Received 15/01/18

SOME WEIGHTED INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL WHEN A FUNCTION IS BOUNDED

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we provide some simple ways to approximate the Riemann-Stieltjes integral of a product of two functions $\int_a^b f(t)\,g(t)\,dv(t)$ by the use of simpler quantities and under several assumptions for the functions involved, one of them satisfying the boundedness condition

$$\left|f(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}\left|\Gamma-\gamma\right| \text{ for each } t\in[a,b],$$

where $f:[a,b]\to\mathbb{C}$. Applications for continuous functions of selfadjoint operators and functions of unitary operators on Hilbert spaces are also given.

1. Introduction

One can approximate the *Stieltjes integral* $\int_a^b f(t)\,du(t)$ with the following simpler quantities:

$$\frac{1}{b-a}\left[u(b)-u(a)\right]\cdot\int_a^b f(t)\,dt \qquad ([25],\,[26]) \tag{1.1}$$

$$f(x)\left[u(b)-u(a)\right] \qquad ([15],\,[16]) \tag{1.2}$$

or with

$$\left[u(b)-u(x)\right]f(b)+\left[u(x)-u(a)\right]f(a) \qquad ([24]), \tag{1.3}$$

where $x\in[a,b]$.

In order to provide *a priory* sharp bounds for the *approximation error*, consider the functionals:

$$D(f,u;a,b):=\int_a^b f(t)\,du(t)-\frac{1}{b-a}\left[u(b)-u(a)\right]\cdot\int_a^b f(t)\,dt,$$

$$\Theta(f,u;a,b,x):=\int_a^b f(t)\,du(t)-f(x)\left[u(b)-u(a)\right]$$

and

$$T(f,u;a,b,x):=\int_a^b f(t)\,du(t)-\left[u(b)-u(x)\right]f(b)-\left[u(x)-u(a)\right]f(a).$$

If the *integrand* f is *Riemann integrable* on $[a,b]$ and the *integrator* $u:[a,b]\to\mathbb{R}$ is $L-$*Lipschitzian*, i.e.,

$$\left|u(t)-u(s)\right|\leq L\left|t-s\right| \qquad \text{for each } \ t,s\in[a,b], \tag{1.4}$$

1991 *Mathematics Subject Classification.* 26D15, 26D10, 26D07, 26A33.

Key words and phrases. Riemann-Stieltjes integral, Continuous functions, Functions of bounded variation, Hilbert spaces, Selfadjoint operators, Unitary operators.

RGMIA Res. Rep. Coll. 21 (2018), Art. 4, 23 pp.

then the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and, as pointed out in [25],

$$|D(f,u;a,b)| \le L\int_a^b \left| f(t) - \int_a^b \frac{1}{b-a} f(s)\,ds \right| dt. \tag{1.5}$$

The inequality (1.5) is sharp in the sense that the multiplicative constant $C=1$ in front of L cannot be replaced by a smaller quantity. Moreover, if there exists the constants $m,\ M \in \mathbb{R}$ such that $m \le f(t) \le M$ for a.e. $t \in [a,b]$, then [25]

$$|D(f,u;a,b)| \le \frac{1}{2} L (M-m)(b-a). \tag{1.6}$$

The constant $\frac{1}{2}$ is best possible in (1.6).

A different approach in the case of integrands of bounded variation were considered by the same authors in 2001, [26], where they showed that

$$|D(f,u;a,b)| \le \max_{t\in[a,b]} \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \bigvee_a^b (u), \tag{1.7}$$

provided that f is continuous and u is of bounded variation. Here $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. The inequality (1.7) is sharp.

If we assume that f is K-Lipschitzian, then [26]

$$|D(f,u;a,b)| \le \frac{1}{2} K (b-a) \bigvee_a^b (u), \tag{1.8}$$

with $\frac{1}{2}$ the best possible constant in (1.8).

For various bounds on the error functional $D(f,u;a,b)$ where f and u belong to different classes of function for which the Stieltjes integral exists, see [21], [20], [19], and [8] and the references therein.

For the functional $\theta(f,u;a,b,x)$ we have the bound [15]:

$$\begin{aligned} &|\theta(f,u;a,b,x)| \\ &\le H\left[(x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f)\right] \\ &\le H \times \begin{cases} \left[(x-a)^r + (b-x)^r\right]\left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right]; \\ \left[(x-a)^{qr} + (b-x)^{qr}\right]^{\frac{1}{q}} \left[\left(\bigvee_a^x (f)\right)^p + \left(\bigvee_x^b (f)\right)^p\right]^{\frac{1}{p}} \\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (f), \end{cases} \end{aligned} \tag{1.9}$$

provided f is of bounded variation and u is of *r-H-Hölder type*, i.e.,

$$|u(t) - u(s)| \le H|t-s|^r \qquad \text{for each } \ t,s\in[a,b], \tag{1.10}$$

with given $H>0$ and $r\in(0,1]$.

If f is of q-K-Hölder type and u is of bounded variation, then [16]

$$|\theta(f,u;a,b,x)| \leq K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^q \bigvee_a^b (u), \tag{1.11}$$

for any $x \in [a,b]$.

If u is monotonic nondecreasing and f of q-K-Hölder type, then the following refinement of (1.11) also holds [8]:

$$\begin{aligned}|\theta(f,u;a,b,x)| &\leq K\Bigg[(b-x)^q u(b)-(x-a)^q u(a) \\ &\qquad + q\left\{\int_a^x \frac{u(t)\,dt}{(x-t)^{1-q}}-\int_x^b \frac{u(t)\,dt}{(t-x)^{1-q}}\right\}\Bigg] \\ &\leq K\left[(b-x)^q[u(b)-u(x)]+(x-a)^q[u(x)-u(a)]\right] \\ &\leq K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^q [u(b)-u(a)],\end{aligned} \tag{1.12}$$

for any $x \in [a,b]$.

If f is monotonic nondecreasing and u is of r-H-Hölder type, then [8]:

$$\begin{aligned}&|\theta(f,u;a,b,x)| \\ &\leq H\Bigg[[(x-a)^r-(b-x)^r]f(x) \\ &\qquad + r\left\{\int_a^x \frac{f(t)\,dt}{(b-t)^{1-r}}-\int_x^b \frac{f(t)\,dt}{(t-r)^{1-r}}\right\}\Bigg] \\ &\leq H\left\{(b-x)^r[f(b)-f(x)]+(x-a)^r[f(x)-f(a)]\right\} \\ &\leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r [f(b)-f(a)],\end{aligned} \tag{1.13}$$

for any $x \in [a,b]$.

The error functional $T(f,u;a,b,x)$ satisfies similar bounds, see [24], [8], [3] and [2] and the details are omitted.

Motivated by the above results, in this paper we provide some simple ways to approximate the Riemann-Stieltjes integral of a product of two functions $\int_a^b f(t)\,g(t)\,dv(t)$ by the use of simpler quantities and under several assumptions for the functions involved. Applications for continuous functions of selfadjoint operators and continuous functions of unitary operators on Hilbert spaces are also given.

