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On Hadamard's Inequality for Trigonometrically ρ - Convex Functions

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Abstract

The aim of this paper is to derive Hadamard's inequality for trigonometrically ρ - convex functions. Furthermore, we establish some integral inequalities for higher powers of trigonometrically ρ - convex functions.

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1 Introduction

Trigonometrically ρ - convex functions have interesting applications in the theory of entire functions (of order $0<\rho<\infty$) and in the theory of cavitational diagrams for hydroprofiles, see for example [1, 2, 3]. In what follows, we shall be concerned with real finite functions defined on a finite or infinite interval $I \subset \mathbb{R}$. The relation, see for example [4]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

is known in the literature as Hermite - Hadamard's inequality or Hadamard's inequality, which holds for convex functions $f: I \to \mathbb{R}$, and $a, b \in I$ with $a<b$. This inequality has been generalized and applied in many various aspects, see for example [5, 6, 7].

2 Definitions and Preliminary results

In this section we present the basic definitions and results which will be used in the sequel , for more information see [1, 2].

Definition 2.1 *A function* $f: I \to \mathbb{R}$ *is said to be* ***trigonometrically*** ρ ***- convex,*** *if for any arbitrary closed subinterval* $[u, v]$ *of* I *such that* $0<\rho(v-u)<\pi$, *the graph of* $f(x)$

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for $x \in [u, v]$ *lies nowhere above the* ρ *- trigonometric function, determined by the equation:*

$$H(x) = H(x; u, v, f) = A\cos\rho x + B\sin\rho x,$$

where A *and* B *are chosen such that* $H(u) = f(u)$, *and* $H(v) = f(v)$.
Equivalently, if for all $x \in [u, v]$

$$f(x) \le H(x) = \frac{f(u)\sin\rho(v-x) + f(v)\sin\rho(x-u)}{\sin\rho(v-u)}. \tag{2.1}$$

Definition 2.2 *A function*

$$T_u(x) = A\cos\rho x + B\sin\rho x$$

is said to be ***supporting function*** *for* $f(x)$ *at the point* $u \in I$, *if*

$$T_u(u) = f(u), \quad \text{and} \quad T_u(x) \le f(x) \quad \forall\ x \in I. \tag{2.2}$$

That is, if $f(x)$ *and* $T_u(x)$ *agree at* $x = u$, *and the graph of* $f(x)$ *does not lie under the support curve.*

Theorem 2.1 [8] *A function* $f : I \to \mathbb{R}$ *is trigonometrically* ρ *- convex on* I *if and only if there exists a supporting function for* $f(x)$ *at each point* $x \in I$.

Theorem 2.2 *A trigonometrically* ρ *- convex function* $f : I \to \mathbb{R}$ *has finite left and right derivatives* $f'_-(x)$, $f'_+(x)$ *at every point* $x \in I$, *and* $f'_-(x) \le f'_+(x)$ *for all* $x \in I$.

Property 2.1 *Under the assumptions of Theorem 2.2, the function* f *is continuously differentiable on* I *with the exception of an at most countable set.*

Property 2.2 [8] *If* $f : I \to \mathbb{R}$ *is differentiable trigonometrically* ρ *- convex function, then the supporting function for* $f(x)$ *at the point* $u \in I$ *has the form:*

$$T_u(x) = f(u)\cos\rho(x-u) + \frac{f'(u)}{\rho}\sin\rho(x-u). \tag{2.3}$$

3 Main Results

Theorem 3.1 *Suppose* $f : I \to \mathbb{R}$ *is trigonometrically* ρ *- convex function, and* $a, b \in I$ *with* $a < b$, *such that* $0 < \rho(b-a) < \pi$. *Then, one has the inequality:*

$$\frac{2}{\rho} f\left(\frac{a+b}{2}\right)\sin\rho\left(\frac{b-a}{2}\right) \le \int_a^b f(x)\,dx \le \frac{1}{\rho}\left[f(a) + f(b)\right]\tan\rho\left(\frac{b-a}{2}\right). \tag{3.1}$$

