SOME INEQUALITIES OF FEJÉR TYPE FOR TRIGONOMETRICALLY ρ-CONVEX FUNCTIONS

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ABSTRACT. In this paper we establish several Fejér type integral inequalities for trigonometrically ρ-convex functions.

1. INTRODUCTION

The following integral inequality

$$(1.1) \qquad f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2},$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, for which we would like to refer the reader to the monograph [8], the recent survey paper [7] and the references therein.

In 1906, Fejér [9], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 1 (Fejér's Inequality). *Consider the integral $\int_a^b h(x)\,w(x)\,dx$, where h is a convex function in the interval $[a,b]$ and g is a positive function in the same interval such that*

$$g(x) = g(a+b-x),\ x \in [a,b],$$

i.e., $y = g(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b),0\right)$ and is normal to the x-axis. Under those conditions the following inequalities are valid:

$$(1.2) \qquad h\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \int_a^b h(x)\,g(x)\,dx \le \frac{h(a)+h(b)}{2}\int_a^b g(x)\,dx.$$

If h is concave on $[a,b]$, then the inequalities reverse in (1.2).

Clearly, for $g(x) \equiv 1$ on $[a,b]$ we get (1.1).

Let I be a finite or infinite open interval of real numbers and $\rho > 0$.

In the following we present the basic definitions and results concerning the class of trigonometrically ρ-convex function, see for example [11], [12] and [3], [5], [6], [10], [13], [14] and [15].

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Following [1], we say that a function $f : I \to \mathbb{R}$ is *trigonometrically ρ-convex* on I if for any closed subinterval $[a, b]$ of I with $0 < b - a < \frac{\pi}{\rho}$ we have

$$(1.3) \qquad f(x) \leq \frac{\sin[\rho(b-x)]}{\sin[\rho(b-a)]} f(a) + \frac{\sin[\rho(x-a)]}{\sin[\rho(b-a)]} f(b)$$

for all $x \in [a, b]$.

If the inequality (1.3) holds with " $\geq$ ", then the function will be called *trigonometrically ρ-concave* on I.

Geometrically speaking, this means that the graph of f on $[a, b]$ lies nowhere above the ρ-trigonometric function determined by the equation

$$H(x) = H(x; a, b, f) := A \cos(\rho x) + B \sin(\rho x)$$

where A and B are chosen such that $H(a) = f(a)$ and $H(b) = f(b)$.

If we take $x = (1 - t)a + tb \in [a, b]$, $t \in [0, 1]$, then the condition (1.3) becomes

$$(1.4) \qquad f((1-t)a + tb) \leq \frac{\sin[\rho(1-t)(b-a)]}{\sin[\rho(b-a)]} f(a) + \frac{\sin[\rho t(b-a)]}{\sin[\rho(b-a)]} f(b)$$

for any $t \in [0, 1]$.

We have the following properties of trigonometrically ρ-convex on I, [1].

(i) A trigonometrically ρ-convex function $f : I \to \mathbb{R}$ has finite right and left derivatives $f'_+(x)$ and $f'_-(x)$ at every point $x \in I$ and $f'_-(x) \leq f'_+(x)$. The function f is differentiable on I with the exception of an at most countable set.

(ii) A necessary and sufficient condition for the function $f : I \to \mathbb{R}$ to be trigonometrically ρ-convex function on I is that it satisfies the gradient inequality

$$(1.5) \qquad f(y) \geq f(x) \cos[\rho(y-x)] + K_{x,f} \sin[\rho(y-x)]$$

for any $x, y \in I$ where $K_{x,f} \in [f'_-(x), f'_+(x)]$. If f is differentiable at the point x then $K_{x,f} = f'(x)$.

(iii) A necessary and sufficient condition for the function f to be a trigonometrically ρ-convex in I, is that the function

$$\varphi(x) = f'(x) + \rho^2 \int_a^x f(t)\, dt$$

is nondecreasing on I, where $a \in I$.