2. General Results

We have the simple equality of interest for what follows:

Lemma 1. *Let* $f, g, v : [a,b] \to \mathbb{C}$, $\lambda, \mu \in \mathbb{C}$ *and* $x \in [a,b]$. *If* $fg, g \in \mathcal{R}_{\mathbb{C}}(v,[a,x]) \cap \mathcal{R}_{\mathbb{C}}(v,[x,b])$, *then* $fg, g \in \mathcal{R}_{\mathbb{C}}(v,[a,b])$ *and*

$$\begin{aligned}
\int_a^b f(t)\,g(t)\,dv(t) &= \lambda \int_a^x g(t)\,dv(t) + \mu \int_x^b g(t)\,dv(t) \\
&\quad + \int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\mu]\,g(t)\,dv(t) \\
&= \mu \int_a^b g(t)\,dv(t) + (\lambda-\mu)\int_a^x g(t)\,dv(t) \\
&\quad + \int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\mu]\,g(t)\,dv(t).
\end{aligned} \tag{2.1}$$

In particular, for $\mu = \lambda$, *we have*

$$\begin{aligned}
\int_a^b f(t)\,g(t)\,dv(t) &= \lambda \int_a^b g(t)\,dv(t) \\
&\quad + \int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\lambda]\,g(t)\,dv(t) \\
&= \lambda \int_a^b g(t)\,dv(t) + \int_a^b [f(t)-\lambda]\,g(t)\,dv(t).
\end{aligned} \tag{2.2}$$

Proof. The integrability follows by Theorem 7. 4 from [1] which says that if a function is Riemann-Stieltjes integrable on the intervals $[a,x]$, $[x,b]$ with $x \in [a,b]$, then it is integrable on the whole interval $[a,b]$.

Using the properties of the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^x [f(t)-\lambda]\,g(t)\,dv(t) + \int_x^b [f(t)-\mu]\,g(t)\,dv(t) \\
&= \int_a^x f(t)\,g(t)\,dv(t) - \lambda \int_a^x g(t)\,dv(t) + \int_x^b f(t)\,g(t)\,dv(t) - \mu \int_x^b g(t)\,dv(t) \\
&= \int_a^b f(t)\,g(t)\,dv(t) - \lambda \int_a^x g(t)\,dv(t) - \mu \int_x^b g(t)\,dv(t),
\end{aligned}$$

which is equivalent to the first equality in (2.1).

The rest is obvious. □

Corollary 1. *Assume that* $f, v : [a,b] \to \mathbb{C}$ *and* $x \in [a,b]$ *are such that* $f \in \mathcal{R}_{\mathbb{C}}(v,[a,x]) \cap \mathcal{R}_{\mathbb{C}}(v,[x,b])$. *Then for any* $\lambda, \mu \in \mathbb{C}$ *we have the equality*

$$\begin{aligned}
\int_a^b f(t)\,dv(t) &= \lambda[v(x)-v(a)] + \mu[v(b)-v(x)] \\
&\quad + \int_a^x [f(t)-\lambda]\,dv(t) + \int_x^b [f(t)-\mu]\,dv(t).
\end{aligned} \tag{2.3}$$

In particular, for $\mu = \lambda$, we have

$$\begin{aligned} \int_a^b f(t)\,dv(t) &= \lambda\left[v(b) - v(a)\right] \\ &\quad + \int_a^x \left[f(t) - \lambda\right] dv(t) + \int_x^b \left[f(t) - \lambda\right] dv(t) \\ &= \lambda\left[v(b) - v(a)\right] + \int_a^b \left[f(t) - \lambda\right] dv(t). \end{aligned} \tag{2.4}$$

The proof follows by Lemma 1 for $g(t) = 1$, $t \in [a,b]$.

Remark 1. *We observe that, see* [1, Theorem 7.27]*, if f, $g \in \mathcal{C}_{\mathbb{C}}[a,b]$, namely, are continuous on $[a,b]$ and $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, namely of bounded variation on $[a,b]$, then for any $x \in [a,b]$ the Riemann-Stieltjes integrals in Lemma 1 exist and the equalities (2.1) and (2.2) hold.*

Now, for γ, $\Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} |\operatorname{Re}\left[(\Gamma - f(t))\left(\overline{f(t)} - \overline{\gamma}\right)\right] \geq 0 \text{ for each } t \in [a,b] \right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} |\ \left| f(t) - \frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma| \text{ for each } t \in [a,b] \right\}.$$

This family of functions is a particular case of the class introduced in [22]

$$\begin{aligned} &\bar{\Delta}_{[a,b],g}(\gamma,\Gamma) \\ &:= \left\{ f : [a,b] \to \mathbb{C} |\ \left| f(t) - \frac{\gamma+\Gamma}{2} g(t)\right| \leq \frac{1}{2}|\Gamma - \gamma|\,|g(t)| \text{ for each } t \in [a,b] \right\}, \end{aligned}$$

where $g : [a,b] \to \mathbb{C}$.

The following representation result may be stated.

Proposition 1. *For any γ, $\Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{2.5}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma+\Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right] \geq 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma - \gamma|^2 - \left| z - \frac{\gamma+\Gamma}{2}\right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.5) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 2. *For any* $\gamma,\ \Gamma \in \mathbb{C},\ \gamma \neq \Gamma,$*we have that*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \{f : [a,b] \to \mathbb{C} \mid (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \geq 0 \text{ for each } t \in [a,b]\}. \tag{2.6}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\gamma,\Gamma) := \{f : [a,b] \to \mathbb{C} \mid \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \text{ and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for each } t \in [a,b]\}. \tag{2.7}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{2.8}$$

We consider the following functional

$$P(f,g,v;\gamma,\Gamma,a,b) := \int_a^b f(t)\,g(t)\,dv(t) - \frac{\gamma+\Gamma}{2}\int_a^b g(t)\,dv(t) \tag{2.9}$$

for the complex valued functions $f,\ g,\ v$ defined on $[a,b]$ and such that the involved Riemann-Stieltjes integrals exist, and for $\gamma,\ \Gamma \in \mathbb{C}$.

Theorem 1. *Let* $f,\ g \in \mathcal{C}_{\mathbb{C}}[a,b]$ *and* $\gamma,\ \Gamma \in \mathbb{C},\ \gamma \neq \Gamma$ *such that* $f \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$.

(i) *If* $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, *then*

$$|P(f,g,v;\gamma,\Gamma,a,b)| \leq \frac{1}{2}|\Gamma-\gamma|\int_a^b |g(t)|\,d\left(\bigvee_a^t(v)\right) \leq \frac{1}{2}|\Gamma-\gamma|\max_{t\in[a,b]}|g(t)|\bigvee_a^b(v). \tag{2.10}$$

(ii) *If* $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$, *namely,* v *is Lipschitzian with the constant* $L>0$,

$$|v(t)-v(s)| \leq L|t-s| \text{ for any } t,\ s \in [a,b],$$

then we also have

$$|P(f,g,v;\gamma,\Gamma,a,b)| \leq \frac{1}{2}|\Gamma-\gamma|\,L\int_a^b |g(t)|\,dt \leq \frac{1}{2}|\Gamma-\gamma|(b-a)\max_{t\in[a,b]}|g(t)|. \tag{2.11}$$

(iii) *If* $v \in \mathcal{M}^{\nearrow}[a,b]$, *namely,* v *is monotonic increasing on* $[a,b]$, *then we have*

$$|P(f,g,v;\gamma,\Gamma,a,b)| \leq \frac{1}{2}|\Gamma-\gamma|\int_a^b |g(t)|\,dv(t) \leq \frac{1}{2}|\Gamma-\gamma|[v(b)-v(a)]\max_{t\in[a,b]}|g(t)|. \tag{2.12}$$

Proof. (i) It is well known that if $p \in \mathcal{R}(u,[a,b])$ where $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$ then we have [1, p. 177]

$$\left|\int_a^b p(t)\,du(t)\right| \leq \int_a^b |p(t)|\,d\left(\bigvee_a^t(u)\right) \leq \sup_{t\in[a,b]}|p(t)|\bigvee_a^b(u). \tag{2.13}$$

By the equality (2.2) we have

$$\int_a^b f(t)\,g(t)\,dv(t)-\frac{\gamma+\Gamma}{2}\int_a^b g(t)\,dv(t)=\int_a^b\left[f(t)-\frac{\gamma+\Gamma}{2}\right]g(t)\,dv(t). \tag{2.14}$$

Since $f\in\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ then by (2.13) and (2.14) we have

$$\begin{aligned}\left|\int_a^b f(t)\,g(t)\,dv(t)-\frac{\gamma+\Gamma}{2}\int_a^b g(t)\,dv(t)\right| &\le \int_a^b\left|\left[f(t)-\frac{\gamma+\Gamma}{2}\right]g(t)\right|d\left(\bigvee_a^t(v)\right)\\ &=\int_a^b\left|f(t)-\frac{\gamma+\Gamma}{2}\right||g(t)|\,d\left(\bigvee_a^t(v)\right)\\ &\le\frac{1}{2}|\Gamma-\gamma|\int_a^b|g(u)|\,d\left(\bigvee_a^t(v)\right)\end{aligned}$$

and the first inequality in (2.10) is proved. The second part is obvious.

