APPROXIMATING THE RIEMANN-STIELTJES INTEGRAL IN TERMS OF INTEGRAL MEANS

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Abstract. In this paper we provide among others some simple error estimates in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the use of the formula

$$\left[u(b)-\frac{1}{b-x}\int_x^b u(s)\,ds\right]f(b)+\left[\frac{1}{x-a}\int_a^x u(s)\,ds-u(a)\right]f(a)$$
$$+\left(\frac{1}{b-x}\int_x^b u(s)\,ds-\frac{1}{x-a}\int_a^x u(s)\,ds\right)f(x),$$

where $x\in(a,b)$, under various assumptions for the functions u and f and such that the involved Riemann-Stieltjes integral exists.

1. Introduction

One can approximate the *Stieltjes integral* $\int_a^b f(t)\,du(t)$ with the following simpler quantities:

$$\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt \qquad ([27],\ [28]) \tag{1.1}$$

$$f(x)[u(b)-u(a)] \qquad ([15],\ [16]) \tag{1.2}$$

or with

$$[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a) \qquad ([26]), \tag{1.3}$$

where $x\in[a,b]$.

In order to provide *a priory* sharp bounds for the *approximation error*, consider the functionals:

$$D(f,u;a,b):=\int_a^b f(t)\,du(t)-\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt,$$

$$\Theta(f,u;a,b,x):=\int_a^b f(t)\,du(t)-f(x)[u(b)-u(a)]$$

and

$$T(f,u;a,b,x):=\int_a^b f(t)\,du(t)-[u(b)-u(x)]f(b)-[u(x)-u(a)]f(a).$$

If the *integrand* f is *Riemann integrable* on $[a,b]$ and the *integrator* $u:[a,b]\to\mathbb{R}$ is $L-$*Lipschitzian*, i.e.,

$$|u(t)-u(s)|\le L|t-s| \qquad \text{for each } t,s\in[a,b], \tag{1.4}$$

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then the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and, as pointed out in [27],

$$|D(f,u;a,b)| \leq L \int_a^b \left| f(t) - \int_a^b \frac{1}{b-a} f(s)\,ds \right| dt. \tag{1.5}$$

The inequality (1.5) is sharp in the sense that the multiplicative constant $C = 1$ in front of L cannot be replaced by a smaller quantity. Moreover, if there exists the constants m, $M \in \mathbb{R}$ such that $m \leq f(t) \leq M$ for a.e. $t \in [a,b]$, then [27]

$$|D(f,u;a,b)| \leq \frac{1}{2} L (M-m)(b-a). \tag{1.6}$$

The constant $\frac{1}{2}$ is best possible in (1.6).

A different approach in the case of integrands of bounded variation were considered by the same authors in 2001, [28], where they showed that

$$|D(f,u;a,b)| \leq \max_{t\in[a,b]} \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| \bigvee_a^b (u), \tag{1.7}$$

provided that f is continuous and u is of bounded variation. Here $\bigvee_a^b (u)$ denotes the total variation of u on $[a,b]$. The inequality (1.7) is sharp.

If we assume that f is K-Lipschitzian, then [28]

$$|D(f,u;a,b)| \leq \frac{1}{2} K (b-a) \bigvee_a^b (u), \tag{1.8}$$

with $\frac{1}{2}$ the best possible constant in (1.8).

For various bounds on the error functional $D(f,u;a,b)$ where f and u belong to different classes of function for which the Stieltjes integral exists, see [22], [21], [20], and [8] and the references therein.

Bounds for the functional $\Theta(f,u;a,b,x)$ can be found in [15], [16] and [8], while for the functional $T(f,u;a,b,x)$ they may be found in [26], [8], [3] and [2]. The details are omitted.