(iv) Let $f : I \to \mathbb{R}$ be a two times continuously differentiable function on I. Then f is trigonometrically ρ-convex on I if and only if for all $x \in I$ we have

$$(1.6) \qquad f''(x) + \rho^2 f(x) \geq 0.$$

For other properties of trigonometrically ρ-convex functions, see [1].

As general examples of trigonometrically ρ-convex functions we can give the indicator function

$$h_F(\theta) := \limsup_{r \to \infty} \frac{\log \left| F\left(re^{i\theta}\right) \right|}{r^\rho}, \ \theta \in (\alpha, \beta),$$

where F is an entire function of order $\rho \in (0, \infty)$.

If $0 < \beta - \alpha < \frac{\pi}{\rho}$, then, it was shown in 1908 by Phragmén and Lindelöf, see [11], that h_F is trigonometrically ρ-convex on (α, β).

Using the condition (1.6) one can also observe that any nonnegative twice differentiable and convex function on I is also trigonometrically ρ-convex on I for any $\rho > 0$.

There exists also concave functions on an interval that are trigonometrically ρ-convex on that interval for some $\rho > 0$.

Consider for example $f(x) = \cos x$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then

$$f''(x) + \rho^2 f(x) = -\cos x + \rho^2 \cos x = (\rho^2 - 1)\cos x$$

which shows that it is trigonometrically ρ-convex on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for all $\rho > 1$ and trigonometrically ρ-concave for $\rho \in (0, 1)$.

Consider the function $f : (0, \infty) \to (0, \infty)$, $f(x) = x^p$ with $p \in \mathbb{R}\backslash\{0\}$. If $p \in (-\infty, 0) \cup [1, \infty)$ the function is convex and therefore trigonometrically ρ-convex for any $\rho > 0$. If $p \in (0, 1)$ then the function is concave and

$$f''(x) + \rho^2 f(x) = \rho^2 x^p - p(1-p)x^{p-2} = \rho^2 x^{p-2}\left(x^2 - \frac{p(1-p)}{\rho^2}\right), \quad x > 0.$$

This shows that for $p \in (0, 1)$ and $\rho > 0$ the function $f(x) = x^p$ is trigonometrically ρ-convex on $\left(\frac{1}{\rho}\sqrt{p(1-p)}, \infty\right)$ and trigonometrically ρ-concave on $\left(0, \frac{1}{\rho}\sqrt{p(1-p)}\right)$.

Consider the concave function $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln x$. We observe that

$$g(x) := f''(x) + \rho^2 f(x) = \rho^2 \ln x - \frac{1}{x^2}, \quad x > 0.$$

We have

$$g'(x) = \frac{2 + \rho^2 x^2}{x^2} > 0 \text{ for } x > 0$$

and

$$\lim_{x \to 0+} g(x) = -\infty, \quad \lim_{x \to \infty} g(x) = \infty$$

showing that the function g is strictly increasing on $(0, \infty)$ and the equation $g(x) = 0$ has a unique solution. Therefore $g(x) < 0$ for $x \in (0, x_\rho)$ and $g(x) > 0$ for $x \in (x_\rho, \infty)$, where x_ρ is the unique solution of the equation $\ln x = \frac{1}{\rho^2 x^2}$.

In conclusion, if $\rho > 0$, then the function $f(x) = \ln x$ is trigonometrically ρ-concave on $(0, x_\rho)$ and trigonometrically ρ-convex on (x_ρ, ∞).

In this paper we establish several Fejér type integral inequalities for trigonometrically ρ-convex functions. Applications for special means are provided as well.