(ii) It is well known that if $p\in\mathcal{R}(u,[a,b])$, where $u\in\mathcal{L}_{L,\mathbb{C}}[a,b]$, then we have

$$\left|\int_a^b p(t)\,dv(t)\right|\le L\int_a^b|p(t)|\,dt. \tag{2.15}$$

By using (2.14) we then get (2.11).

(iii) It is well known that if $p\in\mathcal{R}(u,[a,b])$, where $u\in\mathcal{M}^{\nearrow}[a,b]$, then we have

$$\left|\int_a^b p(t)\,dv(t)\right|\le L\int_a^b|p(t)|\,dv(t). \tag{2.16}$$

By using (2.14) we then get (2.12). □

Remark 2. *We define the simpler functional for $g\equiv 1$ by*

$$P(f,v;\gamma,\Gamma,a,b):=P(f,1,v;\gamma,\Gamma,a,b)=\int_a^b f(t)\,dv(t)-\frac{\gamma+\Gamma}{2}[v(b)-v(a)].$$

Let $f\in\mathcal{C}_{\mathbb{C}}[a,b]$ and $\gamma,\ \Gamma\in\mathbb{C},\ \gamma\neq\Gamma$ such that $f\in\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$. If $v\in\mathcal{BV}_{\mathbb{C}}[a,b]$, then

$$|P(f,v;\gamma,\Gamma,a,b)|\le\frac{1}{2}|\Gamma-\gamma|\bigvee_a^b(v). \tag{2.17}$$

If $v\in\mathcal{L}_{L,\mathbb{C}}[a,b]$, then

$$|P(f,v;\gamma,\Gamma,a,b)|\le\frac{1}{2}L|\Gamma-\gamma|(b-a). \tag{2.18}$$

If $v\in\mathcal{M}^{\nearrow}[a,b]$, then

$$\left|\int_a^b P(f,v;\gamma,\Gamma,a,b)\right|\le\frac{1}{2}|\Gamma-\gamma|[v(b)-v(a)]. \tag{2.19}$$

We observe that, if $f\in\mathcal{C}[a,b]$, namely f is real valued and continuous on $[a,b]$ and if we put $m:=\min_{t\in[a,b]}f(t)$ and $M:=\max_{t\in[a,b]}f(t)$ then by (2.17)-(2.19) we get

$$|P(f,v;m,M,a,b)|\le\frac{1}{2}(M-m)\bigvee_a^b(v)$$

if $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$ *,*

$$|P(f,v;m,M,a,b)| \leq \frac{1}{2}L(M-m)(b-a)$$

if $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$ *and*

$$|P(f,v;m,M,a,b)| \leq \frac{1}{2}(M-m)[v(b)-v(a)]$$

if $v \in \mathcal{M}^{\nearrow}[a,b]$ *, that have been obtained in* [22].

For other results of this type, see [17].

3. Quasi-Grüss Type Inequalities

We consider the functional

$$Q(f,g,v;\gamma,\Gamma,\delta,\Delta,a,b) := \int_a^b f(t)\,g(t)\,dv(t) - \frac{\gamma+\Gamma}{2}\int_a^b g(t)\,dv(t) - \frac{\delta+\Delta}{2}\int_a^b f(t)\,dv(t) + \frac{\gamma+\Gamma}{2}\cdot\frac{\delta+\Delta}{2}[v(b)-v(a)]$$

for the complex valued functions f, g, v defined on $[a,b]$ and such that the involved Riemann-Stieltjes integrals exist, and for γ, Γ, δ, $\Delta \in \mathbb{C}$.

We have the following quasi-Grüss type inequality:

Proposition 2. *Let* f*,* $g \in \mathcal{C}_{\mathbb{C}}[a,b]$ *and* γ*,* Γ*,* δ*,* $\Delta \in \mathbb{C}$*,* $\gamma \neq \Gamma$*,* $\delta \neq \Delta$ *such that* $f \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ *and* $g \in \bar{\Delta}_{[a,b]}(\delta,\Delta)$*. If* $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$*, then*

$$|Q(f,g,v;\gamma,\Gamma,a,b)| \leq \frac{1}{4}|\Gamma-\gamma|\,|\Delta-\delta|\bigvee_a^b(v). \tag{3.1}$$

If $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$*, then*

$$|Q(f,g,v;\gamma,\Gamma,a,b)| \leq \frac{1}{4}|\Gamma-\gamma|\,|\Delta-\delta|\,L(b-a).$$

If $v \in \mathcal{M}^{\nearrow}[a,b]$*, then*

$$|Q(f,g,v;\gamma,\Gamma,a,b)| \leq \frac{1}{4}|\Gamma-\gamma|\,|\Delta-\delta|\,[v(b)-v(a)].$$

Proof. If we replace in (2.10) g by $g - \frac{\delta+\Delta}{2}$, then we get

$$\left|\int_a^b f(t)\,g(t)\,dv(t) - \frac{\gamma+\Gamma}{2}\int_a^b g(t)\,dv(t) - \frac{\delta+\Delta}{2}\int_a^b f(t)\,dv(t) + \frac{\gamma+\Gamma}{2}\cdot\frac{\delta+\Delta}{2}[v(b)-v(a)]\right| \leq \frac{1}{2}|\Gamma-\gamma|\int_a^b\left|g(t)-\frac{\delta+\Delta}{2}\right|d\left(\bigvee_a^t(v)\right).$$

Since $g \in \bar{\Delta}_{[a,b]}(\delta,\Delta)$, then

$$\int_a^b\left|g(t)-\frac{\delta+\Delta}{2}\right|d\left(\bigvee_a^t(v)\right) \leq \frac{1}{2}|\Delta-\delta|\int_a^b d\left(\bigvee_a^t(v)\right) = \frac{1}{2}|\Delta-\delta|\bigvee_a^b(v)$$

and the inequality (3.1) is proved.

The proofs of the other two statements follow in a similar way and we omit the details. □

Proposition 3. *Let* $f,\ g \in \mathcal{C}_{\mathbb{C}}[a,b]$, $g \in \mathcal{BV}_{\mathbb{C}}[a,b]$ *and* $\gamma,\ \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ *such that* $f \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$. *If* $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, *then*

$$|Q(f,g,v;g(a),g(b),a,b)| \leq \frac{1}{4}|\Gamma-\gamma| \bigvee_a^b (g) \bigvee_a^b (v), \tag{3.2}$$

where

$$Q(f,g,v;g(a),g(b),a,b) = \int_a^b f(t)\,g(t)\,dv(t) - \frac{\gamma+\Gamma}{2}\int_a^b g(t)\,dv(t) - \frac{g(a)+g(b)}{2}\int_a^b f(t)\,dv(t) + \frac{\gamma+\Gamma}{2}\cdot\frac{g(a)+g(b)}{2}[v(b)-v(a)].$$

If $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$, *then*

$$|Q(f,g,v;g(a),g(b),a,b)| \leq \frac{1}{4}|\Gamma-\gamma| L(b-a) \bigvee_a^b (g). \tag{3.3}$$

If $v \in \mathcal{M}^{\nearrow}[a,b]$, *then*

$$|Q(f,g,v;g(a),g(b),a,b)| \leq \frac{1}{4}|\Gamma-\gamma| \bigvee_a^b (g)[v(b)-v(a)]. \tag{3.4}$$