Motivated by the above results, in this paper we provide some simple ways to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the use of integral means, namely we establish bounds for the error functionals

$$\begin{aligned} &DM(f,u,a,b,x) \\ &:= \left[u(b) - \frac{1}{b-x}\int_x^b u(s)\,ds\right] f(b) + \left[\frac{1}{x-a}\int_a^x u(s)\,ds - u(a)\right] f(a) \\ &\quad + \left(\frac{1}{b-x}\int_x^b u(s)\,ds - \frac{1}{x-a}\int_a^x u(s)\,ds\right) f(x) - \int_a^b f(t)\,du(t) \end{aligned} \tag{1.9}$$

where $x \in (a,b)$ and

$$\begin{aligned} &M(f,u,a,b) \\ &:= \left[u(b) - \frac{1}{b-a}\int_a^b u(s)\,ds\right] f(b) + \left[\frac{1}{b-a}\int_a^b u(s)\,ds - u(a)\right] f(a) \end{aligned} \tag{1.10}$$

under various assumptions for the functions u and f and such that the involved Riemann-Stieltjes integral exists.

2. The Main Results

We have:

Lemma 1. *Let* $f,\, u : [a,b] \to \mathbb{C}$ *and* $x \in [a,b]$ *such that* $f \in \mathcal{RS}_u[a,x] \cap \mathcal{RS}_u[x,b]$. *Then for any* $\gamma,\, \mu \in \mathbb{C}$,

$$[u(b)-\mu] f(b) + [\gamma - u(a)] f(a) + (\mu-\gamma) f(x) - \int_a^b f(t)\, du(t) = \int_a^x [u(t)-\gamma]\, df(t) + \int_x^b [u(t)-\mu]\, df(t). \tag{2.1}$$

In particular, for $\mu = \gamma$ *we have*

$$[u(b)-\gamma] f(b) + [\gamma - u(a)] f(a) - \int_a^b f(t)\, du(t) = \int_a^b [u(t)-\gamma]\, df(t). \tag{2.2}$$

Proof. Using integration by parts rule for the Riemann-Stieltjes integral, we have

$$\int_a^x [u(t)-\gamma]\, df(t) = [u(x)-\gamma] f(x) - [u(a)-\gamma] f(a) - \int_a^x f(t)\, du(t)$$

and

$$\int_x^b [u(t)-\mu]\, df(t) = [u(b)-\mu] f(b) - [u(x)-\mu] f(x) - \int_x^b f(t)\, du(t)$$

for any $x \in [a,b]$.

If we add these two equalities, we get

$$\begin{aligned}
&\int_a^x [u(t)-\gamma]\, df(t) + \int_x^b [u(t)-\mu]\, df(t) \\
&\quad = [u(b)-\mu] f(b) + [\gamma - u(a)] f(a) + [\mu - u(x)] f(x) \\
&\qquad + [u(x)-\gamma] f(x) - \int_a^x f(t)\, du(t) - \int_x^b f(t)\, du(t) \\
&\quad = [u(b)-\mu] f(b) + [\gamma - u(a)] f(a) + (\mu-\gamma) f(x) - \int_a^b f(t)\, du(t)
\end{aligned}$$

for any $x \in [a,b]$, which proves the desired equality (2.1). □

If in (2.1) we chose $\gamma = \frac{1}{x-a}\int_a^x u(s)\, ds$ and $\mu = \frac{1}{b-x}\int_x^b u(s)\, ds$, we get

$$\begin{aligned}
&DM(f,u,a,b,x) \\
&= \int_a^x \left[u(t) - \frac{1}{x-a}\int_a^x u(s)\, ds\right] df(t) + \int_x^b \left[u(t) - \frac{1}{b-x}\int_x^b u(s)\, ds\right] df(t),
\end{aligned} \tag{2.3}$$

for $x \in (a,b)$.