2. Some Fejér's Type Inequalities

We start with the following lemma of interest in itself:

Lemma 1. *Assume that the function $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I. Then for any a, $b \in I$ with $0 < b - a < \frac{\pi}{\rho}$ and $x \in [a, b]$ we have*

$$(2.1) \quad f\left(\frac{a+b}{2}\right)\cos\left[\rho\left(x - \frac{a+b}{2}\right)\right] \leq \frac{1}{2}\left[f(x) + f(a+b-x)\right]$$

$$\leq \frac{f(a) + f(b)}{2}\frac{\cos\left[\rho\left(x - \frac{a+b}{2}\right)\right]}{\cos\left[\frac{\rho(b-a)}{2}\right]}.$$

Proof. From (1.3) we have by replacing x with $a + b - x$ that

(2.2) $$f(a + b - x) \le \frac{\sin[\rho(x-a)]}{\sin[\rho(b-a)]} f(a) + \frac{\sin[\rho(b-x)]}{\sin[\rho(b-a)]} f(b)$$

for any $x \in [a, b]$.

If we add (1.3) with (2.2) we get

(2.3) $$f(x) + f(a + b - x)$$

$$\le \frac{\sin[\rho(b-x)]}{\sin[\rho(b-a)]} f(a) + \frac{\sin[\rho(x-a)]}{\sin[\rho(b-a)]} f(b)$$

$$+ \frac{\sin[\rho(x-a)]}{\sin[\rho(b-a)]} f(a) + \frac{\sin[\rho(b-x)]}{\sin[\rho(b-a)]} f(b)$$

$$= \frac{\sin[\rho(b-x)] + \sin[\rho(x-a)]}{\sin[\rho(b-a)]} f(a)$$

$$+ \frac{\sin[\rho(b-x)] + \sin[\rho(x-a)]}{\sin[\rho(b-a)]} f(b)$$

$$= \frac{\sin[\rho(b-x)] + \sin[\rho(x-a)]}{\sin[\rho(b-a)]} [f(a) + f(b)]$$

for any $x \in [a, b]$.

Observe that

(2.4) $$\frac{\sin[\rho(b-x)] + \sin[\rho(x-a)]}{\sin[\rho(b-a)]}$$

$$= \frac{2\sin\left[\frac{\rho(b-a)}{2}\right] \cos\left[\rho\left(x - \frac{a+b}{2}\right)\right]}{2\sin\left[\frac{\rho(b-a)}{2}\right] \cos\left[\frac{\rho(b-a)}{2}\right]} = \frac{\cos\left[\rho\left(x - \frac{a+b}{2}\right)\right]}{\cos\left[\frac{\rho(b-a)}{2}\right]}$$

for any $x \in [a, b]$.

Using the equality (2.4) and dividing by 2 in (2.3) we get the second inequality in (2.1).

From (1.4) for $t = \frac{1}{2}$ and $a = u$, $b = v$ we get

$$f\left(\frac{u+v}{2}\right) \le \frac{\sin\left[\rho\left(\frac{v-u}{2}\right)\right]}{\sin[\rho(v-u)]} f(u) + \frac{\sin\left[\rho\left(\frac{v-u}{2}\right)\right]}{\sin[\rho(v-u)]} f(v)$$

$$= \frac{\sin\left[\rho\left(\frac{v-u}{2}\right)\right]}{\sin[\rho(v-u)]} [f(u) + f(v)]$$

$$= \frac{\sin\left[\rho\left(\frac{v-u}{2}\right)\right]}{2\sin\left[\rho\left(\frac{v-u}{2}\right)\right] \cos\left[\rho\left(\frac{v-u}{2}\right)\right]} [f(u) + f(v)]$$

$$= \frac{1}{\cos\left[\rho\left(\frac{v-u}{2}\right)\right]} \frac{f(u) + f(v)}{2},$$

which implies that

(2.5) $$f\left(\frac{u+v}{2}\right) \cos\left[\rho\left(\frac{v-u}{2}\right)\right] \le \frac{f(u) + f(v)}{2}$$

for any $u, v \in I$.