Proof. If we replace in (2.10) g by $g - \frac{g(a)+g(b)}{2}$, then we get

$$\left|\int_a^b f(t)\,g(t)\,dv(t) - \frac{\gamma+\Gamma}{2}\int_a^b g(t)\,dv(t) - \frac{g(a)+g(b)}{2}\int_a^b f(t)\,dv(t) + \frac{\gamma+\Gamma}{2}\cdot\frac{g(a)+g(b)}{2}[v(b)-v(a)]\right| \leq \frac{1}{2}|\Gamma-\gamma| \int_a^b \left|g(t) - \frac{g(a)+g(b)}{2}\right| d\left(\bigvee_a^t (v)\right).$$

Since $g \in \mathcal{BV}_{\mathbb{C}}[a,b]$, hence

$$\left|g(t) - \frac{g(a)+g(b)}{2}\right| = \left|\frac{g(t)-g(a)+g(t)-g(b)}{2}\right| \leq \frac{1}{2}[|g(t)-g(a)|+|g(b)-g(t)|] \leq \frac{1}{2}\bigvee_a^b (g)$$

for any $t \in [a,b]$.

Therefore

$$\int_a^b \left|g(t) - \frac{g(a)+g(b)}{2}\right| d\left(\bigvee_a^t (v)\right) \leq \frac{1}{2}\bigvee_a^b (g) \int_a^b d\left(\bigvee_a^t (v)\right) = \frac{1}{2}\bigvee_a^b (g) \bigvee_a^b (v)$$

and the inequality (3.2) is proved.

The proofs of the other statements follow in a similar way and we omit the details. □

Proposition 4. *Let f, $g \in \mathcal{C}_{\mathbb{C}}[a,b]$ and $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ such that $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$. If $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, then*

$$\left|Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds, \frac{1}{b-a}\int_a^b g(s)\,ds, a, b\right)\right| \tag{3.5}$$
$$\leq \frac{1}{2}|\Gamma-\gamma|\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right| d\left(\bigvee_a^t(v)\right)$$
$$\leq \frac{1}{2}|\Gamma-\gamma|\max_{t\in[a,b]}\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|\bigvee_a^b(v),$$

where

$$Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds, \frac{1}{b-a}\int_a^b g(s)\,ds, a, b\right)$$
$$= \int_a^b f(t)\,g(t)\,dv(t) - \frac{\gamma+\Gamma}{2}\int_a^b g(t)\,dv(t)$$
$$-\int_a^b f(t)\,dv(t)\,\frac{1}{b-a}\int_a^b g(t)\,dt + [v(b)-v(a)]\,\frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\Bigg|.$$

If $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$, then

$$\left|Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds, \frac{1}{b-a}\int_a^b g(s)\,ds, a, b\right)\right| \tag{3.6}$$
$$\leq \frac{1}{2}|\Gamma-\gamma|\,L\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right| dt$$
$$\leq \frac{1}{2}|\Gamma-\gamma|\,L\,(b-a)\max_{t\in[a,b]}\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right| dt.$$

If $v \in \mathcal{M}^{\nearrow}[a,b]$, then

$$\left|Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds, \frac{1}{b-a}\int_a^b g(s)\,ds, a, b\right)\right| \tag{3.7}$$
$$\leq \frac{1}{2}|\Gamma-\gamma|\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right| dv(t)$$
$$\leq \frac{1}{2}|\Gamma-\gamma|\max_{t\in[a,b]}\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|[v(b)-v(a)].$$

Proof. The first inequality follows by Theorem 1 by replacing g with $g-\frac{1}{b-a}\int_a^b g(s)\,ds$. The second part follows by the fact that

$$\begin{aligned}
&\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right| d\left(\bigvee_a^t(v)\right) \\
&\leq \max_{t\in[a,b]}\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right| \int_a^b d\left(\bigvee_a^t(v)\right) \\
&= \max_{t\in[a,b]}\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right| \bigvee_a^b(v).
\end{aligned}$$

The proofs of the other statements follow in a similar way and we omit the details. □

Remark 3. *We observe that the quantity*

$$\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|,\ t\in[a,b]$$

is the left hand side in Ostrowski type inequalities for various classes of functions g. For a recent survey on these inequalities, see [12]*. Therefore, if*

$$\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right| \leq M_{g,[a,b]}(t),\ t\in[a,b]$$

is such of inequality, then from (3.5) we get

$$\begin{aligned}
&\left|Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds,\frac{1}{b-a}\int_a^b g(s)\,ds,a,b\right)\right| \\
&\qquad\qquad \leq \frac{1}{2}|\Gamma-\gamma|\int_a^b M_{g,[a,b]}(t)\,d\left(\bigvee_a^t(v)\right)
\end{aligned}\tag{3.8}$$

if $v\in\mathcal{BV}_{\mathbb{C}}[a,b]$, from (3.6) we get

$$\begin{aligned}
&\left|Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds,\frac{1}{b-a}\int_a^b g(s)\,ds,a,b\right)\right| \\
&\qquad\qquad \leq \frac{1}{2}|\Gamma-\gamma|\,L\int_a^b M_{g,[a,b]}(t)\,dt
\end{aligned}\tag{3.9}$$

if $v\in\mathcal{L}_{L,\mathbb{C}}[a,b]$ and from (3.7) we get

$$\begin{aligned}
&\left|Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds,\frac{1}{b-a}\int_a^b g(s)\,ds,a,b\right)\right| \\
&\qquad\qquad \leq \frac{1}{2}|\Gamma-\gamma|\int_a^b M_{g,[a,b]}(t)\,dv(t),
\end{aligned}\tag{3.10}$$

if $v\in\mathcal{M}^{\nearrow}[a,b]$.

For instance, if $g:[a,b]\to\mathbb{C}$ is of bounded variation, then we have, see [10] *and* [11]

$$\left|g(t)-\frac{1}{b-a}\int_a^b g(t)\,dt\right|\le\left[\frac{1}{2}+\frac{\left|t-\frac{a+b}{2}\right|}{b-a}\right]\bigvee_a^b(g) \tag{3.11}$$

for any $t\in[a,b]$. The constant $\frac{1}{2}$ is the best possible one.

Observe that

$$\begin{aligned}
&\int_a^b\left[\frac{1}{2}+\frac{\left|t-\frac{a+b}{2}\right|}{b-a}\right]d\left(\bigvee_a^t(v)\right)\\
&=\frac{1}{2}\bigvee_a^b(v)+\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|d\left(\bigvee_a^t(v)\right)\\
&=\frac{1}{2}\bigvee_a^b(v)+\frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)d\left(\bigvee_a^t(v)\right)\\
&+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)d\left(\bigvee_a^t(v)\right)\\
&=\frac{1}{2}\bigvee_a^b(v)+\frac{1}{b-a}\left[\left(\frac{a+b}{2}-t\right)\bigvee_a^t(v)\bigg|_a^{\frac{a+b}{2}}+\int_a^{\frac{a+b}{2}}\bigvee_a^t(v)\,dt\right]\\
&+\frac{1}{b-a}\left[\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\bigg|_{\frac{a+b}{2}}^b-\int_{\frac{a+b}{2}}^b\bigvee_a^t(v)\,dt\right]\\
&=\bigvee_a^b(v)+\frac{1}{b-a}\int_a^{\frac{a+b}{2}}\bigvee_a^t(v)\,dt-\frac{1}{b-a}\int_{\frac{a+b}{2}}^b\bigvee_a^t(v)\,dt\\
&=\bigvee_a^b(v)+\frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}}\bigvee_a^t(v)\,dt-\int_{\frac{a+b}{2}}^b\bigvee_a^t(v)\,dt\right)\\
&=\bigvee_a^b(v)-\frac{1}{b-a}\int_a^b\operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt.
\end{aligned}$$

Then by (3.8) we get

$$\begin{aligned}
&\left|Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds,\frac{1}{b-a}\int_a^b g(s)\,ds,a,b\right)\right|\\
&\le\frac{1}{2}|\Gamma-\gamma|\bigvee_a^b(g)\left[\bigvee_a^b(v)-\frac{1}{b-a}\int_a^b\operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt\right]\\
&\qquad\le\frac{1}{2}|\Gamma-\gamma|\bigvee_a^b(g)\bigvee_a^b(v)
\end{aligned} \tag{3.12}$$

if $v,\ g\in\mathcal{BV}_{\mathbb{C}}[a,b]$.

