B are positive and invertible, then by (3.3) we get

$$\mathcal{P}_{\exp}\left(A^{-1},B^{-1}\right)=A^{-1}\mathcal{P}_{\exp}(B,A)B^{-1}. \tag{3.5}$$

Similar equalities hold for the trigonometric functions $\sin z$, $\cos z$ or for hyperbolic functions $\sinh z$ and $\cosh z$. The details are omitted.

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