Received 06/01/18

# APPROXIMATION OF INTEGRAL OPERATORS FOR ABSOLUTELY CONTINUOUS FUNCTIONS WHOSE DERIVATIVES ARE ESSENTIALLY BOUNDED

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Abstract. Approximations via Ostrowski type inequalities for the integral transform with the kernel $K(.,.)$ of absolutely continuous functions $g:[a,b]\to\mathbb{R}$ whose derivative $g':[a,b]\to\mathbb{R}$ belongs to $L_\infty[a,b]$ are obtained. Applications for particular integral transforms such as the Finite Mellin transform, the Finite Sine and Cosine transforms are also given .

## 1. Introduction

Let $K:\mathbb{R}^2\to\mathbb{K}$ be a Lebesgue measurable function. Define the *integral operator*

$$A(g)(t):=\int_a^b K(t,s)\,g(s)\,ds,\ \ t\in[a,b] \tag{1.1}$$

for $g$ a measurable function and such that the Lebesgue integral on the finite interval $[a,b]$ exists for almost every real number $t$ from $[a,b]$.

Such examples of integral operators are the finite Fourier transform, the finite Mellin transform and other well known finite transform from Mathematical Physics (see for instance [5]).

The finite Fourier transform, as a specific Integral Operator of type (1.1), has long been a principle analytical tool in such diverse fields as linear systems, optics, random process modelling, probability theory, quantum physics and boundary-value problems [5].

In what follows we briefly mention some approximation results for the finite Fourier transform whose proofs have employed recent techniques and facts from integral inequalities theory of Ostrowski type.

Let $g:[a,b]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ be a Lebesgue integrable mapping defined on the finite interval $[a,b]$ and $\mathcal{F}(g)$ its finite Fourier transform, i.e.,

$$\mathcal{F}(g)(t):=\int_a^b g(s)\,e^{-2\pi its}ds \text{ with } t\in[a,b].$$

The following inequality was obtained in [6].

1991 *Mathematics Subject Classification.* 26D15, 26D10, 41A55.

*Key words and phrases.* Integral Operators, Finite Fourier transform, Finite Mellin transform, Finite Cos and Sine transforms, Ostrowski type inequalities.



RGMIA Res. Rep. Coll. 21 (2018), Art. 2, 11 pp.

**Theorem 1.** *Let $g : [a, b] \to \mathbb{K}$ be an absolutely continuous mapping on $[a, b]$. Then we have the inequality*

$$\left|\mathcal{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(t)\,dt\right| \leq \begin{cases} \frac{1}{3}\|g'\|_\infty (b-a)^2 & \text{if } g' \in L_\infty[a,b]; \\ \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\|g'\|_p & \text{if } g' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ (b-a)\|g'\|_1, \end{cases}$$

*for all $x \in [a, b]$, $(x \neq 0)$ where $E$ is the exponential mean of two complex numbers, that is,*

$$E(z,w) := \begin{cases} \dfrac{e^z - e^w}{z-w} & \text{if } z \neq w, \\ \exp(w) & \text{if } z = w, \end{cases} \quad z, w \in \mathbb{C}.$$

For functions of bounded variation, the following result holds as well (see [7]):

**Theorem 2.** *Let $g : [a, b] \to \mathbb{K}$ be a mapping of bounded variation on $[a, b]$. Then we have the inequality*

$$\left|\mathcal{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(s)\,ds\right| \leq \frac{3}{4}(b-a)\bigvee_a^b(g),$$

*for all $x \in [a, b]$, $x \neq 0$, where $\bigvee_a^b(g)$ is the total variation of $g$ on $[a, b]$.*

Finally, we mention the following result obtained in [8] providing an approximation of the Fourier transform for Lebesgue integrable functions:

**Theorem 3.** *Let $g : [a, b] \to \mathbb{K}$ be a measurable function on $[a, b]$. Then we have the estimates:*

$$\left|\mathcal{F}(g)(x) - E(-2\pi ixa, -2\pi ixb)\int_a^b g(s)\,ds\right| \leq \begin{cases} \frac{2\pi}{3}|x|(b-a)^2\|g\|_\infty & \text{if } g \in L_\infty[a,b]; \\ \frac{2^{1+\frac{1}{q}}(b-a)^{1+\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}\pi|x|\|g\|_p & \text{if } g \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ 2\pi|x|(b-a)\|g\|_1, & \text{if } g \in L_1[a,b] \end{cases}$$

*for all $x \in [a, b]$, $x \neq 0$.*

For other inequalities for integral transform, see [1]-[4], [9] and [10].