Proof. Let u be an arbitrary point in (a, b). As $f(x)$ is trigonometrically ρ - convex function, then from Definition 2.1, and Definition 2.2, we observe that the graph of $f(x)$ lies nowhere above the ρ - trigonometric chord:

$$H(x) = \frac{f(a)\sin\rho(b-x) + f(b)\sin\rho(x-a)}{\sin\rho(b-a)}, \tag{3.2}$$

joining $(a, f(a))$ and $(b, f(b))$, and nowhere below any support curve at $(u, f(u))$. The supporting function $T_u(x)$ for $f(x)$ at the point $u \in (a, b)$ can be written as follows:

$$T_u(x) = f(u)\cos\rho(x-u) + K_{u,f}\sin\rho(x-u)$$
$$= K\sin\rho(x+\alpha-u), \tag{3.3}$$
where $K_{u,f}$ is a fixed real number depends on u and f,
$$K = \sqrt{f^2(u)+K_{u,f}^2}, \qquad \text{and} \qquad \tan\rho\alpha = \frac{f(u)}{K_{u,f}}.$$
Hence,
$$T_u(x) \le f(x) \le H(x) \qquad x\in[a,b],$$
and thus,
$$\int_a^b T_u(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b H(x)\,dx. \tag{3.4}$$
Using (3.2), one has
$$\int_a^b H(x)dx = \frac{1}{\sin\rho(b-a)}\left[f(a)\int_a^b \sin\rho(b-x)dx + f(b)\int_a^b \sin\rho(x-a)dx\right]$$
$$= \frac{1}{\rho}\left[f(a)+f(b)\right]\left[\frac{1-\cos\rho(b-a)}{\sin\rho(b-a)}\right]$$
$$= \frac{1}{\rho}\left[f(a)+f(b)\right]\tan\rho\left(\frac{b-a}{2}\right). \tag{3.5}$$
Using (3.3), one obtains

$$\int_a^b T_u(x)dx = K\int_a^b \sin\rho(x+\alpha-u)dx$$
$$= \frac{2}{\rho}K\sin\rho\left[\left(\frac{a+b}{2}\right)+\alpha-u\right]\sin\rho\left(\frac{b-a}{2}\right)$$
$$= \frac{2}{\rho}T_u\left(\frac{a+b}{2}\right)\sin\rho\left(\frac{b-a}{2}\right). \tag{3.6}$$
But from Definition 2.2, we observe that:
$$T_u\left(\frac{a+b}{2}\right) \le f\left(\frac{a+b}{2}\right) \quad \text{for all } u\in(a,b). \tag{3.7}$$
Taking the maximum of the term $\int_a^b T_u(x)\,dx \le \int_a^b f(x)\,dx$ in (3.4) and (3.6) for $u\in(a,b)$,

and from (3.7), then it follows that:
$$\int_a^b f(x)\,dx \ge \max_{a<u<b}\left\{\int_a^b T_u(x)\,dx\right\}$$
$$= \frac{2}{\rho}\max_{a<u<b}\left\{T_u\left(\frac{a+b}{2}\right)\right\}\sin\rho\left(\frac{b-a}{2}\right)$$
$$= \frac{2}{\rho}f\left(\frac{a+b}{2}\right)\sin\rho\left(\frac{b-a}{2}\right). \tag{3.8}$$
Hence, from (3.4), (3.5), and (3.8), the claim follows. □

Remark 3.1 For a trigonometrically ρ - convex function $f : I \to \mathbb{R}$, the constant $K_{u,f}$ in the above theorem is equal to $\dfrac{f'(u)}{\rho}$ if f is differentiable at the point $u \in I$, otherwise, $K_{u,f} \in [\dfrac{f'_-(u)}{\rho}, \dfrac{f'_+(u)}{\rho}]$.

Now, we give an estimation for the inequality (3.1) of the power function $f^n(x)$ instead of $f(x)$.

Theorem 3.2 *Let* $f : I \to \mathbb{R}$ *be a non-negative trigonometrically* ρ *- convex function,* $n \in \mathbb{N}$, *and* $a, b \in I$ *with* $a < b$, *such that* $0 < \rho(b-a) < \pi$. *Then, one has:*

$$\int_a^b f^n(x)dx \le \sin^{-n}\rho(b-a) \sum_{r=0}^{n} \binom{n}{r} [f(a)]^{n-r} [f(b)]^r \times \times \int_a^b \sin^r \rho(x-a) \sin^{n-r}\rho(b-x)dx. \tag{3.9}$$