The last inequality in (3.12) follows by Chebyshev's inequality for monotonic functions that gives that

$$\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t (v)\,dt \geq \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\frac{1}{b-a}\int_a^b \bigvee_a^t (v)\,dt = 0.$$

Observe also that

$$\int_a^b \left[\frac{1}{2}+\frac{\left|t-\frac{a+b}{2}\right|}{b-a}\right]dt = \frac{3}{4}(b-a),$$

then by (3.9) we get

$$\left|Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds,\frac{1}{b-a}\int_a^b g(s)\,ds,a,b\right)\right| \leq \frac{3}{8}|\Gamma-\gamma|\,L(b-a)\bigvee_a^b (g) \tag{3.13}$$

if $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$ and $g \in \mathcal{BV}_{\mathbb{C}}[a,b]$.

Finally, since

$$\int_a^b \left[\frac{1}{2}+\frac{\left|t-\frac{a+b}{2}\right|}{b-a}\right]dv(t) = v(b)-v(a) - \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)[v(t)-v(a)]\,dt,$$

then we get by (3.10) that

$$\begin{aligned}&\left|Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds,\frac{1}{b-a}\int_a^b g(s)\,ds,a,b\right)\right| \\ &\leq \frac{1}{2}|\Gamma-\gamma|\bigvee_a^b (g)\left[v(b)-v(a)-\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)[v(t)-v(a)]\,dt\right] \\ &\leq \frac{1}{2}|\Gamma-\gamma|\bigvee_a^b (g)\,[v(b)-v(a)]\end{aligned} \tag{3.14}$$

if $v \in \mathcal{M}^{\nearrow}[a,b]$ and $g \in \mathcal{BV}_{\mathbb{C}}[a,b]$.

4. Grüss Type Inequalities

Consider the *Grüss type functional*

$$G(f,g,v;a,b) := \int_a^b f(t)\,g(t)\,dv(t) - \frac{1}{v(b)-v(a)}\int_a^b f(t)\,dv(t)\int_a^b g(t)\,dv(t) \tag{4.1}$$

for the complex valued functions f, g, v defined on $[a,b]$ and such that the involved Riemann-Stieltjes integrals exist and $v(b) \neq v(a)$.

We have:

Proposition 5. *Let $f, g \in \mathcal{C}_{\mathbb{C}}[a,b]$ and $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ such that $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$. If $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$ with $v(b) \neq v(a)$, then*

$$\begin{aligned} |G(f,g,v;a,b)| &\leq \frac{1}{2}|\Gamma-\gamma| \int_a^b \left| g(t) - \frac{1}{v(b)-v(a)} \int_a^b g(s)\,dv(s) \right| d\left(\bigvee_a^t (v) \right) \\ &\leq \frac{1}{2}|\Gamma-\gamma| \bigvee_a^b (v) \max_{t\in[a,b]} \left| g(t) - \frac{1}{v(b)-v(a)} \int_a^b g(s)\,dv(s) \right| dt. \end{aligned} \tag{4.2}$$

If $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$, then

$$\begin{aligned} |G(f,g,v;a,b)| &\leq \frac{1}{2}|\Gamma-\gamma| L \int_a^b \left| g(t) - \frac{1}{v(b)-v(a)} \int_a^b g(s)\,dv(s) \right| dt \\ &\leq \frac{1}{2}|\Gamma-\gamma| L(b-a) \max_{t\in[a,b]} \left| g(t) - \frac{1}{v(b)-v(a)} \int_a^b g(s)\,dv(s) \right|. \end{aligned} \tag{4.3}$$

If $v \in \mathcal{M}^{\nearrow}[a,b]$, then

$$\begin{aligned} |G(f,g,v;a,b)| &\leq \frac{1}{2}|\Gamma-\gamma| \int_a^b \left| g(t) - \frac{1}{v(b)-v(a)} \int_a^b g(s)\,dv(s) \right| dv(t) \\ &\leq \frac{1}{2}|\Gamma-\gamma| [v(b)-v(a)] \max_{t\in[a,b]} \left| g(t) - \frac{1}{v(b)-v(a)} \int_a^b g(s)\,dv(s) \right|. \end{aligned} \tag{4.4}$$

Proof. By Theorem 1, on replacing g with $g - \frac{1}{v(b)-v(a)} \int_a^b g(s)\,dv(s)$ we get

$$\begin{aligned} &\left| \int_a^b f(t) \left[g(t) - \frac{1}{v(b)-v(a)} \int_a^b g(s)\,dv(s) \right] dv(t) \right. \\ &\qquad \left. - \frac{\gamma+\Gamma}{2} \int_a^b \left[g(t) - \frac{1}{v(b)-v(a)} \int_a^b g(s)\,dv(s) \right] dv(t) \right| \\ &\qquad\qquad \leq \frac{1}{2}|\Gamma-\gamma| \int_a^b \left| g(t) - \frac{1}{v(b)-v(a)} \int_a^b g(s)\,dv(s) \right| d\left(\bigvee_a^t (v) \right). \end{aligned}$$

Since

$$\begin{aligned} &\int_a^b f(t) \left[g(t) - \frac{1}{v(b)-v(a)} \int_a^b g(s)\,dv(s) \right] dv(t) \\ &\qquad = \int_a^b f(t)\,g(t)\,dv(t) - \frac{1}{v(b)-v(a)} \int_a^b f(t)\,dv(t) \int_a^b g(t)\,dv(t) \end{aligned}$$

and

$$\int_a^b \left[g(t) - \frac{1}{v(b)-v(a)}\int_a^b g(s)\,dv(s)\right] dv(t) = 0,$$

hence the first inequality (4.2) is obtained. The second inequality is obvious.

The rest follow in a similar way and we omit the details. □

Remark 4. *If g is of K-Lipschitzian and v is of bounded variation, then* [16]

$$\left|g(t)[v(b)-v(a)] - \int_a^b g(s)\,dv(s)\right| \le K\left[\frac{1}{2}(b-a) + \left|t - \frac{a+b}{2}\right|\right]\bigvee_a^b(v),$$

for any $t \in [a,b]$.