Now, if we in (2.5) take $v = x$ and $u = a + b - x$, then we get the first inequality in (2.1). □

Remark 1. *By taking* $x = (1 - t)\, a + tb$ *in (2.1) we get the equivalent double inequality*

(2.6) $\quad f\left(\dfrac{a+b}{2}\right) \cos\left[\rho\left(t - \dfrac{1}{2}\right)(b - a)\right]$

$$\leq \dfrac{1}{2}\left[f\left((1 - t)\, a + tb\right) + f\left(ta + (1 - t)\, b\right)\right]$$

$$\leq \dfrac{f(a) + f(b)}{2}\, \dfrac{\cos\left[\rho\left(t - \frac{1}{2}\right)(b - a)\right]}{\cos\left[\frac{\rho(b-a)}{2}\right]}$$

for any $a,\, b \in I$ *with* $0 < |b - a| < \frac{\pi}{\rho}$ *and* $t \in [0, 1]$.

We have:

Theorem 2. *Assume that the function* $f : I \to \mathbb{R}$ *is trigonometrically* ρ-*convex on* I *and* $a,\, b \in I$ *with* $0 < b - a < \frac{\pi}{\rho}$. *Assume also that* $w : [a, b] \to \mathbb{R}$ *is a positive, symmetric and integrable function on* $[a, b]$, *then we have*

(2.7) $\quad f\left(\dfrac{a+b}{2}\right) \displaystyle\int_a^b \cos\left[\rho\left(x - \dfrac{a+b}{2}\right)\right] w(x)\, dx$

$$\leq \int_a^b f(x)\, w(x)\, dx$$

$$\leq \dfrac{f(a) + f(b)}{2} \sec\left[\dfrac{\rho(b-a)}{2}\right] \int_a^b \cos\left[\rho\left(x - \dfrac{a+b}{2}\right)\right] w(x)\, dx.$$

Proof. We multiply the inequality (2.1) by $w(x) \geq 0$ and integrate to get

(2.8) $\quad f\left(\dfrac{a+b}{2}\right) \displaystyle\int_a^b \cos\left[\rho\left(x - \dfrac{a+b}{2}\right)\right] w(x)\, dx$

$$\leq \dfrac{1}{2}\int_a^b \left[f(x) + f(a + b - x)\right] w(x)\, dx$$

$$\leq \dfrac{f(a) + f(b)}{2}\, \dfrac{\int_a^b \cos\left[\rho\left(x - \frac{a+b}{2}\right)\right] w(x)\, dx}{\cos\left[\frac{\rho(b-a)}{2}\right]}.$$

By using the change of variable $y = a + b - x$ and the symmetry of w we have

$$\int_a^b f(a + b - x)\, w(x)\, dx = \int_a^b f(y)\, w(a + b - y)\, dy = \int_a^b f(y)\, w(y)\, dy,$$

therefore

$$\dfrac{1}{2}\int_a^b \left[f(x) + f(a + b - x)\right] w(x)\, dx = \int_a^b f(x)\, w(x)\, dx$$

and by (2.8) we get (2.7). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Corollary 1. *With the assumption of Theorem 2 for* f *we have*

(2.9) $\quad (b - a)\, f\left(\dfrac{a+b}{2}\right) \leq \displaystyle\int_a^b f(x) \sec\left[\rho\left(x - \dfrac{a+b}{2}\right)\right] dx$

$$\leq \dfrac{f(a) + f(b)}{2}\, \dfrac{b - a}{\cos\left[\frac{\rho(b-a)}{2}\right]}$$

and

(2.10) $\quad \dfrac{1}{2}\left[b-a+\dfrac{1}{\rho}\sin\left[\rho\left(b-a\right)\right]\right]f\left(\dfrac{a+b}{2}\right)$

$$\leq \int_a^b f\left(x\right)\cos\left[\rho\left(x-\dfrac{a+b}{2}\right)\right]dx$$

$$\leq \dfrac{b-a+\frac{1}{\rho}\sin\left[\rho\left(b-a\right)\right]}{2\cos\left[\frac{\rho(b-a)}{2}\right]}\dfrac{f\left(a\right)+f\left(b\right)}{2}.$$

Proof. The inequality (2.9) follows by (2.7) for $w\left(x\right)=\sec\left[\rho\left(x-\frac{a+b}{2}\right)\right]$, $x\in\left[a,b\right]$.

If we take in (2.7) $w\left(x\right)=\cos\left[\rho\left(x-\frac{a+b}{2}\right)\right]$, then we get

(2.11) $\quad f\left(\dfrac{a+b}{2}\right)\displaystyle\int_a^b\cos^2\left[\rho\left(x-\dfrac{a+b}{2}\right)\right]dx$

$$\leq \int_a^b f\left(x\right)\cos\left[\rho\left(x-\dfrac{a+b}{2}\right)\right]dx$$

$$\leq \dfrac{f\left(a\right)+f\left(b\right)}{2}\dfrac{\int_a^b\cos^2\left[\rho\left(x-\frac{a+b}{2}\right)\right]dx}{\cos\left[\frac{\rho(b-a)}{2}\right]}.$$

Since

$$\int_a^b\cos^2\left[\rho\left(x-\dfrac{a+b}{2}\right)\right]dx=\dfrac{1}{2}\left[\left(b-a\right)+\int_a^b\cos\left[2\rho\left(x-\dfrac{a+b}{2}\right)\right]dx\right]$$

and

$$\int_a^b\cos\left[2\rho\left(x-\dfrac{a+b}{2}\right)\right]dx=\dfrac{1}{2\rho}\sin\left[2\rho\left(x-\dfrac{a+b}{2}\right)\right]\Big|_a^b$$

$$=\dfrac{1}{2\rho}\sin\left[2\rho\left(b-\dfrac{a+b}{2}\right)\right]-\dfrac{1}{2\rho}\sin\left[2\rho\left(a-\dfrac{a+b}{2}\right)\right]$$

$$=\dfrac{1}{\rho}\sin\left[\rho\left(b-a\right)\right],$$

hence

$$\int_a^b\cos^2\left[\rho\left(x-\dfrac{a+b}{2}\right)\right]dx=\dfrac{1}{2}\left[\left(b-a\right)+\dfrac{1}{\rho}\sin\left[\rho\left(b-a\right)\right]\right]$$

and by (2.11) we get (2.10). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 2. *Assume that the function* $f:I\rightarrow\mathbb{R}$ *is trigonometrically* ρ-*convex on* I *and* $a,b\in I$ *with* $0<b-a<\frac{\pi}{\rho}$. *Assume also that* $p:\left[a,b\right]\rightarrow\mathbb{R}$ *is a positive, symmetric and integrable function on* $\left[a,b\right]$, *then we have*

(2.12) $\quad f\left(\dfrac{a+b}{2}\right)\leq\dfrac{1}{\int_a^b p\left(x\right)dx}\displaystyle\int_a^b f\left(x\right)p\left(x\right)\sec\left[\rho\left(x-\dfrac{a+b}{2}\right)\right]dx$

$$\leq\dfrac{f\left(a\right)+f\left(b\right)}{2}\sec\left[\dfrac{\rho\left(b-a\right)}{2}\right].$$

The proof follows by Theorem 2 for the positive symmetric mean $w\left(x\right)=p\left(x\right)\sec\left[\rho\left(x-\frac{a+b}{2}\right)\right]$, $x\in\left[a,b\right]$.