Motivated by the above results and utilising the Montgomery identity that plays a crucial role in obtaining different Ostrowski type inequalities we consider in the following the problem of approximating the type of Integral Operators considered in (1.1) and apply the obtained results for some particular instances such as the Mellin Transform and the Sine and Cosine transforms.

## 2. Integral Inequalities

We start with the following integral inequality.

**Theorem 4.** *Let $g:[a,b]\to\mathbb{K}$ be an absolutely continuous mapping on $[a,b]$ with $g'\in L_\infty[a,b]$. Then we have the inequality*

$$\begin{aligned}(2.1)\quad &\left|A(g)(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\int_a^b K(t,s)\,ds\right| \\ &\leq\left[\frac{b-a}{4}\int_a^b|K(t,s)|\,ds+\frac{1}{b-a}\int_a^b\left(s-\frac{a+b}{2}\right)^2|K(t,s)|\,ds\right]\|g'\|_\infty \\ &\leq\frac{b-a}{2}\|g'\|_\infty\int_a^b|K(t,s)|\,ds\end{aligned}$$

*for all $t\in[a,b]$.*

*Proof.* Using the integration by parts formula for absolutely continuous mappings on $[a,b]$, we have

$$\int_a^x(s-a)\,g'(s)\,ds=(x-a)\,g(x)-\int_a^x g(s)\,ds \qquad (2.2)$$

and

$$\int_x^b(s-b)\,g'(s)\,ds=(b-x)\,g(x)-\int_x^b g(s)\,ds \qquad (2.3)$$

for all $x\in[a,b]$.
Adding (2.2) and (2.3) we obtain Montgomery's identity:

$$g(x)=\frac{1}{b-a}\int_a^b g(s)\,ds+\frac{1}{b-a}\int_a^b p(x,s)\,g'(s)\,ds,\ x\in[a,b], \qquad (2.4)$$

where the kernel $p:[a,b]^2\to\mathbb{R}$ is defined by

$$p(z,v):=\begin{cases} v-a \text{ if } v\in[a,z], \\ v-b \text{ if } v\in(z,b]\end{cases}.$$

Now, using the definition of the integral operator $A$, we obtain

$$\begin{aligned}(2.5)\quad &A(g)(t) \\ &=\int_a^b\left[\frac{1}{b-a}\int_a^b g(z)\,dz+\frac{1}{b-a}\int_a^b p(s,z)\,g'(z)\,dz\right]K(t,s)\,ds \\ &=\frac{1}{b-a}\int_a^b g(z)\,dz\int_a^b K(t,s)\,ds+\frac{1}{b-a}\int_a^b\int_a^b p(s,z)\,K(t,s)\,g'(z)\,dzds.\end{aligned}$$

Now, if we use the representation (2.5) and the properties of modulus, we obtain

$$\begin{aligned}(2.6)\quad &\left|A(g)(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\int_a^b K(t,s)\,ds\right| \\ &\leq\frac{1}{b-a}\int_a^b\int_a^b|p(s,z)|\,|K(t,s)|\,|g'(z)|\,dzds.\end{aligned}$$

If we assume that $g' \in L_\infty [a,b]$, then we have

$$\begin{aligned}
&\int_a^b \int_a^b |p(s,z)| |K(t,s)| |g'(z)| dzds \\
&\le \|g'\|_\infty \int_a^b \int_a^b |p(s,z)| |K(t,s)| dzds \\
&= \|g'\|_\infty \int_a^b \left[ \int_a^s |p(s,z)| dz + \int_s^b |p(s,z)| dz \right] |K(t,s)| ds \\
&= \|g'\|_\infty \int_a^b \left[ \int_a^s (z-a) dz + \int_s^b (b-z) dz \right] |K(t,s)| ds \\
&= \|g'\|_\infty \int_a^b \left[ \frac{(s-a)^2 + (b-s)^2}{2} \right] |K(t,s)| ds \\
&= \|g'\|_\infty \int_a^b \left[ \frac{(b-a)^2}{4} + \left( s - \frac{a+b}{2} \right)^2 \right] |K(t,s)| ds \\
&= \left[ \frac{(b-a)^2}{4} \int_a^b |K(t,s)| ds + \int_a^b \left( s - \frac{a+b}{2} \right)^2 |K(t,s)| ds \right] \|g'\|_\infty
\end{aligned}$$