By (4.2) we then have

$$\begin{aligned}
&|G(f,g,v;a,b)| \\
&\le \frac{1}{2}\frac{|\Gamma-\gamma|}{|v(b)-v(a)|}\int_a^b \left|g(t)[v(b)-v(a)] - \int_a^b g(s)\,dv(s)\right| d\left(\bigvee_a^t(v)\right) \\
&\le \frac{1}{2}\frac{|\Gamma-\gamma|}{|v(b)-v(a)|}K\bigvee_a^b(v)\int_a^b \left|\frac{1}{2}(b-a) + \left|t-\frac{a+b}{2}\right|\right| d\left(\bigvee_a^t(v)\right).
\end{aligned}$$

Since, as above

$$\begin{aligned}
&\int_a^b \left[\frac{1}{2}(b-a) + \left|t-\frac{a+b}{2}\right|\right] d\left(\bigvee_a^t(v)\right) \\
&= (b-a)\bigvee_a^b(v) - \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt \le (b-a)\bigvee_a^b(v),
\end{aligned}$$

then we get the following upper bounds for the magnitude of $G(f,g,v;a,b)$

$$\begin{aligned}
(4.5)\quad &|G(f,g,v;a,b)| \\
&\le \frac{1}{2}\frac{|\Gamma-\gamma|}{|v(b)-v(a)|}K\bigvee_a^b(v)\left[(b-a)\bigvee_a^b(v) - \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt\right] \\
&\le \frac{1}{2}K\frac{|\Gamma-\gamma|(b-a)}{|v(b)-v(a)|}\left(\bigvee_a^b(v)\right)^2.
\end{aligned}$$

Any other upper bounds for $|\theta(g,v;a,b,t)|$ with $t \in [a,b]$, see for instance the survey [9]*, will provide the corresponding bounds for $|G(f,g,v;a,b)|$. The details are left to the interested reader.*

5. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \le \lambda, \\ \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_{\lambda} := \varphi_{\lambda}(A) \tag{5.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [27, p. 256]:

Theorem 2 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_{\lambda}\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_{\lambda} \le E_{\lambda'}$ *for* $\lambda \le \lambda'$;
b) $E_{a-0} = 0, E_b = I$ *and* $E_{\lambda+0} = E_{\lambda}$ *for all* $\lambda \in \mathbb{R}$;
c) *We have the representation*

$$A = \int_{a-0}^{b} \lambda dE_{\lambda}.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A) \in \mathcal{B}(H)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the inequality

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda'_k) \left[E_{\lambda_k} - E_{\lambda_{k-1}} \right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < a = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = b, \\ \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \\ \lambda'_k \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda) \, dE_{\lambda}, \tag{5.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 3. *With the assumptions of Theorem 2 for A, E_{λ} and φ we have the representations*

$$\varphi(A) x = \int_{a-0}^{b} \varphi(\lambda) \, dE_{\lambda} x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A) x, y \rangle = \int_{a-0}^{b} \varphi(\lambda) \, d\langle E_{\lambda} x, y \rangle \quad \text{for all } x, \ y \in H. \tag{5.3}$$

In particular,

$$\langle \varphi(A) x, x \rangle = \int_{a-0}^{b} \varphi(\lambda) \, d\langle E_{\lambda} x, x \rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{a-0}^{b} |\varphi(\lambda)|^2 \, d\|E_\lambda x\|^2 \quad \textit{for all } x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [23].

Lemma 2. *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any x, $y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta} \left(\langle E_{(\cdot)} x, y\rangle\right)\right]^2 \le \langle (E_\beta - E_\alpha) x, x\rangle \langle (E_\beta - E_\alpha) y, y\rangle, \tag{5.4}$$

where $\bigvee_{\alpha}^{\beta} \left(\langle E_{(\cdot)} x, y\rangle\right)$ denotes the total variation of the function $\langle E_{(\cdot)} x, y\rangle$ on $[\alpha, \beta]$.

Remark 5. *For $\alpha = a - \varepsilon$ with $\varepsilon > 0$ and $\beta = b$ we get from (5.4) the inequality*

$$\bigvee_{a-\varepsilon}^{b} \left(\langle E_{(\cdot)} x, y\rangle\right) \le \langle (I - E_{a-\varepsilon}) x, x\rangle^{1/2} \langle (I - E_{a-\varepsilon}) y, y\rangle^{1/2} \tag{5.5}$$

for any x, $y \in H$.

This implies, for any x, $y \in H$, that

$$\bigvee_{a-0}^{b} \left(\langle E_{(\cdot)} x, y\rangle\right) \le \|x\| \|y\|, \tag{5.6}$$

where $\bigvee_{a-0}^{b} \left(\langle E_{(\cdot)} x, y\rangle\right)$ denotes the limit $\lim_{\varepsilon\to 0+} \left[\bigvee_{a-\varepsilon}^{b} \left(\langle E_{(\cdot)} x, y\rangle\right)\right]$.

We can state the following result for functions of selfadjoint operators:

Theorem 3. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda | \lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda | \lambda \in Sp(A)\} =: \max Sp(A)$. Also, assume that $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator A and f, $g : I \to \mathbb{C}$ are continuous on I, $[a, b] \subset \mathring{I}$ (the interior of I). If γ, $\Gamma \in \mathbb{C}$, $\gamma \ne \Gamma$ such that $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$, then*

$$\left|\langle f(A) g(A) x, y\rangle - \frac{\gamma + \Gamma}{2} \langle g(A) x, y\rangle\right| \tag{5.7}$$
$$\le \frac{1}{2} |\Gamma - \gamma| \max_{t\in[a,b]} |g(t)| \bigvee_{a-0}^{b} \left(\langle E_{(\cdot)} x, y\rangle\right) \le \frac{1}{2} |\Gamma - \gamma| \max_{t\in[a,b]} |g(t)| \|x\| \|y\|$$

for any x, $y \in H$.

Proof. Using the inequality (2.10), we have

$$\left|\int_{a-\varepsilon}^{b} f(t) g(t) \, d\langle E_t x, y\rangle - \frac{\gamma + \Gamma}{2} \int_{a-\varepsilon}^{b} g(t) \, d\langle E_t x, y\rangle\right|$$
$$\le \frac{1}{2} |\Gamma - \gamma| \max_{t\in[a-\varepsilon,b]} |g(t)| \bigvee_{a-\varepsilon}^{b} \left(\langle E_{(\cdot)} x, y\rangle\right)$$

for small $\varepsilon > 0$ and for any $x, y \in H$.

Taking the limit over $\varepsilon \to 0+$ and using the continuity of f, g and the Spectral Representation Theorem, we deduce the desired result (5.7). $\square$

Corollary 4. *With the assumptions of Theorem 3 and if $\gamma, \Gamma, \delta, \Delta \in \mathbb{C}$, $\gamma \neq \Gamma$, $\delta \neq \Delta$ such that $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ and $g \in \bar{\Delta}_{[a,b]}(\delta, \Delta)$, then*

$$\begin{aligned}(5.8)\quad &\left|\langle f(A) g(A) x, y\rangle - \frac{\gamma+\Gamma}{2}\langle g(A) x, y\rangle \right.\\ &\left. \qquad - \frac{\delta+\Delta}{2}\langle f(A) x, y\rangle + \frac{\gamma+\Gamma}{2}\frac{\delta+\Delta}{2}\langle x, y\rangle\right| \\ &\qquad \leq \frac{1}{4}|\Gamma-\gamma||\Delta-\delta| \bigvee_{a-0}^{b}\left(\langle E_{(\cdot)} x, y\rangle\right) \leq \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\|x\|\|y\|\end{aligned}$$

for any $x, y \in H$.

Corollary 5. *With the assumptions of Theorem 3 and if $g \in \mathcal{BV}_{\mathbb{C}}[a,b]$ and γ, $\Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ such that $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$, then*

$$\begin{aligned}(5.9)\quad &\left|\langle f(A) g(A) x, y\rangle - \frac{\gamma+\Gamma}{2}\langle g(A) x, y\rangle \right.\\ &\left. \qquad - \frac{g(a)+g(b)}{2}\langle f(A) x, y\rangle + \frac{\gamma+\Gamma}{2}\frac{g(a)+g(b)}{2}\langle x, y\rangle\right| \\ &\qquad \leq \frac{1}{4}|\Gamma-\gamma| \bigvee_{a}^{b}(g) \bigvee_{a-0}^{b}\left(\langle E_{(\cdot)} x, y\rangle\right) \leq \frac{1}{4}|\Gamma-\gamma|\bigvee_{a}^{b}(g)\|x\|\|y\|\end{aligned}$$

for any $x, y \in H$.