Proof. Since $f(x)$ is trigonometrically ρ - convex function, then it is clear from Definition 2.1, that

$$f(x) \le H(x) \qquad \forall x \in [a,b].$$

As $f(x)$ is non-negative, we infer that:

$$f^n(x) \le H^n(x) \qquad \forall n \in \mathbb{N}.$$

Thus, using (3.2), one obtains

$$\begin{aligned} \int_a^b f^n(x)dx &\le \int_a^b H^n(x)dx \\ &= \frac{1}{\sin^n \rho(b-a)} \int_a^b [f(a)\sin\rho(b-x) + f(b)\sin\rho(x-a)]^n dx \\ &= \sin^{-n}\rho(b-a) \sum_{r=0}^{n} \binom{n}{r} [f(a)]^{n-r} [f(b)]^r \times \\ &\qquad \times \int_a^b \sin^r \rho(x-a) \sin^{n-r}\rho(b-x)dx. \end{aligned}$$

Hence, the theorem follows. □

Theorem 3.3 *Assume* $f : I \to \mathbb{R}$ *is trigonometrically* ρ *- convex function,* $n \in \mathbb{N}$, *and* $a, b \in I$ *with* $a < b$, *such that* $0 < \rho(b-a) < \pi$. *Then, one has the inequality:*

$$\int_a^b f^{2n-1}(x)dx \ge 4\left(\frac{\lambda}{2}\right)^{2n-1} \sum_{r=0}^{n-1} \frac{(-1)^{n+r-1}}{\rho m} \binom{2n-1}{r} \sin \rho m \left(\frac{b-a}{2}\right) \sin \rho m \beta, \tag{3.10}$$

where $m = 2n - 2r - 1$, $\lambda = \sqrt{f^2\left(\frac{a+b}{2}\right) + K^2_{\frac{a+b}{2},f}}$, *and* $\tan \rho\beta = \dfrac{f\left(\frac{a+b}{2}\right)}{K_{\frac{a+b}{2},f}}$.

Proof. As $f(x)$ is trigonometrically ρ - convex function, then from Definition 2.2, we have

$$f(x) \ge T_u(x) \qquad \forall x \in [a,b],$$
and consequently,
$$f^{2n-1}(x) \ge T_u^{2n-1}(x) \qquad \forall n \in \mathbb{N}.$$
Thus, using (3.3), one has
$$\begin{aligned}
\int_a^b f^{2n-1}(x)\,dx &\ge \int_a^b T_u^{2n-1}(x)\,dx \\
&= K^{2n-1}\int_a^b \sin^{2n-1}\rho(x+\alpha-u)\,dx \\
&= 2\left(\frac{K}{2}\right)^{2n-1}\sum_{r=0}^{n-1}(-1)^{n+r-1}\binom{2n-1}{r}\int_a^b \sin\rho(2n-2r-1)(x+\alpha-u)\,dx \\
&= 4\left(\frac{K}{2}\right)^{2n-1}\sum_{r=0}^{n-1}\frac{(-1)^{n+r-1}}{\rho m}\binom{2n-1}{r}\sin\rho m\left(\frac{b-a}{2}\right)\times \\
&\qquad \times\sin\rho m\left[\left(\frac{a+b}{2}\right)+\alpha-u\right],
\end{aligned} \tag{3.11}$$
where, $m = 2n-2r-1.$

The best possible bound of (3.11) can be obtained by taking the maximum of this inequality for $u \in (a,b)$. In particular, if $u = \dfrac{a+b}{2}$, we get
$$\int_a^b f^{2n-1}(x)\,dx \ge 4\left(\frac{\lambda}{2}\right)^{2n-1}\sum_{r=0}^{n-1}\frac{(-1)^{n+r-1}}{\rho m}\binom{2n-1}{r}\sin\rho m\left(\frac{b-a}{2}\right)\sin\rho m\beta,$$
where λ and β are as defined above, and hence the theorem. □

Remark 3.2 Hadamard's inequality (3.1) is an immediate consequence of Theorem 3.2, and Theorem 3.3 by taking $n=1$.

Theorem 3.4 *Let* $f : I \to \mathbb{R}$ *be a differentiable trigonometrically* ρ *- convex function,* $n \in \mathbb{N}$, *and* $a, b \in I$ *with* $a < b$, *such that* $0 < 2\rho(b-a) < \pi$. *If* $f(a) \ge 0$ *and* $f'(a) \ge 0$, *one then has the following inequalities:*
$$\int_a^b f^{2n}(x)\,dx \ge \left(\frac{\mu}{2}\right)^{2n}\left\{\binom{2n}{n}(b-a) + 2\sum_{r=0}^{n-1}\frac{(-1)^{n-r}}{\rho(n-r)}\binom{2n}{r}\sin\rho(n-r)(b-a)\times \atop \times\cos\rho(n-r)(b-a+2\gamma)\right\}, \tag{3.12}$$

$$\int_a^b f^{2n-1}(x)\,dx \ge 4\left(\frac{\mu}{2}\right)^{2n-1}\sum_{r=0}^{n-1}\frac{(-1)^{n+r-1}}{\rho m}\binom{2n-1}{r}\sin\rho m\left(\frac{b-a}{2}\right)\sin\rho m\left[\left(\frac{b-a}{2}\right)+\gamma\right], \tag{3.13}$$

where $m = 2n-2r-1,$ $\quad \mu = \sqrt{f^2(a) + \left(\dfrac{f'(a)}{\rho}\right)^2}$, *and* $\tan\rho\gamma = \dfrac{\rho f(a)}{f'(a)}.$

Proof. As $f(x)$ is trigonometrically ρ - convex function, then from Definition 2.2, it follows that:

$$f(x) \geq T_a(x) \qquad \forall x \in [a,b]. \tag{3.14}$$

Since $f(x)$ is differentiable, then from Property 2.2, $T_a(x)$ can be written in the form:

$$\begin{aligned} T_a(x) &= f(a)\cos\rho(x-a) + \frac{f'(a)}{\rho}\sin\rho(x-a) \\ &= \mu \sin\rho(x+\gamma-a), \end{aligned} \tag{3.15}$$

where $\mu = \sqrt{f^2(a) + \left(\frac{f'(a)}{\rho}\right)^2}$, and $\tan\rho\gamma = \frac{\rho f(a)}{f'(a)}$.

As $f(a) \geq 0$, $f'(a) \geq 0$, and $\rho(b+\gamma-a) < \pi$, then from (3.15), we conclude that

$$T_a(x) \geq 0 \qquad \forall x \in [a,b].$$

Thus, using (3.14), one obtains

$$f^n(x) \geq T_a^n(x) \qquad \forall n \in \mathbb{N}. \tag{3.16}$$

Therefore, from (3.15) and (3.16), the following two cases arise,

Case 1.

$$\begin{aligned} \int_a^b f^{2n}(x)dx &\geq \int_a^b T_a^{2n}(x)dx \\ &= \mu^{2n} \int_a^b \sin^{2n}\rho(x+\gamma-a)dx \\ &= \left(\frac{\mu}{2}\right)^{2n} \int_a^b \left\{ \binom{2n}{n} + \sum_{r=0}^{n-1} (-1)^{n-r}\, 2\binom{2n}{r} \cos 2\rho(n-r)(x+\gamma-a) \right\} dx \\ &= \left(\frac{\mu}{2}\right)^{2n} \left\{ \binom{2n}{n}(b-a) + 2\sum_{r=0}^{n-1} \frac{(-1)^{n-r}}{\rho(n-r)} \binom{2n}{r} \sin\rho(n-r)(b-a) \times \cos\rho(n-r)(b-a+2\gamma) \right\}. \end{aligned}$$

Case 2.

$$\begin{aligned} \int_a^b f^{2n-1}(x)dx &\geq \int_a^b T_a^{2n-1}(x)dx \\ &= \mu^{2n-1} \int_a^b \sin^{2n-1}\rho(x+\gamma-a)dx \\ &= 2\left(\frac{\mu}{2}\right)^{2n-1} \sum_{r=0}^{n-1} (-1)^{n+r-1} \binom{2n-1}{r} \int_a^b \sin\rho(2n-2r-1)(x+\gamma-a)dx \\ &= 4\left(\frac{\mu}{2}\right)^{2n-1} \sum_{r=0}^{n-1} \frac{(-1)^{n+r-1}}{\rho m} \binom{2n-1}{r} \sin\rho m\left(\frac{b-a}{2}\right) \sin\rho m\left[\left(\frac{b-a}{2}\right)+\gamma\right], \end{aligned}$$

where, $m = 2n-2r-1$.

Hence, the theorem. □

Remark 3.2 For the trigonometric expansions in Theorem 3.3, and Theorem 3.4, one can refer to [9].

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