and the first inequality in (2.1) is proved.
For the second part, we observe that $\left(s - \frac{a+b}{2}\right)^2 \le \frac{(b-a)^2}{4}$, and then

$$\int_a^b \left( s - \frac{a+b}{2} \right)^2 |K(t,s)| ds \le \frac{(b-a)^2}{4} \int_a^b |K(t,s)| ds,$$

which proves the desired inequality. □

**Remark 1.** *We observe that the integral*

$$\int_a^b \left( s - \frac{a+b}{2} \right)^2 |K(t,s)| ds$$

*can also be upper-bounded in the following manner by applying Hölder's integral inequality:*

$$\begin{aligned}
&\int_a^b \left( s - \frac{a+b}{2} \right)^2 |K(t,s)| ds \qquad (2.7)\\
&\le \left[ \int_a^b \left| s - \frac{a+b}{2} \right|^{2q} ds \right]^{\frac{1}{q}} \left[ \int_a^b |K(t,s)|^p ds \right]^{\frac{1}{p}} \\
&= \left[ \frac{2 \frac{(b-a)^{2q+1}}{2^{2q+1}}}{2q+1} \right]^{\frac{1}{q}} \left[ \int_a^b |K(t,s)|^p ds \right]^{\frac{1}{p}} = \frac{(b-a)^{2+\frac{1}{q}}}{4(2q+1)^{\frac{1}{q}}} \left[ \int_a^b |K(t,s)|^p ds \right]^{\frac{1}{p}},
\end{aligned}$$

*where* $\frac{1}{p}+\frac{1}{q}=1, p>1$.
*Using* (2.1) *and* (2.7) *we can state that:*

$$\begin{aligned}(2.8)\qquad & \left|A(g)(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\int_a^b K(t,s)\,ds\right| \\ & \leq \left[\frac{b-a}{4}\int_a^b |K(t,s)|\,ds+\frac{1}{b-a}\int_a^b \left(s-\frac{a+b}{2}\right)^2 |K(t,s)|\,ds\right]\|g'\|_\infty \\ & \leq \frac{b-a}{4}\left[\int_a^b |K(t,s)|\,ds+\frac{(b-a)^{\frac{1}{q}}}{(2q+1)^{\frac{1}{q}}}\left[\int_a^b |K(t,s)|^p\,ds\right]^{\frac{1}{p}}\right]\|g'\|_\infty .\end{aligned}$$

*In addition, we have*

$$\begin{aligned}(2.9)\qquad \int_a^b \left(s-\frac{a+b}{2}\right)^2 |K(t,s)|\,ds & \leq \sup_{s\in[a,b]}|K(t,s)|\int_a^b \left(s-\frac{a+b}{2}\right)^2 ds \\ & = \frac{(b-a)^3}{12}\sup_{s\in[a,b]}|K(t,s)| .\end{aligned}$$

*Then, by* (2.1) *and* (2.8), *we have*

$$\begin{aligned}(2.10)\qquad & \left|A(g)(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\int_a^b K(t,s)\,ds\right| \\ & \leq \left[\frac{(b-a)}{4}\int_a^b |K(t,s)|\,ds+\frac{1}{b-a}\int_a^b \left(s-\frac{a+b}{2}\right)^2 |K(t,s)|\,ds\right]\|g'\|_\infty \\ & \leq \left[\frac{(b-a)}{4}\int_a^b |K(t,s)|\,ds+\frac{(b-a)^2}{12}\sup_{s\in[a,b]}|K(t,s)|\right]\|g'\|_\infty \\ & \leq \frac{(b-a)^2}{3}\sup_{s\in[a,b]}|K(t,s)|\,\|g'\|_\infty .\end{aligned}$$

**Remark 2.** *If we use the notations* $\|K(t,\cdot)\|_1 := \int_a^b |K(t,s)|\,ds$ *and* $\|K(t,\cdot)\|_\infty := \sup_{s\in[a,b]}|K(t,s)|$, *then we conclude with the inequality:*

$$(2.11)\qquad \left|A(g)(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\cdot\int_a^b K(t,s)\,ds\right| \leq \begin{cases} \frac{b-a}{2}\|K(t,\cdot)\|_1\|g'\|_\infty , \\ \\ \frac{(b-a)^2}{3}\|K(t,\cdot)\|_\infty\|g'\|_\infty \end{cases}$$

*for all* $t\in[a,b]$.

**Remark 3.** *If we use the second inequality in (2.11) for the kernel* $K(t,s)=e^{-2\pi its}$ *we recapture the first inequality in Theorem 1.*

## 3. A Quadrature Formula

Let us consider the division of the interval $[a,b]$ given by $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and put $h_i = x_{i+1} - x_i$ $(i = 0, ..., n-1)$ and $\nu(h) := \max_{i=\overline{0,n-1}} h_i$.

Define the sum

$$A(g, I_n, t) := \sum_{i=0}^{n-1} \frac{1}{h_i} \int_{x_i}^{x_{i+1}} g(s)\,ds \times \int_{x_i}^{x_{i+1}} K(t,s)\,ds, \tag{3.1}$$

where $t \in [a,b]$.

The following is an approximation result for the integral operator $A$.

**Theorem 5.** *Let $g : [a,b] \to \mathbb{K}$ be an absolutely continuous mapping on $[a,b]$ with $g' \in L_\infty[a,b]$. Then we have*

$$A(g)(t) = A(g, I_n, t) + R(g, I_n, t), \ t \in [a,b], \tag{3.2}$$

*where $A(g, I_n, \cdot)$ is the approximation formula defined in (3.1) and the remainder $R(g, I_n, \cdot)$ satisfies the bound*

$$|R(g, I_n, t)| \le \begin{cases} \frac{1}{2}\nu(h)\|g'\|_\infty \|K(t,\cdot)\|_1, \\ \\ \frac{1}{3}\|g'\|_\infty \|K(t,\cdot)\|_\infty \sum_{i=0}^{n-1} h_i^2 \end{cases} \tag{3.3}$$

$$\le \nu(h) \times \begin{cases} \frac{1}{2}\|g'\|_\infty \|K(t,\cdot)\|_1, \\ \\ \frac{1}{3}(b-a)\|g'\|_\infty \|K(t,\cdot)\|_\infty \end{cases}$$

*for all $t \in [a,b]$.*

*Proof.* Apply (2.1) on the intervals $[x_i, x_{i+1}]$ and for $t \in [a,b]$ to obtain

$$\begin{aligned} &\left| \int_{x_i}^{x_{i+1}} g(s) K(t,s)\,ds - \frac{1}{h_i} \int_{x_i}^{x_{i+1}} g(s)\,ds \int_{x_i}^{x_{i+1}} K(t,s)\,ds \right| \\ &\le \frac{h_i}{2} \|g'\|_\infty \int_{x_i}^{x_{i+1}} |K(t,s)|\,ds \end{aligned}$$

for all $i \in \{0, ..., n-1\}$.

Summing over $i$ from 0 to $n-1$, we deduce

$$\begin{aligned} &|R(g, I_n, t)| \\ &\le \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} g(s) K(t,s)\,ds - \frac{1}{h_i} \int_{x_i}^{x_{i+1}} g(s)\,ds \times \int_{x_i}^{x_{i+1}} K(t,s)\,ds \right| \\ &\le \frac{1}{2}\|g'\|_\infty \sum_{i=0}^{n-1} h_i \int_{x_i}^{x_{i+1}} |K(t,s)|\,ds \le \frac{1}{2}\nu(h)\|g'\|_\infty \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |K(t,s)|\,ds \\ &= \frac{1}{2}\nu(h)\|g'\|_\infty \|K(t,\cdot)\|_1, \end{aligned}$$

and the first part of (3.3) is proved.

Now, if we write the inequality (2.10) on the intervals $[x_i, x_{i+1}]$, we obtain

$$\left| \int_{x_i}^{x_{i+1}} g(s) K(t,s)\,ds - \frac{1}{h_i} \int_{x_i}^{x_{i+1}} g(s)\,ds \cdot \int_{x_i}^{x_{i+1}} K(t,s)\,ds \right| \le \frac{h_i^2}{3} \|g'\|_\infty \|K(t,\cdot)\|_\infty$$

for all $i \in \{0,...,n-1\}$.
Summing over $i$ from 0 to $n-1$ and using the generalized triangle inequality we obtain the second part of (3.3). □

In practical applications it is useful to assume that the partition $I_n$ is an equidistant division of $[a,b]$. That is,

$$I_n : x_i := a + i \cdot \frac{b-a}{n}, i = 0, ..., n.$$

In this particular case, we can consider the sum

$$A_n(g,t) := \frac{n}{b-a} \sum_{i=0}^{n-1} \int_{a+i\frac{b-a}{n}}^{a+(i+1)\frac{b-a}{n}} g(s)\,ds \times \int_{a+i\frac{b-a}{n}}^{a+(i+1)\cdot\frac{b-a}{n}} K(t,s)\,ds. \tag{3.4}$$

The following corollary holds.

**Corollary 1.** *Let $g : [a,b] \to \mathbb{K}$ be as in Theorem 5. Then we have the approximation formula*

$$A(g)(t) = A_n(g,t) + R_n(g,t), \tag{3.5}$$

*where $A_n(g,t)$ is as given in (3.4) and the remainder satisfies the estimate*

$$|R(g,I_n,t)| \le \begin{cases} \frac{b-a}{2n} \|g'\|_\infty \|K(t,\cdot)\|_1 \\ \\ \frac{b-a}{3n} \|g'\|_\infty \|K(t,\cdot)\|_\infty \end{cases}, \tag{3.6}$$

*for all $t \in [a,b]$.*

## 4. Applications for the Mellin Transform

Consider the *Mellin transform* for a mapping $f$ defined on a finite interval $[a,b] \subset \mathbb{R}_+ = [0,\infty)$, i.e.,

$$M(g)(t) := \int_a^b g(s)\,s^{t-1}ds, t \in [a,b].$$

The following approximation result holds.

**Proposition 1.** *Let $g : [a,b] \to \mathbb{K}$ be an absolutely continuous mapping on $[a,b]$ with $g' \in L_\infty[a,b]$. Then*

$$\begin{aligned} &\left| M(g)(t) - L_{t-1}^{t-1}(a,b) \int_a^b g(s)\,ds \right| \\ &\le \left\{ \left[ (b-a)^2 + A^2(a,b) \right] L_{t-1}^{t-1} - 2A(a,b)\, L_t^t(a,b) + L_{t+1}^{t+1}(a,b) \right\} \|g'\|_\infty \\ &\le \frac{(b-a)^2}{2} L_{t-1}^{t-1}(a,b) \end{aligned} \tag{4.1}$$

*for all $t \in [a,b] \setminus \{0,1\}$, where*

$$L_p(a,b) := \left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}}, (b \neq a), p \in \mathbb{R} \setminus \{-1,0\}$$

*is the $p$-logarithmic mean of the positive numbers $a$, $b$ and $A(a,b) = \frac{a+b}{2}$ is the arithmetic mean.*

*Proof.* We use Theorem 4 for the kernel $K(t,s) = s^{t-1}$.
Observing that

$$\frac{1}{b-a}\int_a^b K(t,s)\,ds = \frac{1}{b-a}\int_a^b s^{t-1}ds = L_{t-1}^{t-1}(a,b),$$
$$\int_a^b |K(t,s)|\,ds = (b-a)\,L_{t-1}^{t-1}(a,b),$$

and

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \left(s-\frac{a+b}{2}\right)^2 |K(t,s)|\,ds \\
&= \frac{1}{b-a}\int_a^b \left[s^2 - 2A(a,b)\,s + A^2(a,b)\right] s^{t-1}ds \\
&= \frac{1}{b-a}\int_a^b s^{t+1}ds - 2A(a,b)\frac{1}{b-a}\int_a^b s^t ds + A^2(a,b)\frac{1}{b-a}\int_a^b s^{t-1}ds \\
&= L_{t+1}^{t+1}(a,b) - 2A(a,b)\,L_t^t(a,b) + A^2(a,b)\,L_{t-1}^{t-1}(a,b),
\end{aligned}$$

then from (2.1) we deduce the desired inequality (4.1). □

**Remark 4.** *Now, let us observe that*

$$\|K(t,\cdot)\|_\infty = \sup_{s\in[a,b]}|K(t,s)| = \sup_{s\in[a,b]}\left|s^{t-1}\right| = \begin{cases} a^{t-1} & \text{if } t\in(0,1), \\ b^{t-1} & \text{if } t\in(1,\infty). \end{cases}$$

*Consequently, by using Remark 2, we can also state the inequality*

$$\begin{aligned}
&\left|M(g)(t) - L_{t-1}^{t-1}(a,b)\int_a^b g(s)\,ds\right| \\
&\leq \frac{(b-a)^2}{3}\|g'\|_\infty \times \begin{cases} a^{t-1} & \text{if } t\in(0,1), \\ b^{t-1} & \text{if } t\in(1,\infty) \end{cases}
\end{aligned} \tag{4.2}$$

*for all* $t\in[a,b]\setminus\{0,1\}$.

For a given division $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of the interval $[a,b]$, consider the sum

$$M(g,I_n,t) := \sum_{i=0}^{n-1} L_{t-1}^{t-1}(x_i,x_{i+1}) \times \int_{x_i}^{x_{i+1}} g(s)\,ds. \tag{4.3}$$

Using Theorem 5, we can state the following approximation result for the Mellin transform.

**Proposition 2.** *Let* $g:[a,b]\to\mathbb{K}$ *be an absolutely continuous mapping on* $[a,b]$ *and* $g'\in L_\infty[a,b]$. *Then*

$$M(g)(t) = M(g,I_n,t) + R_M(g,I_n,t), \tag{4.4}$$

*where $M(g, I_n, \cdot)$, given by (4.3), approximates the Mellin transform and $R_M(g, I_n, \cdot)$ is the remainder of this transform that satisfies the bound*

$$|R_M(g, I_n, t)| \leq \begin{cases} \frac{1}{2}\nu(h)\|g'\|_\infty (b-a) L_{t-1}^{t-1}(a,b) \\ \\ \frac{1}{3}\|g'\|_\infty \sum_{i=0}^{n-1} h_i^2 \times \begin{cases} a^{t-1} & \text{if } t \in (0,1), \\ \\ b^{t-1} & \text{if } t \in (1,\infty) \end{cases} \end{cases} \tag{4.5}$$

*for all $t \in [a,b] \setminus \{0,1\}$.*

The proof is obvious from Theorem 5 applied for the kernel $K(t,s) = s^{t-1}$, $(s,t) \in [a,b]^2$.

## 5. Applications for Sine and Cosine Transforms

Let us consider the *Sine* and *Cosine Transforms* of an absolutely continuous mapping $g$ on $[a,b] \subset (0,\infty)$

$$S(g)(t) := \int_a^b g(s)\sin(2\pi ts)\,ds,$$

$$C(g)(t) := \int_a^b g(s)\cos(2\pi ts)\,ds,$$

where $t \in [a,b]$.

Also, consider the *special trigonometric means*

$$SIN(a,b) := \begin{cases} \cos a & \text{if} \quad b = a; \\ \\ \frac{\sin b - \sin a}{b-a} & \text{if} \quad b \neq a; \end{cases}$$

$$COS(a,b) := \begin{cases} -\sin a & \text{if} \quad b = a; \\ \\ \frac{\cos b - \cos a}{b-a} & \text{if} \quad b \neq a. \end{cases}$$

The following proposition holds:

**Proposition 3.** *Let $g : [a,b] \to \mathbb{K}$ be an absolutely continuous mapping on $[a,b]$. Then*

$$\begin{aligned} &\left| S(g)(t) + (2\pi t)^2 COS(2\pi tb, 2\pi ta) \int_a^b g(s)\,ds \right| \\ &\leq \pi t \cdot \frac{b-a}{2} \int_{2\pi ta}^{2\pi tb} |\sin v|\,dv + \frac{1}{b-a} \int_a^b \left(s - \frac{a+b}{2}\right)^2 |\sin(2\pi ts)|\,ds \\ &\leq \pi t (b-a)\|g'\|_\infty \int_{2\pi ta}^{2\pi tb} |\sin v|\,dv \leq 2(\pi t)^2 (b-a)^2 \|g'\|_\infty \end{aligned} \tag{5.1}$$

*and*

$$\left|C(g)(t)+(2\pi t)^2 SIN(2\pi tb,2\pi ta)\int_a^b g(s)\,ds\right| \tag{5.2}$$
$$\leq \pi t\cdot\frac{b-a}{2}\int_{2\pi ta}^{2\pi tb}|\cos v|\,dv+\frac{1}{b-a}\int_a^b\left(s-\frac{a+b}{2}\right)^2|\cos(2\pi ts)|\,ds$$
$$\leq \pi t(b-a)\|g'\|_\infty\int_{2\pi ta}^{2\pi tb}|\cos v|\,dv\leq 2(\pi t)^2(b-a)^2\|g'\|_\infty$$

*for* $t\in[a,b]$.

*Proof.* Choose in Theorem 4, $K(t,s):=\sin(2\pi ts)$. Then

$$\int_a^b \sin(2\pi ts)\,ds = 2\pi t\int_{2\pi ta}^{2\pi tb}\sin v dv = 2\pi t\left(-\cos v|_{2\pi ta}^{2\pi tb}\right)$$
$$=(-2\pi t)\frac{\cos 2\pi tb-\cos 2\pi ta}{2\pi t(b-a)}\cdot 2\pi t(b-a)$$
$$=-(2\pi t)^2(b-a)\,COS(2\pi tb,2\pi ta),$$

$$\int_a^b|\sin(2\pi ts)|\,ds=2\pi t\int_{2\pi ta}^{2\pi tb}|\sin v|\,dv\leq 2\pi tb(2\pi tb-2\pi ta)$$
$$=(2\pi t)^2(b-a).$$

Now, by the inequality (2.1), we deduce (5.1).
A similar proof applies to the second inequality, and we omit the details. □

For a given partition $I_n: a=x_0<x_1<...<x_{n-1}<x_n=b$ of the interval $[a,b]$, consider the sums

$$C(g,I_n,t):=(2\pi t)^2\sum_{i=0}^{n-1}COS(2\pi tx_{i+1},2\pi tx_i)\times\int_{x_i}^{x_{i+1}}g(s)\,ds$$

and

$$S(g,I_n,t):=(2\pi t)^2\sum_{i=0}^{n-1}SIN(2\pi tx_{i+1},2\pi tx_i)\times\int_{x_i}^{x_{i+1}}g(s)\,ds,$$

where $t\in[a,b]$.

Using Theorem 5, we can state the following approximation result for the Sine and Cosine transforms.

**Proposition 4.** *Let $g:[a,b]\to\mathbb{K}$ be an absolutely continuous mapping on $[a,b]$ and $g'\in L_\infty[a,b]$. Then*

$$S(g)(t)=-C(g,I_n,t)+R_1(g,I_n,t)\ \text{ and }\ C(g)(t)=S(g,I_n,t)+R_2(g,I_n,t),$$

*where*

$$|R_1(g,I_n,t)|\leq\pi t\|g'\|_\infty\,\nu(h)\int_{2\pi ta}^{2\pi tb}|\sin v|\,dv \tag{5.3}$$

*and*

$$|R_2(g,I_n,t)|\leq\pi t\|g'\|_\infty\,\nu(h)\int_{2\pi ta}^{2\pi tb}|\cos v|\,dv, \tag{5.4}$$

*where* $h_i:=x_{i+1}-x_i$, $(i=0,...,n-1)$, $\nu(h):=\max_{i=\overline{0,n-1}}h_i$ *and* $t\in[a,b]$.

*Proof.* Apply inequality (5.1) on the intervals $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ to obtain

$$\left| \int_{x_i}^{x_{i+1}} g(s) \sin(2\pi ts)\, ds + (2\pi t)^2\, COS(2\pi t x_{i+1}, 2\pi t x_i) \int_{x_i}^{x_{i+1}} g(s)\, ds \right|$$
$$\leq \pi t (x_{i+1} - x_i) \|g'\|_\infty \sum_{i=0}^{n-1} \int_{2\pi t x_i}^{2\pi t x_{i+1}} |\sin v|\, dv$$

for all $t \in [a, b]$. Summing over $i$ from 0 to $n-1$ and using the generalized triangle inequality, we obtain

$$|R_1(g, I_n, t)|$$
$$\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} g(s) \sin(2\pi ts)\, ds + (2\pi t)^2\, COS(2\pi t x_{i+1}, 2\pi t x_i) \int_{x_i}^{x_{i+1}} g(s)\, ds \right|$$
$$\leq \pi t \|g'\|_\infty \sum_{i=0}^{n-1} h_i \int_{2\pi t x_i}^{2\pi t x_{i+1}} |\sin v|\, dv \leq \pi t \|g'\|_\infty \nu(h) \int_{2\pi t b}^{2\pi t b} |\sin v|\, dv.$$

The proof of second part is similar. □

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