Corollary 6. *With the assumptions of Theorem 3 and if $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ such that $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$,*

$$\begin{aligned}(5.10)\quad &\left|\langle f(A) g(A) x, y\rangle - \frac{\gamma+\Gamma}{2}\langle g(A) x, y\rangle \right.\\ &\left. \qquad - \langle f(A) x, y\rangle \frac{1}{b-a}\int_a^b g(t)\,dt + \langle x, y\rangle \frac{\gamma+\Gamma}{2}\frac{1}{b-a}\int_a^b g(t)\,dt\right| \\ &\qquad \leq \frac{1}{2}|\Gamma-\gamma| \max_{t\in[a,b]}\left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| \bigvee_{a-0}^{b}\left(\langle E_{(\cdot)} x, y\rangle\right) \\ &\qquad\qquad \leq \frac{1}{2}|\Gamma-\gamma| \max_{t\in[a,b]}\left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|\|x\|\|y\|\end{aligned}$$

for any $x, y \in H$.

Finally, by the use of the inequality (4.2) in the form

$$\begin{aligned}(5.11)\quad & \left|[v(b)-v(a)]\int_a^b f(t)\,g(t)\,dv(t)-\int_a^b f(t)\,dv(t)\int_a^b g(t)\,dv(t)\right| \\ & \leq \frac{1}{2}|\Gamma-\gamma|\bigvee_a^b(v)\max_{t\in[a,b]}\left|g(t)[v(b)-v(a)]-\int_a^b g(s)\,dv(s)\right|,\end{aligned}$$

provided $v\in\mathcal{BV}_{\mathbb{C}}[a,b]$, $f,\ g\in\mathcal{C}_{\mathbb{C}}[a,b]$ and $\gamma,\ \Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$ such that $f\in\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, we have

Corollary 7. *With the assumptions of Theorem 3 and if* $\gamma,\ \Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$ *such that* $f\in\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, *then*

$$\begin{aligned}(5.12)\quad & |\langle f(A)\,g(A)\,x,y\rangle\langle x,y\rangle-\langle f(A)\,x,y\rangle\langle g(A)\,x,y\rangle| \\ & \leq\frac{1}{2}|\Gamma-\gamma|\max_{t\in[a,b]}|g(t)\langle x,y\rangle-\langle g(A)\,x,y\rangle|\bigvee_{a-0}^b\left(\langle E_{(\cdot)}x,y\rangle\right) \\ & \leq\frac{1}{2}|\Gamma-\gamma|\max_{t\in[a,b]}|g(t)\langle x,y\rangle-\langle g(A)\,x,y\rangle|\,\|x\|\,\|y\|\end{aligned}$$

for any $x,\ y\in H$.

6. Applications for Unitary Operators

A unitary operator is a bounded linear operator $U:H\to H$ on a Hilbert space H satisfying

$$U^*U=UU^*=1_H$$

where U^* is the adjoint of U, and $1_H:H\to H$ is the identity operator. This property is equivalent to the following:

(i) U preserves the inner product $\langle\cdot,\cdot\rangle$ of the Hilbert space, i.e., for all vectors x and y in the Hilbert space, $\langle Ux,Uy\rangle=\langle x,y\rangle$ and
(ii) U is surjective.

The following result is well known [27, p. 275 - p. 276]:

Theorem 4 (Spectral Representation Theorem)**.** *Let U be a unitary operator on the Hilbert space H. Then there exists a family of projections $\{P_\lambda\}_{\lambda\in[0,2\pi]}$, called the spectral family of U, with the following properties*

a) $P_\lambda\leq P_{\lambda'}$ *for* $\lambda\leq\lambda'$;
b) $P_0=0, P_{2\pi}=I$ *and* $P_{\lambda+0}=P_\lambda$ *for all* $\lambda\in[0,2\pi)$;
c) *We have the representation*

$$U=\int_0^{2\pi}\exp(i\lambda)\,dP_\lambda.$$

More generally, for every continuous complex-valued function φ defined on the unit circle $\mathcal{C}(0,1)$ there exists a unique operator $\varphi(U)\in\mathcal{B}(H)$ such that for every $\varepsilon>0$ there exists a $\delta>0$ satisfying the inequality

$$\left\|\varphi(U)-\sum_{k=1}^n\varphi\left(\exp\left(i\lambda_k'\right)\right)\left[P_{\lambda_k}-P_{\lambda_{k-1}}\right]\right\|\leq\varepsilon$$

whenever

$$\begin{cases} 0 = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = 2\pi, \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$\varphi(U) = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda, \tag{6.1}$$

where the integral is of Riemann-Stieltjes type.

Corollary 8. *With the assumptions of Theorem 4 for U, P_λ and φ we have the representations*

$$\varphi(U)\, x = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(U)\, x, y\rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, y\rangle \quad \text{for all } x, y \in H. \tag{6.2}$$

In particular,

$$\langle \varphi(U)\, x, x\rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(U)\, x\|^2 = \int_0^{2\pi} |\varphi(\exp(i\lambda))|^2\, d\|P_\lambda x\|^2 \quad \text{for all } x \in H.$$

On making use of an argument similar to the one in [23, Theorem 6], we have:

Lemma 3. *Let $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ be the spectral family of the unitary operator U on the Hilbert space H. Then for any x, $y \in H$ and $0 \leq \alpha < \beta \leq 2\pi$ we have the inequality*

$$\bigvee_\alpha^\beta \left(\langle P_{(\cdot)} x, y\rangle\right) \leq \langle (P_\beta - P_\alpha)\, x, x\rangle^{1/2} \langle (P_\beta - P_\alpha)\, y, y\rangle^{1/2}, \tag{6.3}$$

where $\bigvee_\alpha^\beta \left(\langle P_{(\cdot)} x, y\rangle\right)$ denotes the total variation of the function $\langle P_{(\cdot)} x, y\rangle$ on $[\alpha, \beta]$.

In particular,

$$\bigvee_0^{2\pi} \left(\langle P_{(\cdot)} x, y\rangle\right) \leq \|x\|\, \|y\| \tag{6.4}$$

for any x, $y \in H$.

Theorem 5. *Let U be a unitary operator on the Hilbert space H and $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ the spectral family of projections of U. Also, assume that f, $g : \mathcal{C}(0,1) \to \mathbb{C}$ are* *continuous on* $\mathcal{C}(0,1)$*. If* $\phi,\ \Phi\in\mathbb{C},\ \phi\neq\Phi$ *are such that* $f\circ\exp(i\cdot)\in\bar{\Delta}_{[0,2\pi]}(\phi,\Phi)$, *then*

$$\left|\langle f(U)\,g(U)\,x,y\rangle-\frac{\phi+\Phi}{2}\langle g(U)\,x,y\rangle\right| \le\frac{1}{2}|\Phi-\phi|\max_{t\in[0,2\pi]}|g(\exp(it))|\bigvee_{0}^{2\pi}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right) \le\frac{1}{2}|\Phi-\phi|\max_{t\in[0,2\pi]}|g(\exp(it))|\,\|x\|\,\|y\| \tag{6.5}$$

for any $x,\ y\in H$.

The proof follows by Theorem 1 and the Spectral Representation Theorem for unitary operators in a similar way with the proof of Theorem 3 and we omit the details.

Corollary 9. *With the assumptions of Theorem 5 and if* $\phi,\ \Phi,\ \psi,\ \Psi\in\mathbb{C},\ \phi\neq\Phi$ *such that* $f\circ\exp(i\cdot)\in\bar{\Delta}_{[0,2\pi]}(\phi,\Phi)$, $g\circ\exp(i\cdot)\in\bar{\Delta}_{[0,2\pi]}(\psi,\Psi)$ *then*

$$\left|\langle f(U)\,g(U)\,x,y\rangle-\frac{\phi+\Phi}{2}\langle g(U)\,x,y\rangle -\frac{\psi+\Psi}{2}\langle f(U)\,x,y\rangle+\frac{\phi+\Phi}{2}\frac{\psi+\Psi}{2}\langle x,y\rangle\right| \le\frac{1}{4}|\Phi-\phi|\,|\Psi-\psi|\bigvee_{0}^{2\pi}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right)\le\frac{1}{4}|\Phi-\phi|\,|\Psi-\psi|\,\|x\|\,\|y\| \tag{6.6}$$

for any $x,\ y\in H$.