3. Related Results

We have:

Theorem 3. *Assume that the function $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I and a, $b \in I$ with $0 < b - a < \frac{\pi}{\rho}$. Assume also that $w : [a, b] \to \mathbb{R}$ is a positive, integrable function on $[a, b]$ and such that $\int_a^b \cos(\rho y) w(y) \, dy \neq 0$ and*

$$(3.1) \qquad \overline{x}_\rho := \frac{1}{\rho} \arctan \left(\frac{\int_a^b \sin(\rho y) w(y) \, dy}{\int_a^b \cos(\rho y) w(y) \, dy} \right) \in [a, b],$$

then we have

$$(3.2) \qquad \int_a^b f(t) w(t) \, dt \geq f(\overline{x}_\rho) \int_a^b \cos[\rho(t - \overline{x}_\rho)] w(t) \, dt.$$

Proof. From the gradient inequality (1.5) we have, by multiplying with $w(y) \geq 0$ and integrating on $[a, b]$, we have

$$(3.3) \quad \int_a^b f(y) w(y) \, dy \geq f(x) \int_a^b \cos[\rho(y - x)] w(y) \, dy$$

$$+ K_{x,f} \int_a^b \sin[\rho(y - x)] w(y) \, dy$$

for any $x \in [a, b]$.

We have

$$\int_a^b \sin[\rho(y - x)] w(y) \, dy$$

$$= \int_a^b \sin[\rho(y - x)] w(y) \, dy$$

$$= \int_a^b [\sin(\rho y) \cos(\rho x) - \sin(\rho x) \cos(\rho y)] w(y) \, dy$$

$$= \cos(\rho x) \int_a^b \sin(\rho y) w(y) \, dy - \sin(\rho x) \int_a^b \cos(\rho y) w(y) \, dy$$

$$= \cos(\rho x) \int_a^b \cos(\rho y) w(y) \, dy \left(\frac{\int_a^b \sin(\rho y) w(y) \, dy}{\int_a^b \cos(\rho y) w(y) \, dy} - \tan(\rho x) \right)$$

for any $x \in [a, b]$.

From (3.1) we have

$$\tan(\rho \overline{x}_\rho) = \frac{\int_a^b \sin(\rho y) w(y) \, dy}{\int_a^b \cos(\rho y) w(y) \, dy},$$

which implies that

$$\int_a^b \sin[\rho(y - \overline{x}_\rho)] w(y) \, dy = 0.$$

Therefore, by taking $\overline{x}_\rho$ in (3.3) we get the desired result (3.1). $\square$

The follwing result also hold.

Theorem 4. *Assume that the function $f : I \to \mathbb{R}$ is trigonometrically ρ-convex on I and $a, b \in I$ with $0 < b - a < \frac{\pi}{\rho}$. Assume also that $w : [a, b] \to \mathbb{R}$ is a positive, integrable function on $[a, b]$, then*

$$(3.4) \quad \int_a^b f(x) w(x) \, dx$$

$$\leq \frac{f(a) + f(b)}{2} \sec \left[\frac{\rho(b-a)}{2} \right] \int_a^b \cos \left[\rho \left(x - \frac{a+b}{2} \right) \right] w(x) \, dx$$

$$+ \frac{f(b) - f(a)}{2} \csc \left[\frac{\rho(b-a)}{2} \right] \int_a^b \sin \left[\rho \left(x - \frac{a+b}{2} \right) \right] w(x) \, dx.$$

Proof. We have

$$\frac{\sin[\rho(b-x)] f(a) + \sin[\rho(x-a)] f(b)}{\sin[\rho(b-a)]}$$

$$- \frac{\sin[\rho(b-x)] + \sin[\rho(x-a)]}{\sin[\rho(b-a)]} \frac{f(a) + f(b)}{2}$$

$$= \frac{\sin[\rho(b-x)]}{\sin[\rho(b-a)]} \left(f(a) - \frac{f(a) + f(b)}{2} \right)$$

$$+ \frac{\sin[\rho(x-a)]}{\sin[\rho(b-a)]} \left(f(b) - \frac{f(a) + f(b)}{2} \right)$$

$$= \frac{f(b) - f(a)}{2} \left[\frac{\sin[\rho(x-a)] - \sin[\rho(b-x)]}{\sin[\rho(b-a)]} \right]$$

$$= \frac{f(b) - f(a)}{2} \frac{2 \sin\left[\rho\left(x - \frac{a+b}{2}\right)\right] \cos\left[\frac{\rho(b-a)}{2}\right]}{2 \sin\left[\frac{\rho(b-a)}{2}\right] \cos\left[\frac{\rho(b-a)}{2}\right]}$$

$$= \frac{f(b) - f(a)}{2} \frac{\sin\left[\rho\left(x - \frac{a+b}{2}\right)\right]}{\sin\left[\frac{\rho(b-a)}{2}\right]}$$

and

$$\frac{\sin[\rho(b-x)] + \sin[\rho(x-a)]}{\sin[\rho(b-a)]}$$

$$= \frac{2 \sin\left[\frac{\rho(b-a)}{2}\right] \cos\left[\rho\left(x - \frac{a+b}{2}\right)\right]}{2 \sin\left[\frac{\rho(b-a)}{2}\right] \cos\left[\frac{\rho(b-a)}{2}\right]} = \frac{\cos\left[\rho\left(x - \frac{a+b}{2}\right)\right]}{\cos\left[\frac{\rho(b-a)}{2}\right]}$$

for any $x \in [a, b]$.

Therefore

$$(3.5) \quad \frac{\sin[\rho(b-x)] f(a) + \sin[\rho(x-a)] f(b)}{\sin[\rho(b-a)]}$$

$$= \frac{\cos\left[\rho\left(x - \frac{a+b}{2}\right)\right]}{\cos\left[\frac{\rho(b-a)}{2}\right]} \frac{f(a) + f(b)}{2} + \frac{f(b) - f(a)}{2} \frac{\sin\left[\rho\left(x - \frac{a+b}{2}\right)\right]}{\sin\left[\frac{\rho(b-a)}{2}\right]},$$

for any $x \in [a, b]$.

If we multiply (3.5) by $w(x) \geq 0$ and integrate, then we get

$$(3.6) \quad \frac{f(a) \int_a^b \sin[\rho(b-x)] w(x) dx + f(b) \int_a^b \sin[\rho(x-a)] w(x) dx}{\sin[\rho(b-a)]}$$

$$= \frac{f(a) + f(b)}{2} \sec\left[\frac{\rho(b-a)}{2}\right] \int_a^b \cos\left[\rho\left(x - \frac{a+b}{2}\right)\right] w(x) dx$$

$$+ \frac{f(b) - f(a)}{2} \csc\left[\frac{\rho(b-a)}{2}\right] \int_a^b \sin\left[\rho\left(x - \frac{a+b}{2}\right)\right] w(x) dx.$$

Now, if we multiply the definition of trigonometrically ρ-convex functions by $w(x) \geq 0$ and integrate, then we get

$$\int_a^b f(x) w(x) dx$$

$$\leq \frac{f(a) \int_a^b \sin[\rho(b-x)] w(x) dx + f(b) \int_a^b \sin[\rho(x-a)] w(x) dx}{\sin[\rho(b-a)]}$$

and by (3.6) we obtain the desired result (3.4). $\qquad\square$

Corollary 3. *With the assumption of Theorem 4 and if*

$$(3.7) \quad \int_a^b \sin\left[\rho\left(x - \frac{a+b}{2}\right)\right] w(x) dx = 0,$$

then the second inequality in (2.7) holds.

Remark 2. *Since $g(x) = \sin\left[\rho\left(x - \frac{a+b}{2}\right)\right]$ is antisymmetric on $[a, b]$, then, if w is symmetric on $[a, b]$, the condition (3.7) holds and then the second inequality in (2.7) is valid. The condition (3.7) for the positive weight w is a more general condition for the second inequality in (2.7) to hold then the usual symmetry considered in Fejér's type inequalities.*

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