If we take $x=\frac{a+b}{2}$ in (2.3), then we get

$$
\begin{aligned}
(2.4)\quad & DM\left(f,u,a,b,\frac{a+b}{2}\right) \\
&=\left[u(b)-\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}u(s)\,ds\right]f(b)+\left[\frac{2}{b-a}\int_{a}^{\frac{a+b}{2}}u(s)\,ds-u(a)\right]f(a) \\
&\quad+\frac{2}{b-a}f\left(\frac{a+b}{2}\right)\int_{b}^{b}\operatorname{sgn}\left(s-\frac{a+b}{2}\right)u(s)\,ds-\int_{a}^{b}f(t)\,du(t) \\
&=\int_{a}^{x}\left[u(t)-\frac{2}{b-a}\int_{a}^{\frac{a+b}{2}}u(s)\,ds\right]df(t)+\int_{\frac{a+b}{2}}^{b}\left[u(t)-\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}u(s)\,ds\right]df(t).
\end{aligned}
$$

If we take in (2.2) $\gamma=\frac{1}{b-a}\int_a^b u(s)\,ds$ then we get

$$
M(f,u,a,b)=\int_{a}^{b}\left[u(t)-\frac{1}{b-a}\int_{a}^{b}u(s)\,ds\right]df(t). \tag{2.5}
$$

We have:

Theorem 1. *Assume that $u:[a,b]\to\mathbb{C}$ is continuous and $f:[a,b]\to\mathbb{C}$ is of bounded variation. Then*

$$
\begin{aligned}
(2.6)\quad |M(f,u,a,b)| &\le \int_{a}^{b}\left|u(t)-\frac{1}{b-a}\int_{a}^{b}u(s)\,ds\right|d\left(\bigvee_{a}^{t}(f)\right) \\
&\le\begin{cases}\max_{t\in[a,b]}\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right|\bigvee_{a}^{b}(f), \\[2ex] \left(\int_a^b\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right|^{p}d\left(\bigvee_{a}^{t}(f)\right)\right)^{1/p}\left(\bigvee_{a}^{b}(f)\right)^{1/q}\end{cases}
\end{aligned}
$$

for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. It is well known that, if $p:[a,b]\to\mathbb{C}$ is continuous and $v:[a,b]\to\mathbb{C}$ of bounded variation, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and [1]

$$
\left|\int_{a}^{b}p(t)\,dv(t)\right|\le\int_{a}^{b}|p(t)|\,d\left(\bigvee_{a}^{t}(v)\right)\le\max_{t\in[a,b]}|p(t)|\bigvee_{a}^{b}(v). \tag{2.7}
$$

Using (2.5) and (2.7) we get

$$
\begin{aligned}
|M(f,u,a,b)| &= \left|\int_{a}^{b}\left[u(t)-\frac{1}{b-a}\int_{a}^{b}u(s)\,ds\right]df(t)\right| \\
&\le\int_{a}^{b}\left|u(t)-\frac{1}{b-a}\int_{a}^{b}u(s)\,ds\right|d\left(\bigvee_{a}^{t}(f)\right).
\end{aligned}
$$

Using the Hölder inequality for monotonic integrators, we have

$$
\begin{aligned}
&\int_a^b \left| u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds \right| d\left(\bigvee_a^t (f)\right) \\
&\leq \begin{cases} \max_{t\in[a,b]} \left| u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds \right| \bigvee_a^b (f) \\[2ex] \left(\int_a^b \left| u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds \right|^p d\left(\bigvee_a^t (f)\right)\right)^{1/p} \left(\int_a^b d\left(\bigvee_a^t (f)\right)\right)^{1/q} \end{cases} \\
&= \begin{cases} \max_{t\in[a,b]} \left| u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds \right| \bigvee_a^b (f) \\[2ex] \left(\int_a^b \left| u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds \right|^p d\left(\bigvee_a^t (f)\right)\right)^{1/p} \left(\bigvee_a^b (f)\right)^{1/q} \end{cases}
\end{aligned}
$$

for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. □

We also have:

Theorem 2. *Assume that $u : [a,b] \to \mathbb{C}$ is Riemann integrable and $f : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, namely*

$$
|f(t) - f(s)| \leq L\,|t-s| \quad \textit{for all } t,\ s \in [a,b]. \tag{2.8}
$$

Then

$$
\begin{aligned}
|M(f,u,a,b)| &\leq L \int_a^b \left| u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds \right| dt \\
&\leq L \times \begin{cases} (b-a)\max_{t\in[a,b]} \left| u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds \right|, \\[2ex] (b-a)^{1/q} \left(\int_a^b \left| u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds \right|^p dt\right)^{1/p} \end{cases}
\end{aligned} \tag{2.9}
$$

for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, we have

$$
\begin{aligned}
|M(f,u,a,b)| &\leq L \int_a^b \left| u(t) - \frac{1}{b-a}\int_a^b u(s)\,ds \right| dt \\
&\leq L(b-a)\left(\frac{1}{b-a}\int_a^b |u(t)|^2\,dt - \left|\frac{1}{b-a}\int_a^b u(s)\,ds\right|^2\right)^{1/2}.
\end{aligned} \tag{2.10}
$$

Proof. It is well known that, if $p : [a,b] \to \mathbb{C}$ is Riemann integrable and $v : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$, then the Riemann-Stieltjes integral $\int_a^b p(t)\,dv(t)$ exists and

$$
\left| \int_a^b p(t)\,dv(t) \right| \leq L \int_a^b |p(t)|\,dt. \tag{2.11}
$$

Using (2.5) and (2.11) we get

$$\begin{aligned}|M(f,u,a,b)| &= \left|\int_a^b \left[u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right] df(t)\right| \\ &\le L\int_a^b \left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right| dt.\end{aligned}$$

By Hölder's inequality we also have

$$\begin{aligned}&\int_a^b \left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right| dt \\ &\le \begin{cases} (b-a)\max_{t\in[a,b]}\left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right|, \\ \\ (b-a)^{1/q}\left(\int_a^b \left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right|^p dt\right)^{1/p}\end{cases}\end{aligned}$$

for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

For $p=2$ we get

$$\begin{aligned}&\left(\frac{1}{b-a}\int_a^b \left|u(t)-\frac{1}{b-a}\int_a^b u(s)\,ds\right|^2 dt\right)^{1/2} \\ &=\left(\frac{1}{b-a}\int_a^b |u(t)|^2\,dt-\left|\frac{1}{b-a}\int_a^b u(s)\,ds\right|^2\right)^{1/2},\end{aligned}$$

which proves (2.10). □

Now, some bounds for the functional $DM(f,u,a,b,x)$.

Theorem 3. *Assume that* $u:[a,b]\to\mathbb{C}$ *is continuous and* $f:[a,b]\to\mathbb{C}$ *is of bounded variation. Then for all* $x\in(a,b)$

$$\begin{aligned}&|DM(f,u,a,b,x)| \\ &\le \begin{cases} \max_{t\in[a,x]}\left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|\bigvee_a^x(f), \\ \\ \left(\int_a^x \left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|^p d\left(\bigvee_a^t(f)\right)\right)^{1/p}\left(\bigvee_a^x(f)\right)^{1/q}\end{cases} \\ &\quad+\begin{cases} \max_{t\in[x,b]}\left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|\bigvee_x^b(f), \\ \\ \left(\int_x^b \left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|^p d\left(\bigvee_x^t(f)\right)\right)^{1/p}\left(\bigvee_x^b(f)\right)^{1/q}.\end{cases}\end{aligned} \tag{2.12}$$

$$\leq \begin{cases} \max\left\{\max_{t\in[a,x]}\left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|,\max_{t\in[x,b]}\left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|\right\} \\ \times \bigvee_a^b (f), \\ \\ \left(\int_a^x \left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|^p d\left(\bigvee_a^t (f)\right)+\int_x^b \left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|^p d\left(\bigvee_x^t (f)\right)\right)^{1/p} \\ \times \left(\bigvee_a^b (f)\right)^{1/q} \end{cases}$$

for p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using the representation (2.3) we have

$$|DM(f,u,a,b,x)| \leq \left|\int_a^x \left[u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right]df(t)\right| + \left|\int_x^b \left[u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right]df(t)\right| \leq \int_a^x \left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right| d\left(\bigvee_a^t (f)\right) + \int_x^b \left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right| d\left(\bigvee_x^t (f)\right) \tag{2.13}$$

$$\leq \begin{cases} \max_{t\in[a,x]}\left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|\bigvee_a^x (f), \\ \\ \left(\int_a^x \left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|^p d\left(\bigvee_a^t (f)\right)\right)^{1/p}\left(\bigvee_a^x (f)\right)^{1/q} \end{cases} + \begin{cases} \max_{t\in[x,b]}\left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|\bigvee_x^b (f), \\ \\ \left(\int_x^b \left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|^p d\left(\bigvee_x^t (f)\right)\right)^{1/p}\left(\bigvee_x^b (f)\right)^{1/q} \end{cases}.$$

Now, observe that

$$\max_{t\in[a,x]}\left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|\bigvee_a^x (f)+\max_{t\in[x,b]}\left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|\bigvee_x^b (f)$$
$$\leq \max\left\{\max_{t\in[a,x]}\left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|,\max_{t\in[x,b]}\left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|\right\}\bigvee_a^b (f)$$

and, by the elementary inequality

$$\alpha\beta+\gamma\delta \leq (\alpha^p+\gamma^p)^{1/p}(\beta^q+\delta^q)^{1/q}$$

where α, β, $\gamma, \delta \geq 0$ and p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\left(\int_a^x \left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|^p d\left(\bigvee_a^t(f)\right)\right)^{1/p}\left(\bigvee_a^x(f)\right)^{1/q} + \left(\int_x^b \left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|^p d\left(\bigvee_x^t(f)\right)\right)^{1/p}\left(\bigvee_x^b(f)\right)^{1/q}$$

$$\leq \left\{\int_a^x \left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|^p d\left(\bigvee_a^t(f)\right) + \int_x^b \left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|^p d\left(\bigvee_x^t(f)\right)\right\}^{1/p} \times \left(\bigvee_a^x(f)+\bigvee_x^b(f)\right)^{1/q}$$

$$= \left\{\int_a^x \left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|^p d\left(\bigvee_a^t(f)\right) + \int_x^b \left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|^p d\left(\bigvee_x^t(f)\right)\right\}^{1/p} \times \left(\bigvee_a^b(f)\right)^{1/q}.$$

This prove the desired result (2.12). □

We also have:

Theorem 4. *Assume that $u:[a,b]\to\mathbb{C}$ is Riemann integrable and $f:[a,b]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$. Then*

$$|DM(f,u,a,b,x)| \tag{2.14}$$

$$\leq L\times\begin{cases} \max_{t\in[a,x]}\left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|(x-a), \\ \\ \left(\int_a^x \left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|^p dt\right)^{1/p}(x-a)^{1/q} \end{cases}$$

$$+L\times\begin{cases} \max_{t\in[x,b]}\left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|(b-x), \\ \\ \left(\int_x^b \left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|^p dt\right)^{1/p}(b-x)^{1/q}. \end{cases}$$

$$\leq L\times\begin{cases} \max\left\{\max_{t\in[a,x]}\left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|,\max_{t\in[x,b]}\left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|\right\} \\ \times(b-a), \\ \\ \left(\int_a^x \left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|^p dt+\int_x^b \left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|^p dt\right)^{1/p} \\ \times(b-a)^{1/q} \end{cases}$$

for all $x\in(a,b)$, where p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

The proof is similar to the ones from Theorems 2 and 3 and we omit the details.

Remark 1. *For $p=q=2$ we get from (2.14) that*

$$|DM(f,u,a,b,x)| \tag{2.15}$$
$$\leq L(x-a)\left(\frac{1}{x-a}\int_a^x \left|u(t)-\frac{1}{x-a}\int_a^x u(s)\,ds\right|^2 dt\right)^{1/2}$$
$$+L(b-x)\left(\frac{1}{b-x}\int_x^b \left|u(t)-\frac{1}{b-x}\int_x^b u(s)\,ds\right|^2 dt\right)^{1/2}$$
$$=L(x-a)\left(\frac{1}{x-a}\int_a^x |u(t)|^2\,dt-\left|\frac{1}{x-a}\int_a^x u(s)\,ds\right|^2\right)^{1/2}$$
$$+L(b-x)\left(\frac{1}{b-x}\int_x^b |u(t)|^2\,dt-\left|\frac{1}{b-x}\int_x^b u(s)\,ds\right|^2\right)^{1/2}$$

for all $x\in(a,b)$.

3. Some Further Bounds

In [14] we proved amongst other that, if $g:[a,b]\to\mathbb{C}$ is of bounded variation, then

$$\left[\frac{1}{b-a}\int_a^b |g(t)|^2\,dt-\left|\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2\right]^{1/2}\leq\frac{1}{2}\bigvee_a^b(g) \tag{3.1}$$

with the constant $\frac{1}{2}$ as best possible.

Now, for γ, $\Gamma\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma):=\left\{g:[a,b]\to\mathbb{C}|\operatorname{Re}\left[(\Gamma-g(t))\left(\overline{g(t)}-\overline{\gamma}\right)\right]\geq 0\ \text{ for each }\ t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma):=\left\{g:[a,b]\to\mathbb{C}|\ \left|g(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|\ \text{ for each }\ t\in[a,b]\right\}.$$

This family of functions is a particular case of the class introduced in [23]

$$\bar{\Delta}_{[a,b],g}(\gamma,\Gamma)$$
$$:=\left\{\ g:[a,b]\to\mathbb{C}|\ \left|g(t)-\frac{\gamma+\Gamma}{2}g(t)\right|\leq\frac{1}{2}|\Gamma-\gamma|\,|g(t)|\ \text{ for each }\ t\in[a,b]\right\},$$

where $g:[a,b]\to\mathbb{C}$.

The following representation result may be stated.

Proposition 1. *For any $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, we have that $\bar{U}_{[a,b]}(\gamma,\Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{3.2}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma + \Gamma}{2} \right| \le \frac{1}{2} |\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma}) \right] \ge 0.$$

This follows by the equality

$$\frac{1}{4} |\Gamma - \gamma|^2 - \left| z - \frac{\gamma + \Gamma}{2} \right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma}) \right]$$

that holds for any $z \in \mathbb{C}$.

The equality (3.2) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that*

$$\begin{aligned} \bar{U}_{[a,b]}(\gamma, \Gamma) = \{ g : [a,b] \to \mathbb{C} \mid & \ (\operatorname{Re}\Gamma - \operatorname{Re} g(t))(\operatorname{Re} g(t) - \operatorname{Re}\gamma) \\ & + (\operatorname{Im}\Gamma - \operatorname{Im} g(t))(\operatorname{Im} g(t) - \operatorname{Im}\gamma) \ge 0 \text{ for each } t \in [a,b] \}. \end{aligned}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \ge \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \ge \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned} \bar{S}_{[a,b]}(\gamma, \Gamma) := \{ g : [a,b] \to \mathbb{C} \mid & \ \operatorname{Re}(\Gamma) \ge \operatorname{Re} g(t) \ge \operatorname{Re}(\gamma) \\ & \text{and } \operatorname{Im}(\Gamma) \ge \operatorname{Im} g(t) \ge \operatorname{Im}(\gamma) \text{ for each } t \in [a,b] \}. \end{aligned}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma, \Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma, \Gamma) \subseteq \bar{U}_{[a,b]}(\gamma, \Gamma).$$

If $g \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ is square integrable on $[a,b]$, then we have the following Grüss type inequality (see for instance [13])

$$\left[\frac{1}{b-a} \int_a^b |g(t)|^2 dt - \left| \frac{1}{b-a} \int_a^b g(s)\, ds \right|^2 \right]^{1/2} \le \frac{1}{2} |\Gamma - \gamma|. \tag{3.3}$$

Proposition 2. *Assume that $u : [a,b] \to \mathbb{C}$ is of bounded variation and $f : [a,b] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$.*

We have

$$|M(f,u,a,b)| \le \frac{1}{2} L (b-a) \bigvee_a^b (u). \tag{3.4}$$

If $x \in (a,b)$, then

$$\begin{aligned} |DM(f,u,a,b,x)| &\le \frac{1}{2} L \left[(x-a) \bigvee_a^x (u) + (b-x) \bigvee_x^b (u) \right] \\ &\le \frac{1}{2} L \begin{cases} \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (u) \\ \\ \frac{1}{2} \left[\bigvee_a^b (u) + \left| \bigvee_a^x (u) - \bigvee_x^b (u) \right| \right] (b-a) \end{cases} \le \frac{1}{2} L (b-a) \bigvee_a^b (u). \end{aligned} \tag{3.5}$$

In particular,

$$\left|DM\left(f,u,a,b,\frac{a+b}{2}\right)\right|\leq\frac{1}{4}L\left(b-a\right)\bigvee_{a}^{b}\left(u\right). \tag{3.6}$$

If $p\in[a,b]$ *is such that* $\bigvee_{a}^{p}(u)=\bigvee_{p}^{b}(u)$, *then also*

$$\left|DM\left(f,u,a,b,p\right)\right|\leq\frac{1}{4}L\left(b-a\right)\bigvee_{a}^{b}\left(u\right). \tag{3.7}$$

The proof follows by the inequalities (2.10), (2.15) and (3.1).

Proposition 3. *Assume that* $u:[a,b]\to\mathbb{C}$ *is Riemann integrable and there exists* $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, *such that* $u\in\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ *and* $f:[a,b]\to\mathbb{C}$ *is Lipschitzian with the constant* $L>0$. *Then*

$$\left|M\left(f,u,a,b\right)\right|\leq\frac{1}{2}L\left(b-a\right)\left|\Gamma-\gamma\right|. \tag{3.8}$$

If $x\in(a,b)$ *and* $u\in\bar{\Delta}_{[a,x]}(\gamma_1,\Gamma_1)\cap\bar{\Delta}_{[x,b]}(\gamma_2,\Gamma_2)$, *then*

$$\begin{aligned}\left|DM\left(f,u,a,b,x\right)\right| &\leq\frac{1}{2}L\left[\left(x-a\right)\left|\Gamma_1-\gamma_1\right|+\left(b-x\right)\left|\Gamma_2-\gamma_2\right|\right]\\ &\leq\frac{1}{2}L\begin{cases}\left[\frac{1}{2}\left(b-a\right)+\left|x-\frac{a+b}{2}\right|\right]\left[\left|\Gamma_1-\gamma_1\right|+\left|\Gamma_2-\gamma_2\right|\right],\\ \max\left\{\left|\Gamma_1-\gamma_1\right|,\left|\Gamma_2-\gamma_2\right|\right\}\left(b-a\right).\end{cases}\end{aligned} \tag{3.9}$$

In particular, if $x=\frac{a+b}{2}$, *then*

$$\left|DM\left(f,u,a,b,\frac{a+b}{2}\right)\right|\leq\frac{1}{4}L\left[\left|\Gamma_1-\gamma_1\right|+\left|\Gamma_2-\gamma_2\right|\right]\left(b-a\right). \tag{3.10}$$

The proof follows by the inequalities (2.10), (2.15) and (3.3).

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