Corollary 10. *With the assumptions of Theorem 5 and if* $\phi,\ \Phi\in\mathbb{C},\ \phi\neq\Phi$ *such that* $f\circ\exp(i\cdot)\in\bar{\Delta}_{[0,2\pi]}(\phi,\Phi)$, *then*

$$\Bigg|\langle f(U)\,g(U)\,x,y\rangle-\frac{\phi+\Phi}{2}\langle g(U)\,x,y\rangle -\langle f(U)\,x,y\rangle\frac{1}{2\pi}\int_0^{2\pi}g(\exp(it))\,dt+\langle x,y\rangle\frac{\phi+\Phi}{2}\frac{1}{2\pi}\int_0^{2\pi}g(\exp(it))\,dt\Bigg| \le\frac{1}{2}|\Phi-\phi|\max_{t\in[0,2\pi]}\left|g(\exp(it))-\frac{1}{2\pi}\int_0^{2\pi}g(\exp(is))\,ds\right|\bigvee_{0}^{2\pi}\left(\left\langle P_{(\cdot)}x,y\right\rangle\right) \le\frac{1}{2}|\Phi-\phi|\max_{t\in[0,2\pi]}\left|g(\exp(it))-\frac{1}{2\pi}\int_0^{2\pi}g(\exp(is))\,ds\right|\|x\|\,\|y\| \tag{6.7}$$

for any $x,\ y\in H$.

Corollary 11. *With the assumptions of Theorem 5 and if* $\phi,\ \Phi \in \mathbb{C}$, $\phi \neq \Phi$ *such that* $f \circ \exp(i\cdot) \in \bar{\Delta}_{[0,2\pi]}(\phi, \Phi)$, *then*

$$\begin{aligned}(6.8)\quad & |\langle f(U)\, g(U)\, x, y\rangle \langle x, y\rangle - \langle f(U)\, x, y\rangle \langle g(U)\, x, y\rangle| \\ & \leq \frac{1}{2}|\Phi - \phi| \max_{t\in[0,2\pi]} |g(\exp(it)) \langle x, y\rangle - \langle g(U)\, x, y\rangle| \bigvee_0^{2\pi} \left(\langle P_{(\cdot)} x, y\rangle\right) \\ & \leq \frac{1}{2}|\Phi - \phi| \max_{t\in[0,2\pi]} |g(\exp(it)) \langle x, y\rangle - \langle g(U)\, x, y\rangle| \, \|x\| \, \|y\|\end{aligned}$$

for any $x,\ y \in H$.

References

[1] T. M. Apostol, *Mathematical Analysis,* Addison-Wesley Publishing Company, Second Edition, 1981.

[2] N. S. Barnett, W. S. Cheung, S. S. Dragomir and A. Sofo, Ostrowski and trapezoid type inequalities for the Stieltjes integral with Lipschitzian integrands or integrators, *Comput. Math. Appl.* **57** (2009), no. 2, 195–201. Preprint *RGMIA Res. Rep. Coll.* 9(2006), No. 4, Article 9.

[3] P. Cerone, W. S. Cheung and S. S. Dragomir, On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation, *Comput. Math. Appl.* **54** (2007), no. 2, 183–191. Preprint *RGMIA Res. Rep. Coll.* **9**(2006), No. 2, Article 14. [ONLINE: http://rgmia.vu.edu.au/v9n2.html].

[4] P. Cerone and S. S. Dragomir, Trapezoid type rules from an inequalities point of view, in *Handbook of Analytic Computational Methods in Applied Mathematics,* Ed. G. Anastassiou, CRC Press, New York, pp. 65-134.

[5] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.* **38** (2007), no. 1, 37–49. Preprint *RGMIA Res. Rep. Coll.,* **5**(2) (2002), Article 14.

[6] P. Cerone, S. S. Dragomir and C. E. M. Pearce, A generalised trapezoid inequality for functions of bounded variation, *Turkish J. Math.,* **24**(2) (2000), 147-163.

[7] X. L. Cheng and J. Sun, A note on the perturbed trapezoid inequality, *J. Ineq. Pure and Appl. Math.,* **3**(2) Art. 29, (2002).

[8] W. S. Cheung and S. S. Dragomir, Two Ostrowski type inequalities for the Stieltjes integral of monotonic functions, *Bull. Austral. Math. Soc.* **75** (2007), no. 2, 299–311., Preprint *RGMIA Res. Rep. Coll.* **9**(2006), No. 3, Article 8.

[9] W. S. Cheung and S. S. Dragomir, A survey on Ostrowski type inequalities for Riemann-Stieltjes integral. *Handbook of functional equations*, 75–104, Springer Optim. Appl., 95,Springer, New York, 2014.

[10] S. S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation. *Bull. Austral. Math. Soc.* **60** (1999), No. 3, 495–508.

[11] S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. Appl.* **4** (2001), No. 1, 59-66. Preprint: *RGMIA Res. Rep. Coll.* **2** (1999), Art. 7.

[12] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 1, 283 pp.

[13] S. S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[14] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Computers and Math. with Applic.,* **38** (1999), 33-37.

[15] S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.*, **7** (2000), 477-485.

[16] S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral $\int_a^b f(t)\, du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM*, **5**(1) (2001), 35-45.

[17] S. S. Dragomir, The median principle for inequalities and applications. *Functional equations, inequalities and applications*, 21–37, Kluwer Acad. Publ., Dordrecht, 2003.

[18] S. S. Dragomir, A companion of the Grüss inequality and applications. *Appl. Math. Lett.* **17** (2004), no. 4, 429–435.
[19] S. S. Dragomir, Inequalities of Grüss type for the Stieltjes integral, *Kragujevac J. Math.*, **26** (2004), 89-122.
[20] S. S. Dragomir, A generalisation of Cerone's identity and applications, *Tamsui Oxf. J. Math. Sci.* **23** (2007), no. 1, 79–90. Preprint *RGMIA Res. Rep. Coll.* **8**(2005), No. 2. Artcile 19.
[21] S. S. Dragomir, Inequalities for Stieltjes integrals with convex integrators and applications, *Appl. Math. Lett.*, **20** (2007), 123-130.
[22] S. S. Dragomir, The perturbed median principle for integral inequalities with applications. Nonlinear analysis and variational problems, 53–63, Springer Optim. Appl., 35, Springer, New York, 2010.
[23] S. S. Dragomir, Some inequalities for continuous functions of selfadjoint operators in Hilbert spaces, *Acta Math Vietnam* (2014) **39**:287–303, DOI 10.1007/s40306-014-0061-4. Preprint *RGMIA Res. Rep. Coll.* **15**(2012), Art. 16.
[24] S. S. Dragomir, C. Buşe, M. V. Boldea and L. Braescu, A generalisation of the trapezoidal rule for the Riemann-Stieltjes integral and applications, *Nonlinear Anal. Forum,* (Korea) **6**(2) (2001), 337-351.
[25] S. S. Dragomir and I. Fedotov, An inequality of Grüss type for the Riemann-Stieltjes integral and applications for special means, *Tamkang J. Math.,* **29**(4) (1998), 287-292.
[26] S. S. Dragomir and I. Fedotov, A Grüss type inequality for mappings of bounded variation and applications for numerical analysis, *Nonlinear Funct. Anal. Appl.,* **6**(3) (2001), 425-433.
[27] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.
[28] Z. Liu, Refinement of an inequality of Grüss type for Riemann-Stieltjes integral, *Soochow J. Math.,* **30**(4) (2004), 483-489.
[29] A. Ostrowski, Uber die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa